VA 2006

- Forelæsninger: tirsdage 13-15, fredage 10-12 i Lille UP1
- Øvelser: 3 hold, tirsdage og fredage (½ uge forskudt)
- Hjemmeside
Linear Programming

- **Tirsdag, 14/11:** *Introduction to LP*, afsnit 29.1 og 29.2
  - Opgaver til 1. øvelsesgang: 29.1-1,-2,-3,4,-5,-6,-7,-8,-9, 29.2-1,-2,-3, BBB (hjemmeopgave, udleveres ved 1. forelæsning, afleveres den 24/11, *ej obligatorisk*)

- **Fredag, 17/11:** *SIMPLEX algorithm*, afsnit 29.3 og 29.5
  - Opgaver til 2. øvelsesgang: 29.3-2,-3,-4,-5,-6, 29.5-3,-4,-5.

- **Tirsdag, 21/11:** *Duality*, afsnit 29.4
  - Opgaver til 3. øvelsesgang: 29.4-1, 29.4-5, 29-2, VA-P2 (hjemmeopgave, udleveres ved 3. forelæsning, afleveres den 1/12, *ej obligatorisk*).
Introduction

- Linear Programming by Example
- Geometric Interpratation
- Linear Programming – Brief History
- Standard and Slack Formulations
- SIMPLEX by Example
Diet Problem (after Chvatal)

• Every day Polly needs:
  – 2000 kcal,
  – 55g protein,
  – 800mg calcium.

• She will get other stuff (e.g., iron and vitamins) by taking pills. Not that this could not be included in the model – we just want to keep it simple.

• She wants a diet that will meet the requirements while being neither expensive nor boring.
# Value and Price per Serving

<table>
<thead>
<tr>
<th>Food</th>
<th>Energy (kcal)</th>
<th>Protein (g)</th>
<th>Calcium (mg)</th>
<th>Price per serving</th>
</tr>
</thead>
<tbody>
<tr>
<td>Oatmeal</td>
<td>110</td>
<td>4</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>Chicken</td>
<td>205</td>
<td>32</td>
<td>12</td>
<td>24</td>
</tr>
<tr>
<td>Eggs</td>
<td>160</td>
<td>13</td>
<td>54</td>
<td>13</td>
</tr>
<tr>
<td>Whole milk</td>
<td>160</td>
<td>8</td>
<td>285</td>
<td>9</td>
</tr>
<tr>
<td>Cherry pie</td>
<td>420</td>
<td>4</td>
<td>22</td>
<td>20</td>
</tr>
<tr>
<td>Pork with beans</td>
<td>260</td>
<td>14</td>
<td>80</td>
<td>19</td>
</tr>
</tbody>
</table>

10 portions of pork with beans would cover her needs! And would cost only 190. But ...
Limits to What Polly Can Stomach

- Oatmeal: at most 4 servings a day.
- Chicken: at most 3 servings a day.
- Eggs: at most 2 servings a day.
- Milk: at most 8 servings a day.
- Cherry pie: at most 2 servings a day.
- Pork with beans: at most 2 servings a day.

8 servings of milk and 2 servings of cherry pie would meet her needs. Boring but she could stomach it. Especially since it would cost 112. Can she find a less expensive diet?
Variables

- $X_1$: number of oatmeal servings.
- $X_2$: number of chicken servings.
- $X_3$: number of eggs servings.
- $X_4$: number of milk servings.
- $X_5$: number of cherry pie servings.
- $X_6$: number of pork and pie servings.
Linear Constraints

\[ x_1, \quad x_2, \quad x_3, \quad x_4, \quad x_5, \quad x_6 \geq 0 \]

\[ \begin{align*}
    x_1 & \leq 4 \\
    x_2 & \leq 3 \\
    x_3 & \leq 2 \\
    x_4 & \leq 8 \\
    x_5 & \leq 2 \\
    x_6 & \leq 2
\end{align*} \]

