Map Theory
Preface, Addendum and Index

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This dissertation in connection with the paper 'Map theory', Theoretical Computer Science 102 (1992) 1–133 has been accepted for public defence for the Danish Dr.Scient. degree by the Faculty of Science of Nature at the University of Copenhagen. Henrik Jeppesen, Head of Faculty, Copenhagen, October 19, 1992.

The defence will take place on Friday, November 20, 1992, at 1:45 p.m. in Auditorium 1 at the Hans Christian Ørsted Institute, Universitetsparken 5, DK 2100, Copenhagen. The dissertation and reprints of the paper 'Map theory' may be acquired free of charge through the Institute four weeks before the public defence, as long as store is available.

Denne afhandling er i forbindelse med artiklen 'Map theory', Theoretical Computer Science 102 (1992) 1–133 af Det naturvidenskabelige Fakultet antaget til offentligt at forsvares for den naturvidenskabelige doktorgrad. København, den 19. oktober, 1992, Henrik Jeppesen, Dekan.

Forsvaret vil finde sted fredag den 20. november, 1992 kl. 13:45 i auditorium 1 på H.C.Ørsted Institutet, Universitetsparken 5, 2100 København Ø. Eksemplarer af afhandlingen samt eksemplarer af artiklen 'Map theory' kan erhverves gratis på institutet fire uger før det offentlige forsvar, så længe lager haves.

Preface

Long have I realized that too much logic at any one time smells suspicious
— Aksel Sandemose, A fugitive crosses his tracks

This dissertation introduces 'map theory' which is a foundation of mathematics based on maps (i.e. functions). Like set theory, map theory is a well-defined, axiomatic theory which, among other, covers all of classical mathematics. Map theory seems to be the first foundation with these properties that makes no reference to sets or related entities. Contrary to set theory, map theory allows unlimited abstraction and has a computer programming language as a subtheory.

The thesis consists of the paper 'Map theory', Theoretical Computer Science 102 (1992) 1–133 plus the present volume which contains a preface in English and Danish, an addendum, an index of Part I and II of the paper and an index of Part III. Note the quick reference list of syntax, axioms and definitions at the end of the volume. The paper 'Map theory' has been reprinted with permission from Elsevier Science Publishers in DIKU report 92/17.

Part I of the paper consists of Sections 1 to 3, and gives an informal introduction to map theory. Section 2 presents the notion of a map and Section 3 outlines possible uses of map theory. Part I is semantic and fairly superficial of nature.

Part II consists of Sections 4 to 8 and gives a formal, syntactic treatment of map theory. Sections 4 to 7 introduce the axioms of map theory and Section 8 models *ZFC* set theory within map theory. The expressive power of the axioms in Section 4, 5, 6 and 7 roughly correspond to λ-calculus, propositional calculus, first order predicate calculus and set theory, respectively. The modeling of *ZFC* in Section 8 verifies that map theory covers all of classical mathematics.

Part III consists of Sections 9 to 15 and proves the consistency of map theory assuming the existence of a strongly inaccessible ordinal. Section 9 introduces the more or less standard *ZFC* notation used in Part III, Section 10 gives an informal overview of the model construction, Section 11 defines the model, Section 12 to 14 proves that the model satisfies the theory, and Section 15 proves additional consistency results. Part III is followed by a concluding section.

Forord

Jeg har længe syntes at alt for megen logik på én gang er mistænkeligt
— Aksel Sandemose, En flygtning krydser sit spor

Denne afhandling introducerer 'funktionslære' ('map theory'). Funktionslære er et matematisk fundament baseret på funktioner ('maps'). Funktionslære er ligesom mængelære en veldefineret, aksiomatisk teori som bl.a. omfatter al klassisk matematik. Funktionslære synes at være det første fundament med disse egenskaber, der ikke refererer til mængder og lignende begreber. Modsat mængdelære har funktionslære ubegrænset abstraktion (komprehension) og indeholder endvidere et programmeringssprog.

Afhandlingen består af artiklen 'Map theory', Theoretical Computer Science 102 (1992) 1–133 samt nærværende bind, der indeholder forord på engelsk og dansk, et addendum, et index for artiklens del I og II samt et index for del III. Bemærk oversigten i kortform over syntax, aksiomer og definitioner bagest i bindet. Artiklen 'Map theory' er optrykt med tilladelse fra Elsevier Science Publishers i DIKU-rapport 92/17.

Artiklens del I består af kapitel 1 til 3 og giver en uformel introduktion til funktionslære. Kapitel 2 introducerer begrebet 'funktion' og kapitel 3 skitserer mulige anvendelser af funktionslære. Del I er semantisk og forholdsvis overfladisk.

Del II består af kapitel 4 til 8 og giver en formel, syntaktisk beskrivelse af funktionslære. Kapitel 4 til 7 introducerer funktionslærens aksiomer. Udtrykskraften af aksiomerne i kapitel 4, 5, 6 og 7 svarer stort set til λ-kalkyle, propositionskalkyle, første ordens prædikatkalkyle henholdsvis mængdelære. Kapitel 8 udvikler *ZFC* mængdelære indenfor funktionslære og beviser dermed, at funktionslæren omfatter al klassisk matematik.

Del III består af kapitel 9 til 15 og beviser, at funktionslæren er konsistent under antagelse af eksistensen af et stærkt unåligt ('strongly inaccessible') ordinaltal. Kapitel 9 gennemgår den anvendte *ZFC* nomenklatur og kapitel 10 giver en uformel gennemgang af den efterfølgende modelkonstruktion. Kapitel 11 definerer en model, kapitel 12 til 14 beviser, at modellen er en model for funktionslære og kapitel 15 beviser yderligere konsistensresultater. Del III efterfølges af et konkluderende kapitel.

Addendum

This addendum explains how the axioms of Section 4, 5, 6 and 7 of the paper 'Map theory' relate to λ-calculus, propositional calculus, first order predicate calculus and set theory, respectively. The addendum also discusses the syntax of map theory.

Let Map_0 be the subtheory of Map whose axioms and inference rules are those stated in Section 4. Likewise, let Map_1 contain the axioms and inference rules of Section 4 to 5 and let Map_2 contain those of Section 4 to 6. Map contains all of map theory, i.e. all axioms and inference rules of Section 4 to 7.

λ-calculus

If two λ-terms $\mathcal{A}$ and $\mathcal{B}$ are β-convertible, then $\mathcal{A} = \mathcal{B}$ is provable in Map_0 (Theorem 4.3.1. T and $(\mathsf{if}\, x\, y\, z)$ are admitted in λ-terms). This shows that the embedding of λ-calculus into Map_0 is 'sound'. Since anything provable in Map_0 is also provable in Map_1, Map_2 and Map, embedding of λ-calculus into Map_1, Map_2 and Map is also sound.

On the contrary, if $\mathcal{A}$ and $\mathcal{B}$ are λ-terms and $\mathcal{A} = \mathcal{B}$ is provable in Map_0, then $\mathcal{A}$ and $\mathcal{B}$ are β-convertible. This is not shown in the thesis, since it is considered out of scope (the scope is to present map theory and to prove its power and consistency). However, the proof is an easy application of Church-Rosser's theorem. This shows that the embedding of λ-calculus into Map_0 is 'complete'.

