Semidefinite programming
— an introduction

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Exercise 1

Consider the positive semidefinite matrix:

\[ A = \begin{pmatrix}
  a_{11} & \cdots & a_{1n} \\
  \vdots & \ddots & \vdots \\
  a_{n1} & \cdots & a_{nn}
\end{pmatrix} \succeq 0 \]

- show that \( a_{ii} \geq 0 \)
- show that if \( a_{ii} = 0 \) then \( a_{ij} = 0 \) for all \( j = 1, \ldots, n \)

Exercise 2

A cone \( \mathcal{K} \) is an object satisfying:

\( \forall x \in \mathcal{K}, \lambda \geq 0 : \lambda x \in \mathcal{K} \)

- Prove that the set of positive semidefinite matrices is a cone.

For a given inner product \( \langle \cdot, \cdot \rangle \) the dual cone of \( \mathcal{K} \) is given by

\[ \mathcal{K}^* = \{ s \in \mathbb{R}^n : \langle s, x \rangle \geq 0, \forall x \in \mathcal{K} \} \]

A cone \( \mathcal{K} \) is self-dual if an inner product can be chosen such that \( \mathcal{K} = \mathcal{K}^* \)

- Show that the set of positive semidefinite matrices is a self-dual cone

Exercise 3

The dense subgraph problem is defined as follows: Given a weighted graph \( G = (V, E, c) \) and a positive integer \( k \). Choose a subset \( U \subseteq V \) of the nodes, with \( |U| = k \) such that the sum of the edge weights in the subgraph spanned by \( U \) is maximized. The problem can formally be defined as follows:

\[
  z = \max \left\{ \sum_{i \in U} \sum_{j \in U} c_{ij} : U \subseteq V, |U| = k \right\}
\]  

(1)

Formulate the dense subgraph problem as a semidefinite optimization problem. Consider various improved formulations.