\[ \begin{align*}
    110x_1 + 205x_2 + 160x_3 + 160x_4 + 420x_5 + 260x_6 & \geq 2000 \\
    4x_1 + 32x_2 + 13x_3 + 8x_4 + 4x_5 + 14x_6 & \geq 55 \\
    2x_1 + 12x_2 + 54x_3 + 285x_4 + 22x_5 + 80x_6 & \geq 800
\end{align*} \]
Linear Programming Problem

\begin{align*}
\text{min} \quad & 3x_1 + 24x_2 + 13x_3 + 9x_4 + 20x_5 + 19x_6 \\
\text{s.t.} \quad & x_1 \leq 4 \\
& x_2 \leq 3 \\
& x_3 \leq 2 \\
& x_4 \leq 8 \\
& x_5 \leq 2 \\
& x_6 \leq 2 \\
& 110x_1 + 205x_2 + 160x_3 + 160x_4 + 420x_5 + 260x_6 \geq 2000 \\
& 4x_1 + 32x_2 + 13x_3 + 8x_4 + 4x_5 + 14x_6 \geq 55 \\
& 2x_1 + 12x_2 + 54x_3 + 285x_4 + 22x_5 + 80x_6 \geq 800 \\
& x_1, x_2, x_3, x_4, x_5, x_6 \geq 0
\end{align*}
Linear Objective Function

\[
\begin{align*}
\text{min} & \quad 3x_1 + 24x_2 + 13x_3 + 9x_4 + 20x_5 + 19x_6 \\
\text{s.t.} & \quad \text{linear constraints}
\end{align*}
\]

- The value of the objective function for a particular set of values for \(x_1, x_2, x_3, x_4, x_5, x_6\) is called its **objective value**.

- If a particular set of values for \(x_1, x_2, x_3, x_4, x_5, x_6\) satisfies all constraints, it is said to be a **feasible solution** (dansk: tillad løsning). The set of all feasible solutions is called the **feasible region** (dansk: tilladt område). It can be shown to be convex.

- A feasible solution that has the maximum (or minimum) objective value is called an **optimal solution**.
General LP Problem

\[ \min \sum_{j=1}^{n} c_j x_j \]
\[ \text{s.t.} \quad m \text{ linear constraints} \]

- Minimization or maximization of a linear objective function with \( n \) real-valued variables.
- An optimal solution must satisfy \( m \) linear constraints (inequalities or equalities).
- Strict inequalities are not allowed.
- "programming" in "linear programming" does not refer to any code. It was chosen before computer programming was born.
Geometric Interpretation

\[ \begin{align*}
\text{max} & \quad x_1 + x_2 \\
\text{s.t.} & \quad 4x_1 - x_2 \leq 8 \\
& \quad 2x_1 + x_2 \leq 10 \\
& \quad 5x_1 - 2x_2 \geq -2 \\
& \quad x_1, x_2 \geq 0
\end{align*} \]
Geometric Interpretation

\[ \text{max} \quad x_1 + x_2 \]
\[ \text{s.t.} \quad 4x_1 - x_2 \leq 8 \]
\[ 2x_1 + x_2 \leq 10 \]
\[ 5x_1 - 2x_2 \geq -2 \]
\[ x_1, \quad x_2 \geq 0 \]
Special Cases of LP

- LP may have no feasible solution (in case of conflicting constraints).
- LP may have feasible solutions but no optimal solution (in case of unboundedness).
- LP may have more than one optimal solution.
Geometric Interpretation in $\mathbb{R}^3$

- 3 variables.
- Each constraint defines a half-space in $\mathbb{R}^3$. The set of feasible solutions is the intersection of these half-spaces. It is convex. Can be unbounded or empty.
- The set of points in which the objective function has the same value $z$ is a plane.
- The value of the objective function increases as the plane is translated in one normal direction and it decreases as it is translated in the other normal direction.
- If the set of feasible solutions is bounded and not empty, then there is an optimal solution in an extreme vertex of the convex set of feasible solutions.
Geometric Interpretation in $\mathbb{R}^d$