Map_1 can prove more about λ-terms than λ-calculus. As an example let $R = \lambda x.\dot{\neg}(x\, x)$ and $R' = (R\, R)$. In Map_1, $R' = \bot$ is provable, so $R' = \bot = (\bot\, \lambda x.x) = (R'\, \lambda x.x)$ is provable even though R' and $(R'\, \lambda x.x)$ are not β-convertible. Hence, the embedding of λ-calculus into Map_1 is incomplete. As a consequence, embedding into Map_2 and Map is also incomplete.

Propositional calculus

For any well-formed formula $\mathcal{A}$ of propositional calculus let $\mathcal{A}'$ be the result of replacing $\Rightarrow$ and $\neg$ of propositional calculus by $\dot{\Rightarrow}$ and $\dot{\neg}$, respectively, of map theory. Let $\mathcal{A}''$ be the equation

$$!x_1, \ldots, !x_n \rightarrow \mathcal{A}'$$

where $x_1, \ldots, x_n$ are the free variables of $\mathcal{A}$ in some, prescribed order. The equation $\mathcal{A}''$ intuitively says 'if $x_1, \ldots, x_n$ are all either true or false, then $\mathcal{A}'$ holds'.

If $\mathcal{A}$ is provable in propositional calculus, then $\mathcal{A}''$ is provable in Map_1 (Theorem 5.2.2). Hence, the embedding of propositional calculus into Map_1 is sound. In consequence, embedding into Map_2 and Map is also sound.

On the contrary, if $\mathcal{A}''$ is provable in Map_1, Map_2 or Map, then $\mathcal{A}$ holds in all 2^n cases in the truth table of $\mathcal{A}$, so $\mathcal{A}$ is provable in propositional calculus (since any tautology is provable in propositional calculus). Hence, the embedding of propositional calculus into Map_1, Map_2 as well as Map is complete.

As other examples of sound and complete embedding, one could translate $\mathcal{A}$ into

$$!x_1 \dot\wedge \cdots \dot\wedge !x_n \rightarrow \mathcal{A}'$$

or

$$\mathcal{A}' = !x_1 \dot\wedge \cdots \dot\wedge !x_n$$

For arbitrary terms $\mathcal{B}_1, \ldots, \mathcal{B}_n$ and $\mathcal{C}$, the equation $\mathcal{B}_1, \ldots, \mathcal{B}_n \rightarrow \mathcal{C}$ is provable in Map_1 (or Map_2 or Map) if and only if $\mathcal{B}_1 \dot\wedge \cdots \dot\wedge \mathcal{B}_n \rightarrow \mathcal{C}$ is provable. This is a consequence of the definition of $\mathcal{B}_1, \ldots, \mathcal{B}_n \rightarrow \mathcal{C}$ and the equation $\mathcal{B}_1 : \ldots : \mathcal{B}_n : \mathcal{C} = (\mathcal{B}_1 \dot\wedge \cdots \dot\wedge \mathcal{B}_n) : \mathcal{C}$ which holds according to Theorem 5.2.2. As a special case, $!x_1 \dot\wedge \cdots \dot\wedge !x_n \rightarrow \mathcal{A}'$ is as good an embedding as $!x_1, \ldots, !x_n \rightarrow \mathcal{A}'$.

The equation $\mathcal{A}' = !x_1 \dot\wedge \cdots \dot\wedge !x_n$ says more than $!x_1, \ldots, !x_n \rightarrow \mathcal{A}'$. The former equation says that $\mathcal{A}'$ is true when $x_1, \ldots x_n$ are all either true or false, and that $\mathcal{A}' = \perp$ when one or more among $x_1, \ldots, x_n$ are $\perp$. The equation $!x_1, \ldots, !x_n \rightarrow \mathcal{A}'$ does not state anything about $\mathcal{A}'$ when one or more among $x_1, \ldots, x_n$ are $\perp$. However, $\mathcal{A}'$ is always $\perp$ when one or more among $x_1, \ldots, x_n$ are $\perp$ since $x_1, \ldots, x_n$ all occur in $\mathcal{A}'$ and $x \dot\Rightarrow y$ and $\dot\neg x$ are strict in their arguments.

First order predicate calculus

For any, well-formed formula $\mathcal{A}$ of first order predicate calculus define $\mathcal{A}'$ as follows: Let $f_1, \ldots, f_m$ and $A_1, \ldots, A_n$ be the function and relation letters, respectively, that occur in $\mathcal{A}$. Let a_i be the arity of f_i for $i \in \{1, \ldots, m\}$ and let b_j be the arity of A_j for $j \in \{1, \ldots, n\}$. Let $y_1, \ldots, y_m, z_1, \ldots, z_n$ be variables that do not occur (free or bound) in $\mathcal{A}$. Now define $\mathcal{A}'$ as the result of replacing $\neg$, $\Rightarrow$, $\forall$, $f_i(\mathcal{B}_1, \ldots, \mathcal{B}_{a_i})$, and $A_j(\mathcal{C}_1, \ldots, \mathcal{C}_{b_j})$ by $\dot\neg$, $\dot\Rightarrow$, $\dot\forall$, $(y_i\, \mathcal{B}'_1 \cdots \mathcal{B}'_{a_i})$, and $(z_j\, \mathcal{C}'_1 \cdots \mathcal{C}'_{b_j})$, respectively. Let $\mathcal{A}''$ be the equation

$$\begin{aligned}
&\phi x_1, \ldots, \phi x_p, \\
&\dot\forall u_1, \ldots, u_{a_1}.\phi(y_1\, u_1 \cdots u_{a_1}), \\
&\vdots \\
&\dot\forall u_1, \ldots, u_{a_m}.\phi(y_m\, u_1 \cdots u_{a_m}), \\
&\dot\forall v_1, \ldots, v_{b_1}.!(z_1\, v_1 \cdots v_{b_1}), \\
&\vdots \\
&\dot\forall v_1, \ldots, v_{b_n}.!(z_n\, v_1 \cdots v_{b_n}) \rightarrow \mathcal{A}'
\end{aligned}$$

where $x_1, \ldots, x_p$ are the free variables of $\mathcal{A}$. The equation of $\mathcal{A}''$ states 'if $x_1, \ldots, x_p$ are well-founded, if $y_1, \ldots, y_m$ map well-founded maps to well-founded maps, and if $z_1, \ldots, z_n$ map well-founded maps to truth values, then $\mathcal{A}'$ holds'.

If $\mathcal{A}$ is a theorem of first order predicate calculus, then $\mathcal{A}''$ is provable in Map_2 (the proof is analogous to but simpler than the development of *ZFC* in Section 8). Hence, embedding of first order predicate calculus into Map_2 and Map is sound.

It is an open question whether the embedding of first order predicate calculus into Map_2 and Map are complete.