- $d$ variables.
- Each constraint defines a half-space in $\mathbb{R}^d$. The set of feasible solutions is the intersection of these half-spaces, called **simplex**. It is convex. Can be unbounded or empty.
- The set of points in which the objective function has the same value $z$ is a **hyperplane**.
- The value of the objective function increases or decreases as the hyperplane is translated.
- If the set of feasible solutions is bounded and not empty, then there is an optimal solution in an extreme vertex of the simplex.
General Idea Behind SIMPLEX Algorithm

- SIMPLEX starts with a feasible solution corresponding to some vertex of the simplex. We will show how to find such a vertex (or decide that the feasible region is empty).
- SIMPLEX keeps "jumping" from a vertex of the simplex to a new vertex if the new vertex offers a feasible solution that is better (or at least not worse). We will show how SIMPLEX "jumps".
- When no more "jumps" are possible, we will show that SIMPLEX is in an optimal vertex (or the LP is unbounded).
History of LP

- L.V. Kantorovich pointed out in 1939 the importance of restricted classes of LPs.
- T.C. Koopmans realized in 1947 the importance of LP for the analysis of classical economic theories.
- G.B. Dantzig designed in 1947 the simplex method to solve LP for U.S. Air Force. Not a polynomial algorithm!
- Many applications followed over the years.
- In 1975 Kantorovich and Koopmans got the Nobel prize.
Applications of LP

• Scheduling problems: airline wishes to schedule its flight crews on all flights while using as few crew members as possible.

• Location problems: Locating drills to maximize the amount of oil that will be extracted under given budget constraints.

• Many network and graph problems can be formulated as LP.

• Integer programming problems.
Shortest Path as LP Problem

- **Given**: Weighted, directed graph $G = (V,E)$ with real-valued weights $w(u,v)$ on all edges $e=(u,v)$ in $E$, a source vertex $s$ and a destination vertex $t$.

- **Find**: Shortest distance $d[t]$ from $s$ to $t$.

- Bellman-Ford algorithm: $d[t]$ is the shortest distance from $s$ to $t$ if and only if
  - no edge $e=(u,v)$ can be relaxed: $d[v] \leq d[u] + w(u,v)$
  - $d[s] = 0$. 
Shortest Path as LP

- maximize $d[t]$

  subject to

  $d[v] \leq d[u] + w(u,v), \forall (u,v) \in E.$

  $d[s] = 0.$

- In particular, at least one of the constraints $d[t] \leq d[u] + w(u,t), (u,t) \in E$ must be tight. So we have to maximize $d[t]$

  $$\max x_t$$

  $$s.t. \quad x_v \leq x_u + w_{uv}, \forall (u,v) \in E$$

  $$x_s = 0$$
Maximum Flow

- **Given**: A directed graph $G = (V,E)$ where each edge $(u,v) \in E$ has a real-valued, nonnegative capacity $c(u,v)$, a source vertex $s$ and a destination vertex $t$.

- **Find**: A maximum flow $f: V \times V \rightarrow \mathbb{R}$ from $s$ to $t$
Flow

• **Given**: A directed graph $G = (V, E)$ where each edge $(u, v) \in E$ has a real-valued, nonnegative **capacity** $c(u, v)$, a **source** vertex $s$ and a **destination** vertex $t$.

• A **flow** from $s$ to $t$ in $G$ is a real-valued function $f: V \times V \rightarrow \mathbb{R}$ satisfying:
  - Capacity constraints: $f(u, v) \leq c(u, v)$, $\forall u, v \in V$.
  - Skew symmetry: $f(u, v) = -f(v, u)$, $\forall u, v \in V$.
  - Flow conservation: $\sum_{v \in V} f(u, v) = 0$, $\forall u \in V \setminus \{s, t\}$

• Flow **value** $|f|$ is defined as

$$\sum_{v \in V} f(s, v)$$
Equivalent LPs

- Two maximization LPs $L$ and $L'$ are equivalent iff for each feasible solution $x$ to $L$ with the objective value $z$, there is a corresponding feasible solution $x'$ to $L'$ with the same objective value $z$, and vice versa.