Set theory

For any well-formed formula of *ZFC* set theory, let $\mathcal{A}'$ be the result of replacing $\Rightarrow$, $\neg$, $\forall$ and $\in$ by $\dot{\Rightarrow}$, $\dot{\neg}$, $\dot{\forall}$ and $\dot{\in}$, respectively. Let $\mathcal{A}''$ be the equation $\phi x_1, \ldots, \phi x_n \to \mathcal{A}'$ where $x_1, \ldots, x_n$ are the free variables of $\mathcal{A}'$. The equation $\mathcal{A}''$ intuitively says 'if $x_1, \ldots, x_n$ all represent sets, then $\mathcal{A}'$ holds'.

If $\mathcal{A}$ is provable in *ZFC*, then $\mathcal{A}''$ is provable in *Map* (Theorem 8.2.1). Hence, the embedding of *ZFC* into *Map* is sound. It is an open question whether the embedding is complete.

Now let ZFC^+ be any extension of *ZFC* (in particular, ZFC^+ can be *ZFC* itself). If $\mathcal{A}$ is provable in ZFC^+, then $\mathcal{A}''$ is provable in $Map^{\circ+}$ where $Map^{\circ+}$ is defined in Section 9 (the definition of $Map^{\circ+}$ depends on ZFC^+ and includes all theorems of ZFC^+ as axioms). Hence, the embedding of ZFC^+ into $Map^{\circ+}$ is sound.

The embedding of ZFC^+ into $Map^{\circ+}$ is also complete as shown in the following. If ZFC^+ is inconsistent, then the completeness of the embedding is trivial. Now assume ZFC^+ is consistent and that $\mathcal{B}$ is not provable in ZFC^+. Since ZFC^+ is consistent and $\mathcal{B}$ is not provable, $ZFC^{++} = ZFC^+ + \{\neg\mathcal{B}\}$ is consistent. $\neg\mathcal{B}$ is provable in ZFC^{++}, so $(\neg\mathcal{B})''$ is provable in $Map^{\circ++}$ where $Map^{\circ++}$ is defined from ZFC^{++} like $Map^{\circ+}$ from ZFC^+. $Map^{\circ++}$ is consistent since ZFC^{++} is consistent (Formula (10) on page 72 and the three lines following the proof of Theorem 15.5.1 on page 129 of 'Map theory'). Hence, $\mathcal{B}''$ is not provable in $Map^{\circ++}$, so $\mathcal{B}''$ is not provable in the subtheory $Map^{\circ+}$ of $Map^{\circ++}$, which proves the completeness.

The syntax of map theory

Several choices were made when defining the syntax of map theory. Some of them are: (1) Should map theory be based on λ-abstraction or the combinators S and K? (2) Should if, ε and ϕ be combinators or syntactic constructs? (3) What should the structure of a well-formed formula be? The last question is covered in Section 16.3.

λ-abstraction was chosen in favor of S and K because expressions built up from S and K tend to be large and incomprehensible. The axioms of S and K are considerably simpler than the corresponding ones for λ-abstraction, so when map theory is considered as an object of study rather than a foundation, S and K are sometimes better than λ-abstraction. This is why the model in Part III is based on S and K, and models λ-abstraction from these combinators.

It is not particularly important whether if, ε and ϕ are combinators or syntactic constructs. In map theory, they are syntactic constructs, i.e., if, ε and ϕ are not maps themselves. However, one may define the corresponding combinators by $\mathsf{IF} = \lambda x.\lambda y.\lambda z.(\mathsf{if}\, x\, y\, z)$, $C = \lambda x.\varepsilon x$ and $W = \lambda x.\phi x$. On the contrary, if if, ε and ϕ were introduced as combinators, then one could easily define the corresponding syntactic constructs.

If if were defined as a combinator, then the axioms could not decide whether (if $\mathsf{T}\,\mathsf{T}$) was T or a proper map. To state that ($\mathsf{if}\, x\, y$) is proper for all x and y would require an additional axiom. Further, the combinators are probably slightly more difficult to understand for novices than the syntactic constructs. For these reasons, and since there are no strong arguments in favor of either approach, the approach with syntactic constructs is chosen.

Again, if map theory is considered as an object of study rather than as a foundation, then combinators are sometimes easier to use. This is why the model in Part III is based on combinators rather than the syntactic constructs.

Index of Part I and II

This is an index of notation used in Part I and II. The notation in Part I and II differs slightly from that of Part III. An index for Part III follows the index for Part I and II.

For each entry, the index displays a mathematical construct, refers to the section where it is defined, and gives a short, informal explanation of the construct.

Constructs that merely occur in Section 3.14

3.14	$\bar{\mathcal{C}}$
3.14	$x{:}y{\rightarrow}z$
3.14	$x \circ y$
3.14	id_x
3.14	p_i
3.14	$x \stackrel{\star}{=} y$
3.14	$\langle x, y \rangle$
3.14	$x \stackrel{\star}{\wedge} y$
3.14	$\stackrel{\star}{\forall} x.\mathcal{A}$
3.14	$x \stackrel{c}{\Rightarrow} y$
3.14	$\star x$
3.14	$\bar{\mathcal{M}}$
3.14	$x \stackrel{\star}{\in} y$
3.14	$f{:}x \stackrel{c}{\rightarrow} y$
3.14	$x \stackrel{c}{\circ} y$
3.14	id^x_y
3.14	cat
3.14	$\bar{F}$
3.14	$\bar{\bar{F}}$
3.14	$func$
3.14	$\bar{id}_x$
3.14	$f \bar{\circ} g$
3.14	Cat

Constructs involving parentheses

2.1	$(f\, x_1 \ldots x_n)$	Shorthand for $(\cdots((f\, x_1)\, x_2) \cdots x_n)$ where $(f\ x)$ denotes f applied to x. The syntactic construct $(f\, x)$ is part of the basic syntax of terms in map theory
3.4	$\langle x_1, \ldots, x_n \rangle$	The tuple consisting of $x_1, \ldots, x_n$ (in that order)
3.13	$\ulcorner \mathcal{A} \urcorner$	The Gödel-number of the term $\mathcal{A}$. Actually, $\ulcorner \mathcal{A} \urcorner$ is a *map* rather than a *number*, so the Gödel-*number* is really a Gödel-*map*

3.13	$x[i]$	The i'th element of the tuple x
3.13	$[\mathcal{A}/x{:=}\mathcal{B}]$	The result of substituting the term $\mathcal{B}$ for all free occurrences of the variable x in the term $\mathcal{A}$

Constructs involving =

3.6	$x \dot{=} y$	Predicate stating that x and y represent the same set
4.3	$\mathcal{A} \stackrel{\bullet}{=} \mathcal{B}$	The statement that the terms $\mathcal{A}$ and $\mathcal{B}$ are identical

Constructs involving <

7.1	$x <_s y$	Predicate stating that x is recognized as being well-founded before y is
7.1	$x <_{s'} y$	A well-founded relation used for transfinite induction
7.3	$x <_{prim} y$	A well-founded relation used to justify the C-Prim Axiom

Constructs involving ≤

2.8	$x \leq_L y$	Predicate stating that the label x equals or contains less information than the label y
2.8	$x \leq y$	Predicate stating that the map x equals or contains less information than the map y