- Similarly for two minimization LPs.

- A minimization LP $L$ and a maximization LP $L'$ are equivalent iff for each feasible solution $x$ to $L$ with the objective value $z$, there is a corresponding feasible solution $x'$ to $L'$ with the objective value $-z$. 
LP in Standard Form

- Maximization of a linear function.
- $n$ non-negative real-valued variables.
- $m$ linear inequalities ("less than or equal to").

\[
\begin{align*}
\text{max} \quad & \sum_{j=1}^{n} c_j x_j \\
\text{s.t.} \quad & \sum_{j=1}^{n} a_{ij} x_j \leq b_i \quad \text{for} \quad i=1,2,\ldots,m \\
& x_j \geq 0 \quad \text{for} \quad j=1,2,\ldots,n
\end{align*}
\]
Converting LP into Standard Form

\[ \begin{align*}
\text{min} \quad & -2x_1 + 3x_2 \\
\text{s.t.} \quad & x_1 + x_2 = 7 \\
& x_1 - 2x_2 \leq 4 \\
& x_1 \geq 0
\end{align*} \]

- Minimization LP is converted to an equivalent maximization problem by negating the coefficients of the objective function.

\[ \begin{align*}
\text{max} \quad & 2x_1 - 3x_2 \\
\text{s.t.} \quad & x_1 + x_2 = 7 \\
& x_1 - 2x_2 \leq 4 \\
& x_1 \geq 0
\end{align*} \]
Converting LP into Standard Form

\[ \begin{align*}
    \text{max} & \quad 2x_1 - 3x_2 \\
    \text{s.t.} & \quad x_1 + x_2 = 7 \\
                  & \quad x_1 - 2x_2 \leq 4 \\
                  & \quad x_1 \geq 0
\end{align*} \]

- Every variable \( x_j \) without non-negativity constraint is replaced by two non-negative variables \( x'_j \) and \( x''_j \) and each occurrence of \( x_j \) is replaced by \( x'_j - x''_j \).

\[ \begin{align*}
    \text{max} & \quad 2x_1 - 3x'_2 + 3x''_2 \\
    \text{s.t.} & \quad x_1 + x'_2 - x''_2 = 7 \\
                  & \quad x_1 - 2x'_2 + 2x''_2 \leq 4 \\
                  & \quad x_1, \quad x'_2, \quad x''_2 \geq 0
\end{align*} \]
Converting LP into Standard Form

$max \quad 2x_1 - 3x'_2 + 3x''_2$
$s.t. \quad x_1 + x'_2 - x''_2 = 7$
$\quad \quad \quad \quad \quad \quad x_1 - 2x'_2 + 2x''_2 \leq 4$
$\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad x_1, \quad x'_2, \quad x''_2 \geq 0$

- Each equality constraint is replaced by a pair of "opposite" inequality constraints.

$max \quad 2x_1 - 3x'_2 + 3x''_2$
$s.t. \quad x_1 + x'_2 - x''_2 \leq 7$
$\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad x_1, \quad x'_2, \quad x''_2 \geq 0$
Converting LP into Standard Form

\[
\begin{align*}
\text{max} & \quad 2x_1 - 3x'_2 + 3x''_2 \\
\text{s.t.} & \quad x_1 + x'_2 - x''_2 \leq 7 \\
& \quad x_1 + x'_2 - x''_2 \geq 7 \\
& \quad x_1 - 2x'_2 + 2x''_2 \leq 4 \\
& \quad x_1, x'_2, x''_2 \geq 0
\end{align*}
\]

- Inequalities are "turned around" by multiplying both sides by -1.