Alphabetic

Greek

7.5	Γ	A syntax class used in the verification of the metatheorem of totality
2.5	εx	Choice construct. εx denotes a well-founded y such that $(x\, y) = \mathsf{T}$ except for certain exceptions. The syntactic construct εx is part of the basic syntax of terms in map theory
2.2	$\tilde{\lambda}$	The label of root nodes of proper maps
1.2	$\lambda x.\mathcal{A}$	The map which maps x to $\mathcal{A}$. The syntactic construct $\lambda x.\mathcal{A}$ is part of the basic syntax of terms in map theory
2.4	σ	Strongly inaccessible ordinal
2.6	Σ	Syntax class used in the metatheorem of totality
2.6	$\bar{\Sigma}$	Syntax class used in the metatheorem of totality
2.5	ϕx	This term equals T if x is well-founded and equals $\bot$ otherwise. The syntactic construct ϕx is part of the basic syntax of terms in map theory

2.4	Φ	The collection of well-founded maps
2.9	$\hat{\Phi}$	An alternative to Φ
3.7	ω	The least, infinite ordinal

C

4.3	*cons*	A pairing construct
7.3	*Curry*	A combinator which transforms a first order function that takes a pair of arguments into a second order function that takes the arguments one at a time

D

8.2	$D(x,y)$	A construct used in proving the axiom of restriction

F

2.8	F	A map that represents falsehood
5.1	F'	A map whose range contains all proper maps. It satisfies

$$(\mathsf{F}'\,x) = \begin{cases} \lambda y.\mathsf{T} & \text{if } x = \mathsf{T} \\ x & \text{if } x \text{ is proper} \\ \lambda y.\bot & \text{if } x = \bot \end{cases}$$

H

5.4	$\overset{h}{\vdash}$	$\mathcal{A} = \mathcal{B} \overset{h}{\vdash} \mathcal{C} = \mathcal{D}$ states that $\mathcal{C} = \mathcal{D}$ is provable if $\mathcal{A} = \mathcal{B}$ is added to the collection of axioms. Furthermore, $\mathcal{A} = \mathcal{B} \overset{h}{\vdash} \mathcal{C} = \mathcal{D}$ states that $\mathcal{C} = \mathcal{D}$ is provable without using the Sub2 inference on variables free in $\mathcal{A}$ or $\mathcal{B}$. The latter restriction corresponds to the usual restriction in the deduction theorem of first order predicate calculus. According to Theorem 5.4.1 and 5.3.1, $\mathcal{A} = \mathsf{T} \overset{h}{\vdash} \mathcal{C} = \mathcal{D}$ if and only if $\mathcal{A} \rightarrow (\mathcal{C} = \mathcal{D})$
3.4	*hd*	Projector function that selects the first element of a pair

I

2.1	I	The identity map $\lambda x.x$
8.2	I'	Map used in proving the power set axiom

2.5	(if $x\ y\ z$)	The construct (if $x\ y\ z$) satisfies $(\text{if}\ x\ y\ z) = \begin{cases} y & \text{if } x = \mathsf{T} \\ z & \text{if } x \text{ is proper} \\ \bot & \text{if } x = \bot \end{cases}$ The syntactic construct (if $x\ y\ z$) is part of the basic syntax of map theory. Note that if itself is not part of the syntax — it is merely allowed when accompanied by three terms x, y and z. In (if $x\ y\ z$), the parentheses are part of the syntactic construct — they do not denote functional application

K

2.1	K	The combinator $\lambda x.\lambda y.x$

M

2.2	M	The collection of all maps
3.6	$Map^{\circ +}$	An axiomatization of map theory from which certain axioms are removed and certain other ones are added. The definition of $Map^{\circ +}$ depends on the definition of ZFC^{+}, where ZFC^{+} can be any extension of ZFC. The consistency of $Map^{\circ +}$ is provable from the assumption that ZFC^{+} is consistent without assuming the existence of a model of ZFC^{+} or assuming the existence of a strongly inaccessible ordinal. $Map^{\circ +}$ has the same expressive power as ZFC^{+}
4.3	*mirror*	An example of a recursive function

N

1.3	NBG	Set theory as defined by von Neumann, Bernays and Gödel
3.4	nil	The empty list

O

3.12	On	The class of all ordinals

P

3.5	P	A pairing construct
3.11	$\mathcal{P}x$	The power set of x

7.3	*Prim*	A combinator for primitive recursive definitions. If $g = (Prim\, f\, a\, b)$ then $(g\,x) = \begin{cases} a & \text{if } x = \mathsf{T} \\ (f\,\lambda u.(g\,(x\,(b\,u)))) & \text{if } x \text{ is proper} \\ \bot & \text{if } x = \bot \end{cases}$

Q

5.3	*QND*	Metatheorem expressing that any map is T, $\bot$ or proper — there is no fourth possibility
5.1	*QND′*	Inference rule expressing that any map is T, $\bot$ or proper — there is no fourth possibility

R

2.2	$r(x)$	The label of the root node of x
2.1	R	A map that is closely related to Russell's paradox
4.3	$\mathcal{A} \xrightarrow{r} \mathcal{B}$	Predicate stating that $\mathcal{A}$ can be reduced to $\mathcal{B}$ in one reduction

S

3.5	$s(x)$	The set represented by the well-founded map x
2.1	S	The combinator $\lambda x.\lambda y.\lambda z.(x\,z\,(y\,z))$
2.4	x^s	The collection of all maps that are recognized as well-founded before x is recognized

T

2.2	T	A map of map theory. Among other, it is used to represent truth and the empty list. The syntactic construct T is part of the basic syntax of terms in map theory
2.2	$\tilde{\mathsf{T}}$	The label of the root node of the map T
2.5	term	The syntax class of terms of map theory
5.3	*TND*	The metatheorem stating that any map which is not $\bot$ represents either truth or falsehood — there is no third possibility
3.4	*tl*	Projector function that selects the second element of a pair

V

2.5	variable	The syntax class of variables of map theory

Y

3.3	Y	A fixed point combinator. Y satisfies $(\mathsf{Y}\, f) = (f\, (\mathsf{Y}\, f))$

Z

1.1	ZFC	Set theory as defined by Zermelo and Fraenkel. ZFC includes the axioms of choice and restriction
3.6	ZFC^{+}	An arbitrary extension of ZFC
8.1	$\mathcal{A}{:}ZFC(x_1, \ldots x_n)$	The statement that $\mathcal{A}$ is a well-formed formula of ZFC whose free variables occur among $x_1, \ldots, x_n$

Other constructs

1.1	$x \in y$	The membership relation of ZFC. The membership relation in map theory is denoted $\dot{\in}$
1.1	$\neg x$	Logical negation in ZFC. Logical negation in map theory is denoted $\dot{\neg}$
1.1	$x \Rightarrow y$	Logical implication in ZFC. Map theory offers two kinds of implication: $\dot{\Rightarrow}$ and $\rightarrow$
1.1	$\forall x : \mathcal{A}$	Universal quantifier in ZFC. The universal quantifier in map theory is denoted $\dot{\forall} x.\mathcal{A}$
1.2	$\bot$	A map that represents infinite looping or undefinedness or total absence of information. The syntactic construct $\bot$ is part of the basic syntax of terms of map theory
2.2	$\tilde{\bot}$	The label of the root node of $\tilde{\bot}$
2.4	G°	The collection of maps that are well-founded w.r.t. the collection G of maps. $f \in G^{\circ}$ iff $\forall x_1, x_2, \ldots \in G \exists n : (f\, x_1\, \ldots\, x_n) = \mathsf{T}$
3.1	$\dot{\neg} x$	Logical negation in map theory. $\dot{\neg} x$ satisfies $\dot{\neg} x = \begin{cases} \mathsf{F} & \text{if } x = \mathsf{T} \\ \mathsf{T} & \text{if } x \text{ is proper} \\ \bot & \text{if } x = \bot \end{cases}$