\[
\begin{align*}
\text{max} & \quad 2x_1 - 3x'_2 + 3x''_2 \\
\text{s.t.} & \quad x_1 + x'_2 - x''_2 \leq 7 \\
& \quad -x_1 - x'_2 + x''_2 \leq -7 \\
& \quad x_1 - 2x'_2 + 2x''_2 \leq 4 \\
& \quad x_1, x'_2, x''_2 \geq 0
\end{align*}
\]
Converting LP into a Standard Form

\[
\begin{align*}
\text{max} & \quad 2x_1 - 3x_2 + 3x_2' \\
\text{s.t.} & \quad x_1 + x_2' - x_2' \leq 7 \\
& \quad -x_1 - x_2' + x_2' \leq -7 \\
& \quad x_1 - 2x_2' + 2x_2' \leq 4 \\
& \quad x_1, x_2', x_2' \geq 0
\end{align*}
\]

- Renaming the variables

\[
\begin{align*}
\text{max} & \quad 2x_1 - 3x_2 + 3x_3 \\
\text{s.t.} & \quad x_1 + x_2 - x_3 \leq 7 \\
& \quad -x_1 - x_2 + x_3 \leq -7 \\
& \quad x_1 - 2x_2 + 2x_3 \leq 4 \\
& \quad x_1, x_2, x_3 \geq 0
\end{align*}
\]
LP in Standard Form

• $n$ variables, $m$ constraints

$$\begin{align*}
    \text{max} & \quad \sum_{j=1}^{n} c_j x_j \\
    \text{s.t.} & \quad \sum_{j=1}^{n} a_{ij} x_j \leq b_i \quad \text{for} \quad i = 1, 2, \ldots, m \\
    & \quad x_j \geq 0 \quad \text{for} \quad j = 1, 2, \ldots, n
\end{align*}$$
Slack Variables (Overskudsværdi)

- Consider one of the constraints, for example

\[ 2x_1 + 3x_2 + x_3 \leq 5 \]

- For every feasible solution \( x_1, x_2, x_3 \), the value of the left-hand side is at most the value of the right-hand side.

- Often there can be a slack between these two values.

- Denote the slack by \( x_4 \).

- By requiring that \( x_4 \geq 0 \), we can replace the inequality by the equality

\[ 2x_1 + 3x_2 + x_3 + x_4 = 5 \]
Slack Variables

\[
\begin{align*}
\text{max} & \quad \sum_{j=1}^{n} c_j x_j \\
\text{s.t.} & \quad \sum_{j=1}^{n} a_{ij} x_j \leq b_i \quad \text{for} \quad i=1,2,\ldots,m \\
& \quad x_j \geq 0 \quad \text{for} \quad j=1,2,\ldots,n
\end{align*}
\]

\[
\begin{align*}
\text{max} & \quad \sum_{j=1}^{n} c_j x_j \\
\text{s.t.} & \quad \sum_{j=1}^{n} a_{ij} x_j + x_{n+i} = b_i \quad \text{for} \quad i=1,2,\ldots,m \\
& \quad x_j \geq 0 \quad \text{for} \quad j=1,2,\ldots,n+m
\end{align*}
\]
LP in Slack Form

\[
\begin{align*}
\text{max} & \quad \sum_{j=1}^{n} c_j x_j \\
\text{s.t.} & \quad \sum_{j=1}^{n} a_{ij} x_j + x_{n+i} = b_i \quad \text{for} \quad i=1,2,\ldots,m \\
& \quad x_j \geq 0 \quad \text{for} \quad j=1,2,\ldots,n+m
\end{align*}
\]

\[
\begin{align*}
\text{max} & \quad z = \sum_{j=1}^{n} c_j x_j \\
\text{s.t.} & \quad x_{n+i} = b_i - \sum_{j=1}^{n} a_{ij} x_j \quad \text{for} \quad i=1,2,\ldots,m \\
& \quad x_j \geq 0 \quad \text{for} \quad j=1,2,\ldots,n+m
\end{align*}
\]