3.1 $\approx x$ Truth values are represented in two ways in map theory. In the 'strong' representation, truth is represented by T and falsehood by F. In the 'weak' representation, T represents truth and any proper map represents falsehood. $\approx x$ denotes the strong representation of the truth value weakly represented by x. $\approx x$ satisfies

$$\approx x = \begin{cases} \mathsf{T} & \text{if } x = \mathsf{T} \\ \mathsf{F} & \text{if } x \text{ is proper} \\ \bot & \text{if } x = \bot \end{cases}$$

As an example of use, the equation $x\dot{\wedge}x = \approx x$ states that $x\dot{\wedge}x$ has the same truth value as x. The equation $x\dot{\wedge}x = x$ fails, e.g., for $x = \lambda x.\bot$ because $\lambda x.\bot$ is a weak but not the strong representation of falsehood. If T, F and $\bot$ are taken to represent the labels $\tilde{\mathsf{T}}$, $\tilde{\lambda}$ and $\tilde{\bot}$, respectively, then $\approx x$ represents $r(x)$

3.1 $!x$ $!x$ is true unless x is $\bot$ in which case $!x$ is $\bot$. $!x$ satisfies

$$!x = \begin{cases} \mathsf{T} & \text{if } x = \mathsf{T} \\ \mathsf{T} & \text{if } x \text{ is proper} \\ \bot & \text{if } x = \bot \end{cases}$$

As an example of use, $x\dot{\vee}\dot{\neg}x = !x$ states that $x\dot{\vee}\dot{\neg}x$ is true unless x is $\bot$ in which case $x\dot{\vee}\dot{\neg}x$ is $\bot$

3.1 ¡x ¡x is false unless x is $\bot$ in which case ¡x is $\bot$. ¡x satisfies

$$¡x = \begin{cases} \mathsf{F} & \text{if } x = \mathsf{T} \\ \mathsf{F} & \text{if } x \text{ is proper} \\ \bot & \text{if } x = \bot \end{cases}$$

As an example of use, $x\dot{\wedge}\dot{\neg}x =$ ¡x states that $x\dot{\wedge}\dot{\neg}x$ is false unless x is $\bot$ in which case $x\dot{\wedge}\dot{\neg}x$ is $\bot$

3.1 $x\dot{\wedge}y$ Logical 'and' in map theory. $x\dot{\wedge}y$ satisfies

$$x\dot{\wedge}y = \begin{cases} \mathsf{T} & \text{if } x = \mathsf{T} \text{ and } y = \mathsf{T} \\ \mathsf{F} & \text{if } x = \mathsf{T} \text{ and } y \text{ is proper} \\ \mathsf{F} & \text{if } x \text{ is proper and } y = \mathsf{T} \\ \mathsf{F} & \text{if } x \text{ and } y \text{ are proper} \\ \bot & \text{otherwise} \end{cases}$$

3.1 $x \dot{\vee} y$ Logical 'or' in map theory. $x \dot{\vee} y$ satisfies

$$x \dot{\vee} y = \begin{cases} \mathsf{T} & \text{if } x = \mathsf{T} \text{ and } y = \mathsf{T} \\ \mathsf{T} & \text{if } x = \mathsf{T} \text{ and } y \text{ is proper} \\ \mathsf{T} & \text{if } x \text{ is proper and } y = \mathsf{T} \\ \mathsf{F} & \text{if } x \text{ and } y \text{ are proper} \\ \bot & \text{otherwise} \end{cases}$$

3.1 $x \dot{\Rightarrow} y$ Logical implication in map theory. $x \dot{\Rightarrow} y$ satisfies

$$x \dot{\Rightarrow} y = \begin{cases} \mathsf{T} & \text{if } x = \mathsf{T} \text{ and } y = \mathsf{T} \\ \mathsf{F} & \text{if } x = \mathsf{T} \text{ and } y \text{ is proper} \\ \mathsf{T} & \text{if } x \text{ is proper and } y = \mathsf{T} \\ \mathsf{T} & \text{if } x \text{ and } y \text{ are proper} \\ \bot & \text{otherwise} \end{cases}$$

The equation $x \dot{\Rightarrow} y = \mathsf{T}$ states that neither x nor y is $\bot$ and that $x = \mathsf{T}$ implies $y = \mathsf{T}$. In contrast, the equation $x \rightarrow y$ states that $x = \mathsf{T}$ implies $y = \mathsf{T}$ without requiring that x and y differ from $\bot$

3.1 $x \dot{\Leftrightarrow} y$ Logical biimplication in map theory. $x \dot{\Leftrightarrow} y$ satisfies

$$x \dot{\Leftrightarrow} y = \begin{cases} \mathsf{T} & \text{if } x = \mathsf{T} \text{ and } y = \mathsf{T} \\ \mathsf{F} & \text{if } x = \mathsf{T} \text{ and } y \text{ is proper} \\ \mathsf{F} & \text{if } x \text{ is proper and } y = \mathsf{T} \\ \mathsf{T} & \text{if } x \text{ and } y \text{ are proper} \\ \bot & \text{otherwise} \end{cases}$$

3.2 $\dot{\exists} x.\mathcal{A}$ The existence quantifier in map theory. $\dot{\exists} x.\mathcal{A}$ satisfies

$$\dot{\exists} x.\mathcal{A} = \begin{cases} \bot & \text{if } \mathcal{A} = \bot \text{ for some } x \in \Phi \\ \mathsf{F} & \text{if } \mathcal{A} \text{ is proper for all } x \in \Phi \\ \mathsf{T} & \text{otherwise} \end{cases}$$

3.2 $\dot{\forall} x.\mathcal{A}$ The universal quantifier in map theory. $\dot{\forall} x.\mathcal{A}$ satisfies

$$\dot{\forall} x.\mathcal{A} = \begin{cases} \bot & \text{if } \mathcal{A} = \bot \text{ for some } x \in \Phi \\ \mathsf{T} & \text{if } \mathcal{A} = \mathsf{T} \text{ for all } x \in \Phi \\ \mathsf{F} & \text{otherwise} \end{cases}$$