\[
\begin{align*}
z & = 0 + \sum_{j=1}^{n} c_j x_j \\
x_{n+i} & = b_i - \sum_{j=1}^{n} a_{ij} x_j \quad \text{for} \quad i=1,2,\ldots,m
\end{align*}
\]
Standard to Slack Form - Example

\[
\begin{align*}
\text{max} & \quad 2x_1 - 3x_2 + 3x_3 \\
\text{s.t.} & \quad x_1 + x_2 - x_3 \leq 7 \\
& \quad -x_1 - x_2 + x_3 \leq -7 \\
& \quad x_1 - 2x_2 + 2x_3 \leq 4 \\
\end{align*}
\]

\[
\begin{align*}
\text{max} & \quad 2x_1 - 3x_2 + 3x_3 \\
\text{s.t.} & \quad x_1 + x_2 - x_3 + x_4 = 7 \\
& \quad -x_1 - x_2 + x_3 + x_5 = -7 \\
& \quad x_1 - 2x_2 + 2x_3 + x_6 = 4 \\
\end{align*}
\]

\[
\begin{align*}
z &= 0 + 2x_1 - 3x_2 + 3x_3 \\
x_4 &= 7 - x_1 - x_2 + x_3 \\
x_5 &= -7 + x_1 + x_2 - x_3 \\
x_6 &= 4 - x_1 + 2x_2 - 2x_3 \\
\end{align*}
\]
Basic Solutions

- Any solution of LP in the standard form yields a solution of LP in the corresponding slack form (with the same objective value) and vice versa.
- Setting right-hand side variables of the slack form to 0 yields a basic solution.
- Left-hand side variables are called basic. Right-hand side variables are called nonbasic.
- The basic variables are said to constitute a basis.
- Note that a basic solution does not need to be feasible.
SIMPLEX - Example

- LP problem in standard form:

\[
\begin{align*}
    \text{max} & \quad 3x_1 + x_2 + 2x_3 \\
    \text{s.t.} & \quad x_1 + x_2 + 3x_3 \leq 30 \\
    & \quad 2x_1 + 2x_2 + 5x_3 \leq 24 \\
    & \quad 4x_1 + x_2 + 2x_3 \leq 36 \\
    & \quad x_1, x_2, x_3 \geq 0
\end{align*}
\]
SIMPLEX – Example Continued

- LP in slack form:

\[
\begin{align*}
  z &= 0 + 3x_1 + x_2 + 2x_3 \\
  x_4 &= 30 - x_1 - x_2 - 3x_3 \\
  x_5 &= 24 - 2x_1 - 2x_2 - 5x_3 \\
  x_6 &= 36 - 4x_1 - x_2 - 2x_3 
\end{align*}
\]

- Set all **nonbasic** variables (right-hand side) to 0.
- Compute values of **basic** variables: \(x_4=30, x_5=24, x_6=36\).
- Compute the objective value \(z ( = 0)\).
- This gives the feasible basic solution \((0,0,0,30,24,36)\).
- It is feasible; not always the case – we were lucky.
SIMPLEX: 1. Pivoting

- Can $x_1$ be increased without violating feasibility?

  \[
  z = 0 + 3x_1 + x_2 + 2x_3 \\
  x_4 = 30 - x_1 - x_2 - 3x_3 \\
  x_5 = 24 - 2x_1 - 2x_2 - 5x_3 \\
  x_6 = 36 - 4x_1 - x_2 - 2x_3
  \]

- If $x_1$ is increased to 1, then $x_4=29$, $x_5=22$, $x_6=32$ while $z=3$. $(1,0,0,29,22,32)$ is a feasible solution.

- If $x_1$ is increased to 2, then $x_4=28$, $x_5=20$, $x_6=28$ while $z=6$. $(2,0,0,28,20,28)$ is a feasible solution.

- If $x_1$ is increased to 3, then $x_4=27$, $x_5=18$, $x_6=24$ while $z=9$. $(3,0,0,27,18,24)$ is a feasible solution.
SIMPLEX: 1. Pivoting

- Can $x_1$ be increased without violating feasibility? By how much?