3.4 $x{::}y$ A pairing construct in map theory

3.4 $x \cdot y$ Tuple concatenation in map theory

3.6	$x \dot{\in} y$	The membership relation in map theory. If x and y are well-founded then $$x \dot{\in} y = \begin{cases} \mathsf{T} & \text{if } s(x) \in s(y) \\ \mathsf{F} & \text{otherwise} \end{cases}$$
3.7	$x \dot{\subseteq} y$	The subset relation in map theory
3.7	$x \cup y$	The union of two sets in map theory
3.9	$\bigcup x$	The union of all elements of x in map theory
4.1	$\vdash$	$\mathcal{A}_1 = \mathcal{B}_1; \ldots; \mathcal{A}_n = \mathcal{B}_n \vdash \mathcal{C} = \mathcal{D}$ states that $\mathcal{C} = \mathcal{D}$ is provable in map theory if $\mathcal{A}_1 = \mathcal{B}_1, \ldots, \mathcal{A}_n = \mathcal{B}_n$ are all provable. Note that the antecedents are separated by semicolons in $\mathcal{A}_1 = \mathcal{B}_1; \ldots; \mathcal{A}_n = \mathcal{B}_n \vdash \mathcal{C} = \mathcal{D}$ whereas the antecedents of $\mathcal{A}_1, \ldots, \mathcal{A}_n \to \mathcal{C} = \mathcal{D}$ are separated by commas. $\vdash$ is used both in inference rules and metatheorems
4.3	$\mathcal{A} \stackrel{*}{\rightharpoondown} \mathcal{B}$	The statement that the term $\mathcal{A}$ can be reduced to $\mathcal{B}$ in a finite number of reductions
4.3	$\models \mathcal{E}$	The statement that the equation $\mathcal{E}$ holds in the model M defined in Part III
5.3	$x{:}y$	$x{:}y$ is y if $x = \mathsf{T}$ and equals some default value if $x \neq \mathsf{T}$. $x{:}y$ satisfies $$x{:}y = \begin{cases} y & \text{if } x = \mathsf{T} \\ \mathsf{T} & \text{if } x \text{ is proper} \\ \bot & \text{if } x = \bot \end{cases}$$ $x{:}y$ is merely used in the definition of $\mathcal{A} \to \mathcal{E}$ and in metatheorems concerning the behavior of $\mathcal{A} \to \mathcal{E}$

5.3 $\mathcal{A}\rightarrow\mathcal{E}$

Let $\mathcal{A}_1,\ldots,\mathcal{A}_n$, $\mathcal{C}$ and $\mathcal{D}$ be terms. The construct

$$\mathcal{A}_1,\ldots,\mathcal{A}_n\rightarrow\mathcal{C}=\mathcal{D}$$

is shorthand for the equation

$$\mathcal{A}_1{:}\ldots{:}\mathcal{A}_n{:}\mathcal{C}=\mathcal{A}_1{:}\ldots{:}\mathcal{A}_n{:}\mathcal{D}$$

and states 'if $\mathcal{A}_1,\ldots,\mathcal{A}_n$ are all true, then $\mathcal{C}=\mathcal{D}$'. The construct

$$\mathcal{A}_1,\ldots,\mathcal{A}_n\rightarrow\mathcal{C}$$

is shorthand for the equation

$$\mathcal{A}_1,\ldots,\mathcal{A}_n\rightarrow\mathcal{C}=\mathsf{T}$$

and states 'if $\mathcal{A}_1,\ldots,\mathcal{A}_n$ are all true, then $\mathcal{C}$ is true. (See $\dot{\Rightarrow}$ and $\vdash$ for other notions of implication). According to theorem 5.4.1 and 5.3.1, $\mathcal{A}\rightarrow\mathcal{C}=\mathcal{D}$ if and only if $\mathcal{A}=\mathsf{T}\overset{h}{\vdash}\mathcal{C}=\mathcal{D}$.

The tautology $x{:}(y{:}z)=(x{:}y){:}z$ shows that parentheses are unnecessary in $\mathcal{A}_1{:}\ldots{:}\mathcal{A}_n{:}\mathcal{C}$. The tautology $\mathcal{A}_1{:}\ldots{:}\mathcal{A}_n{:}\mathcal{C}=(\mathcal{A}_1\dot{\wedge}\cdots\dot{\wedge}\mathcal{A}_n){:}\mathcal{C}$ shows that

$$\mathcal{A}_1,\ldots,\mathcal{A}_n\rightarrow\mathcal{C}=\mathcal{D}$$

if and only if

$$\mathcal{A}_1\dot{\wedge}\cdots\dot{\wedge}\mathcal{A}_n\rightarrow\mathcal{C}=\mathcal{D}$$

The formal definition reads: If $\mathcal{A}$ is a term and $\mathcal{E}$ is an equation, then $\mathcal{A}\rightarrow\mathcal{E}$ is shorthand for $\mathcal{A}{:}\mathcal{C}=\mathcal{A}{:}\mathcal{D}$ where $\mathcal{C}$ and $\mathcal{D}$ are the left and right hand sides of $\mathcal{E}$, respectively. Further, $\mathcal{A}_1,\ldots,\mathcal{A}_n\rightarrow\mathcal{E}$ is shorthand for $\mathcal{A}_1\rightarrow(\mathcal{A}_2\rightarrow(\cdots(\mathcal{A}_n\rightarrow\mathcal{E})\cdots))$

Index of Part III

This is an index of notation used in Part III. The index is kept separate from that of Part I and II since the notation in Part III differs slightly from that of Part I and II.

For each entry, the index displays a mathematical construct, refers to the section where it is defined, and gives a short, informal explanation of the construct. Some constructs both have a formal and an informal definition in the text. In such cases, the index refers to both, and parenthesizes the reference to the informal definition.

Part III introduces light and shade into the notation of Part I and II. In particular, Part III distinguishes between terms and the maps they denote, and Part III constructs several models of parts of map theory that lead up to a model of all of map theory. For this reason, many constructs of map theory have several names: one for each model, and one for the Gödel-number. As an example, $\hat{a}(f,x)$, $\breve{a}(f)(x)$, $\acute{a}(f)(x)$, and $a(f)(x)$ informally all denote the root of $(f\, x_1 \ldots x_n)$ where $\langle x_1, \ldots, x_n \rangle = x$. However, $\hat{a}(f,x)$ applies to elements of a model $\hat{\Phi}$ of well-founded maps, $\breve{a}(f)(x)$ applies to elements of a relativization $\breve{\Phi}$ of that model, $\acute{a}(f)(x)$ applies to elements of $\acute{M}$ which consists of 'terms', i.e. Gödel-numbers, and $a(f)(x)$ applies to elements of a model M of all maps. As can be seen, accents are used to distinguish between different versions of the same concept. Further, each accent refers to a specific model so that constructs with the same accent live within the same model.

Another difference between Part I, II, and III is that Part III uses dot accents (like in $\dot{A}(f,x)$) to denote Gödel-numbers whereas Part I and II use dot accents to distinguish between concepts of map and set theory.

Constructs with a dot accent like $\dot{A}(f,x)$ take Gödel-numbers as arguments and produce Gödel-numbers as their result. In contrast, Constructs with an acute accent like $\acute{a}(f)(x)$ take Gödel-numbers as arguments and produces other than Gödel-numbers as their result. The Gödel-number of a term can be expressed simply by putting a dot accent above each construct in the term.

It is not always easy to see which parts of a construct are variables and which are part of the construct itself. As an example, consider the construct $x \hat{=}^{a}_{G} y$. This is an equivalence relation in x and y parameterized over G. Hence, x, y, and G are variables whereas a is part of the name of the relation. The name of the relation consists of an equal sign, an accent, and the letter a (the relation is related to the function $\hat{a}(f,x)$, but that is another story). When confusion is possible, the index makes clear which parts of a construct are variables by mentioning all the variables explicitly in the explanation of the construct.

Relations that involve stars such as $x \hat{=}^{a*}_{G} y$ are excluded from the index. Decoration of a relation with a star makes the relation work coordinatewise on tuples. See Section 9.6 for details. Further, negations of relations like $x \hat{\neq}^{a}_{G} y$ are excluded from the index.

Constructs involving parentheses

	9.2	$f(x)$	The function f applied to the argument x. Both f and x are sets
	9.2	(x, y)	The ordered pair of x and y
	9.4	$\langle x_1, \ldots, x_n \rangle$	A tuple with the n elements $x_1, \ldots, x_n$
	9.4	$\langle \rangle$	The empty tuple
	9.8	$f\langle\langle x \rangle\rangle$	The well-founded function f applied to the argument x. When f is not well-founded, $f\langle\langle x \rangle\rangle$ forces f to behave somewhat like a well-founded function
(10.1)	11.9	$(f\, x_1 \ldots x_n)$	The map f applied to the maps $x_1, \ldots, x_n$ in turn
	9.9	$\dot{(}f\, x_1 \ldots x_n\dot{)}$	The Gödel number of $(f\, x_1 \ldots x_n)$. Note the dots above the parentheses
	9.9	$\lceil \mathcal{A} \rceil$	The Gödel number of the expression $\mathcal{A}$
	9.10	$\lfloor \mathcal{A} \rfloor$	The relativization of the well-formed formula $\mathcal{A}$ to the model D of ZFC
	12.3	$[\mathcal{A}/x{:=}\mathcal{B}]$	The result of replacing all free occurrences of the variable x in the term $\mathcal{A}$ by the term $\mathcal{B}$
	12.4	$[\![\mathcal{A}]\!]$	The combinator term corresponding to the λ-term $\mathcal{A}$

Constructs involving $=$

	9.5	$G{=_\kappa}H$	Predicate stating the sets G and H have the same cardinality
(10.7)	11.4	$f\acute{=}g$	Predicate stating that the terms f and g denote the same map
	12.5	$\mathcal{A}\dot{=}\mathcal{B}$	Predicate stating that for all values of free variables, the terms $\mathcal{A}$ and $\mathcal{B}$ denote the same map
	11.1	$f\hat{=}^a_G g$	Predicate stating that the maps $f, g \in \hat{\Phi}$ are observationally equivalent when using elements of the set $G \subseteq \hat{\Phi}$ for observations
	11.4	$f\breve{=}^a_G g$	The relativization of $f\hat{=}^a_G g$
	10.2	$f{=}^a_G g$	Predicate stating that the maps f and g are observationally equivalent when using elements of the set G of maps for observations
	11.4	$f\acute{=}^{\Phi}_{v} g$	An approximation to $f\acute{\leq}g$ for $f \in \breve{\Phi}$, $g \in \acute{M}$, and $v \in \acute{M} \rightarrow L$. The relations $f\acute{=}^{\Phi}_{v} g$ and $f\acute{\leq}g$ coincide when $v = \acute{r}$. For $g \in \acute{M}$ and any ordinal α, $f\acute{\leq}g$ holds for at most one $f \in \breve{\Phi}'(\alpha)$ which explains why an equivalence relation can approximate a partial order
	11.4	$f\acute{=}^{Q}_{v,w} g$	An approximation to $f\acute{\leq}g$ to be used in connection with $\acute{=}^{\Phi}_{v}$. In the construct, f, g, v, and w are variables

Constructs involving $\leq$

	9.3	$\alpha{\leq_o}\beta$	Predicate stating that the ordinal α is less than or equal to the ordinal β

	9.5	$G \leq_\kappa H$	Predicate stating that the cardinality of the set G is less than or equal to that of H
	9.7	$x \leq_L y$	A predicate stating that the information contents of the label x is less than or equal to that of y
(10.1)	13.5	$f \leq g$	Predicate stating that the information contents of the map f is less than or equal to that of g
(10.7)	11.4	$f \acute{\leq} g$	Predicate stating that the map denoted by the term f has information contents less than or equal to that of the map denoted by g

Constructs involving <

	9.3	$\alpha <_o \beta$	Predicate stating that the ordinal α is less than the ordinal β
	11.2	$x <_{oo} y$	A well-founded relation which is useful for simultaneous transfinite induction in two variables
	9.5	$G <_\rho H$	Predicate stating that the rank of the set G is less than that of H
	11.1	$x <_{\rho\rho} y$	A well-founded relation which is useful for simultaneous transfinite induction in two variables
	9.5	$G <_\kappa H$	Predicate stating that the cardinality of the set G is less than that of H
	9.8	$f <_w g$	A well-founded relation on well-founded functions f and g which is useful for transfinite induction
(10.1)	11.10	$f <_A g$	A well-founded relation on Φ which is useful for transfinite induction
	11.10	$f <_i g$	A well-founded relation on Φ used for speculations

Alphabetic constructs

Greek

	9.9	$\dot{\varepsilon} x$	The Gödel number of εx
	10.1	$\lambda x.\mathcal{A}$	The map which maps x to $\mathcal{A}$ where x may occur free in $\mathcal{A}$
	9.9	$\dot{\lambda} x.\mathcal{A}$	The Gödel number of $\lambda x.\mathcal{A}$
	12.4	$\grave{\lambda} x.\mathcal{A}$	The combinator term that represents $\lambda x.\mathcal{A}$ where $\mathcal{A}$ is a combinator term
	9.7	$\tilde{\lambda}$	The root of any proper map, i.e. the label of the root node of any proper map
	11.4	ξ	The least ordinal with cardinality greater than $\acute{M} \rightarrow L$
	9.5	$\rho(G)$	The rank of the set G
	9.10	σ	Strongly inaccessible ordinal
	9.9	$\dot{\phi} f$	The Gödel number of ϕf
	14.3	$\dot{\phi}_1 f$	The Gödel number of $\phi_1 f$ where $\phi_1 f = \forall x.\phi(f\, x)$
(10.3)	11.9	Φ	The set of well-founded maps