  $z = 0 + 3x_1 + x_2 + 2x_3$
  $x_4 = 30 - x_1 - x_2 - 3x_3$
  $x_5 = 24 - 2x_1 - 2x_2 - 5x_3$
  $x_6 = 36 - 4x_1 - x_2 - 2x_3$

- If $x_1$ is increased beyond 30 then $x_4$ becomes negative.
- If $x_1$ is increased beyond 12 then $x_5$ becomes negative.
- If $x_1$ is increased beyond 9 then $x_6$ becomes negative.
- Constraint defining $x_6$ is binding.
SIMPLEX: 1. Pivoting

- So $x_1$ can be increased to 9 without losing feasibility. The feasible solution is $(9,0,0,21,6,0)$.

- We will now rewrite the slack form to an equivalent slack form with $x_1$, $x_4$, $x_5$ as basic variables and with $(9,0,0,21,6,0)$ being its feasible basic solution.

- This rewriting is called **pivoting**.

- Binding constraint defining $x_6$ is rewritten so that it has $x_1$ on its left-hand side.

- All occurrences of $x_1$ in other constraints and in the objective function are replaced by the right-hand side of the binding constraint.
SIMPLEX: 1. Pivoting

\[
\begin{align*}
  \text{New basic variables: } x_1 &= 9, \quad x_4 = 21, \quad x_5 = 6 \\
  \text{New objective value } z &= 27
\end{align*}
\]
SIMPLEX: 2. Pivoting

- Can $x_3$ be increased without violating feasibility? By how much?

$$z = 27 + \frac{x_2}{4} + \frac{x_3}{2} - \frac{3x_6}{4}$$
$$x_4 = 21 - \frac{3x_2}{4} - \frac{5x_3}{2} + \frac{x_6}{4}$$
$$x_5 = 6 - \frac{3x_2}{2} - 4x_3 + \frac{x_6}{2}$$
$$x_1 = 9 - \frac{x_2}{4} - \frac{x_3}{2} - \frac{x_6}{4}$$

- If $x_3$ is increased beyond $42/5$ then $x_4 < 0$.
- If $x_3$ is increased beyond $3/2$ then $x_5 < 0$.
- If $x_3$ is increased beyond $9/2$ then $x_1 < 0$.
- Constraint defining $x_5$ is binding.
SIMPLEX: 2. Pivoting

\[ z = 27 + \frac{x_2}{4} + \frac{1}{2} \left( \frac{3}{2} - \frac{3x_2}{8} + \frac{x_6}{8} - \frac{x_5}{4} \right) - \frac{3x_6}{4} \]

\[ x_4 = 21 - \frac{3x_2}{4} - \frac{5}{2} \left( \frac{3}{2} - \frac{3x_2}{8} + \frac{x_6}{8} - \frac{x_5}{4} \right) + \frac{x_6}{4} \]

\[ x_3 = \frac{3}{2} - \frac{3x_2}{8} - \frac{x_5}{4} + \frac{x_6}{8} \]

\[ x_1 = 9 - \frac{x_2}{4} - \frac{1}{2} \left( \frac{3}{2} - \frac{3x_2}{8} + \frac{x_6}{8} - \frac{x_5}{4} \right) - \frac{x_6}{4} \]

\[ z = \frac{111}{4} + \frac{x_2}{16} - \frac{x_5}{8} - \frac{11x_6}{16} \]

\[ x_4 = \frac{69}{4} + \frac{3x_2}{16} + \frac{5x_5}{8} - \frac{x_6}{16} \]

\[ x_3 = \frac{3}{2} - \frac{3x_2}{8} - \frac{x_5}{4} + \frac{x_6}{8} \]

\[ x_1 = \frac{33}{4} - \frac{x_2}{16} + \frac{x_5}{8} - \frac{5x_6}{16} \]

New basic variables: \( x_1 = \frac{33}{4}, \ x_3 = \frac{3}{2}, \ x_4 = \frac{69}{4} \)  
New objective value \( z = 27.75 \)

New feasible basic solution: (\( \frac{33}{4}, 0, \frac{3}{2}, \frac{69}{4}, 0, 0 \))
SIMPLEX: 3. Pivoting

- Can $x_2$ be increased without violating feasibility? By how much?

\[
\begin{align*}
z & = \frac{111}{4} + \frac{x_2}{16} - \frac{x_5}{8} - \frac{11x_6}{16} \\
x_4 & = \frac{69}{4} + \frac{3x_2}{16} + \frac{5x_5}{8} - \frac{x_6}{16} \\
x_3 & = \frac{3}{2} - \frac{3x_2}{8} - \frac{x_5}{4} + \frac{x_6}{8} \\
x_1 & = \frac{33}{4} - \frac{x_2}{16} + \frac{x_5}{8} - \frac{5x_6}{16}
\end{align*}
\]

- If $x_2$ is increased then $x_4$ also increases.

- If $x_2$ is increased beyond 4 then $x_3 < 0$.

- If $x_2$ is increased beyond 132 then $x_1 < 0$.

- Constraint defining $x_3$ is binding.
SIMPLEX: 3. Pivoting

\[
\begin{align*}
z &= 111/4 + \frac{1}{16} (4 - 8x_3/3 - 2x_5/3 + x_6/3) - x_5/8 - 11x_6/16 \\
x_4 &= 69/4 + \frac{1}{16} (4 - 8x_3/3 - 2x_5/3 + x_6/3) + 5x_5/8 - x_6/16 \\
x_2 &= 4 - \frac{1}{16} (4 - 8x_3/3 - 2x_5/3 + x_6/3) - 2x_5/3 + x_6/3 \\
x_1 &= 33/4 - \frac{1}{16} (4 - 8x_3/3 - 2x_5/3 + x_6/3) + x_5/8 - 5x_6/16
\end{align*}
\]

\[
\begin{align*}
z &= 28 - x_3/6 - x_5/6 - 2x_6/3 \\
x_4 &= 18 - x_3/2 + x_5/2 + 0x_6 \\
x_2 &= 4 - 8x_3/3 - 2x_5/3 + x_6/3 \\
x_1 &= 8 + x_3/6 + x_5/6 - x_6/3
\end{align*}
\]

New basic variables: \(x_1 = 8, \ x_2 = 4, \ x_4 = 18\)

New objective value \(z = 28\)

New feasible basic solution: \((8, 4, 0, 18, 0, 0)\) is optimal
Pivoting in General

- **PIVOT(M, B, A, b, c, v, l, e)**
  
  - Compute the coefficients of the bounding constraint so that the **entering** basic variable $x_e$ is expressed as a linear combination of the other variables.

  $$b_e = b_l / a_{le} \quad a_{ej} = a_{lj} / a_{le}, \quad \forall j \in N \setminus e \quad a_{el} = 1 / a_{le}$$

  - Compute the coefficients of the remaining constraints and the objective function (by substituting $x_e$ by the right-hand side of the rewritten binding equation).

  $$b_i = b_i - a_{ie} b_e, \quad \forall i \in B \setminus l \quad a_{ij} = a_{ij} - a_{ie} a_{ej}, \quad \forall j \in N \setminus e \quad a_{il} = -a_{ie} a_{el}$$

  $$v = v + c_e b_e \quad c_j = c_j - c_e a_{ej}, \quad \forall j \in N \setminus e \quad c_l = -c_e a_{el}$$

  - Compute new sets of basic and nonbasic variables (remove $x_e$ from $N$ and add it to $B$, remove $x_i$ from $B$ and add it to $N$).
SIMPLEX – Open Issues

- How to decide that LP is feasible?
- What to do if the initial basic solution is infeasible?
- How to decide that LP is unbounded?
- How to select entering and leaving variables?
- Does SIMPLEX terminate?
- Does it terminate with an optimal solution?