(10.6)	11.1	$\hat{\Phi}$	A model of well-founded maps. This model is the backbone of the model M of all maps
(10.4)	11.3	$\check{\Phi}$	The relativization of $\hat{\Phi}$
	11.4	$\acute{\Phi}$	The image of $\check{\Phi}$ in M under the injection $c(\cdot)$
(10.3)	14.2	$\Phi'(\alpha)$	Stage in forming Φ
	14.2	$\Phi'_\partial(\alpha)$	The boundary (i.e. set of minimal elements) of $\Phi'(\alpha)$
(10.6)	11.1	$\hat{\Phi}'(\alpha)$	Stage in forming $\hat{\Phi}$
(10.4)	11.3	$\check{\Phi}'(\alpha)$	The relativization of $\hat{\Phi}'(\alpha)$
	14.2	$\acute{\Phi}'(\alpha)$	The counterpart of $\check{\Phi}'(\alpha)$ in $\acute{M}$
(10.3)	14.2	$\Phi''(\alpha)$	Stage in forming Φ
	14.2	$\Phi''_\partial(\alpha)$	The boundary (i.e. set of minimal elements) of $\Phi''(\alpha)$
(10.6)	11.1	$\hat{\Phi}''(\alpha)$	Stage in forming $\hat{\Phi}$
(10.4)	11.3	$\check{\Phi}''(\alpha)$	The relativization of $\hat{\Phi}''(\alpha)$
	14.2	$\acute{\Phi}''(\alpha)$	The counterpart of $\check{\Phi}''(\alpha)$ in $\acute{M}$
	9.3	ω	The least infinite ordinal
	9.4	G^ω	The set of infinite sequences of elements of the set G

A

(10.1)	14.2	$a(f)(x)$	The root of $(f\, x_1 \ldots x_n)$ where $\langle x_1, \ldots, x_n\rangle = x$
(10.5)	11.1	$\hat{a}(f, x)$	The root of $(f\, x_1 \ldots x_n)$ where $\langle x_1, \ldots, x_n\rangle = x$ and $f, x_1, \ldots, x_n \in \hat{\Phi}$
(10.8)	11.3	$\check{a}(f)(x)$	The relativization of $\hat{a}(f, x)$
(10.7)	11.4	$\acute{a}(f)(x)$	The root of the map denoted by the term $(f\, x_1 \ldots x_n)$ where $\langle x_1, \ldots, x_n\rangle = x$ and $f, x_1, \ldots, x_n$ are terms
(10.1)	11.9	$A(f, x)$	The map f applied to the map x
	9.9	$\dot{A}(f, x)$	The Gödel number of the map f applied to the map x

B

	12.5	${}^B_d\mathcal{A}$	Predicate stating that the value of the term $\mathcal{A}$ is three-valued Boolean (i.e. has one of the values T, F, or $\perp$) for the assignment $d \in M^V$ of values to free variables
	15.3	$b\dot{e}long$	The Gödel number of a map that expresses the set membership relation

C

(10.7)	11.9	$c(f)$	The map denoted by the term f
(10.7)	14.2	$c^*(f)$	Coordinatewise application of $c(\cdot)$
	14.2	$c''G$	The set of maps denoted by elements of the set G of terms
(10.1)	11.9	C	A particular map
	9.9	$\dot{C}$	The Gödel number of C

9	$Con(x)$	The statement that the axiomatic theory x is consistent
14.3	$\dot{Curry}$	The Gödel number of the map *Curry* (c.f. Section 7.3)

D

9.2	f^d	The domain of the function f
9.2	G^D	The union of domains of functions in the set G of functions. We have $(G \to H)^D = G$ for $H \neq \emptyset$
9.10	D	The domain of a transitive standard model of ZFC. Many definitions depend on D implicitly
12.5	${}^{D}_{d}\mathcal{A}$	Predicate stating that the value of the term $\mathcal{A}$ is defined (i.e. differs from $\perp$) for the assignment $d \in M^V$ of values to free variables
15.4	D^V	The set of assignments of values to variables in the model D of ZFC

E

11.6	$\acute{E}$	The set of polynomials in one variable
15.3	$\dot{equal}$	The Gödel number of a map that expresses the set equality relation

F

12.1	$\dot{f}$	The Gödel number of a particular variable of map theory
12.5	F	The map $\lambda x.\mathsf{T}$ which, by convention, represents falsehood
13.5	$\dot{\mathsf{F}}$	The Gödel number of F (c.f. Section 3.1)
15.4	F^V	The set of assignments of well-founded maps to variables in the model M of map theory
9.2	$fnc(x)$	Predicate stating that the set x is a function
12.3	$free(x, \mathcal{A})$	Predicate stating that the variable x occurs free in the term $\mathcal{A}$
12.3	$freefor(\mathcal{A}, x, \mathcal{B})$	Predicate stating that the substitution $[\mathcal{A}/x{:=}\mathcal{B}]$ is free from variable conflicts

G

12.1	$\dot{g}$	The Gödel number of a particular variable of map theory

H

12.1	$\dot{h}$	The Gödel number of a particular variable of map theory

I

	11.2	$I'_{\alpha,\beta}$	Predicate stating the injectivity of certain functions. Used for proving Corollary 11.2.2 by transfinite induction
	9.9	$(\dot{\mathsf{if}}\, x\, y\, z)$	The Gödel number of $(\mathsf{if}\, x\, y\, z)$

K

(10.1)	11.9	K	A particular map
	9.9	$\dot{K}$	The Gödel number of K

L

	9.7	L	The set of labels, $L = \{\tilde{\mathsf{T}}, \tilde{\lambda}, \tilde{\bot}\}$

M

(10.1)	11.9	$m(f,x)$	The map f applied to the maps $x_1, \ldots, x_n$ in turn where $\langle x_1, \ldots, x_n \rangle = x$
(10.7)	11.4	$\acute{m}(f,x)$	The term $(f\, x_1\, \ldots\, x_n)$ where $\langle x_1, \ldots, x_n \rangle = x$.
(10.1)	11.9	M	A model of map theory, i.e. M may be seen as the set of all maps. The definition of M depends on D implicitly
(10.7)	11.4	$\acute{M}$	Term model of map theory
	12.1	$\grave{M}$	The set of Gödel numbers of λ-terms with free variables
	11.7	$\acute{M}'$	A set used for proving the Root Theorem
	12.1	$\grave{M}'$	The set of Gödel numbers of combinator terms with free variables
	12.5	M^V	The set of assignments of values to variables in the model M of map theory
	9	*Map*	An axiomatic theory of maps
	9	Map°	A reduced version of *Map*
	9	$Map^{\circ+}$	An extension of Map°

O

	9.3	*On*	The class of all ordinals
	9.3	$ord(x)$	Predicate stating that the set x is an ordinal

P

	9.1	$\mathcal{P}G$	The power set of the set G
(10.1)	11.9	P	A particular map
	9.9	$\dot{P}$	The Gödel number of P
	14.3	$\grave{P}$	The Gödel number of the map P when expressed as a λ-term (c.f. Section 7.3)

	14.3	$\dot{Prim}$	The Gödel number of the map *Prim* (c.f. Section 7.3)

Q

	13.2	$q(x)$	A choice function on Φ corresponding to the choice functions $\check{q}(G)$ and $\acute{q}(G)$
	11.4	$\check{q}(G)$	A choice function on $\check{\Phi}$
	13.2	$\acute{q}(G)$	A choice function on $\acute{\Phi}$ corresponding to the choice function $\check{q}(G)$
(10.6)	11.1	$\hat{Q}$	The union of dual stages $\hat{Q}'(\alpha)$ used in forming $\hat{\Phi}$
(10.4)	11.3	$\check{Q}$	The relativization of $\hat{Q}$
(10.3)	14.2	$Q'(\alpha)$	Dual stage in forming Φ
(10.6)	11.1	$\hat{Q}'(\alpha)$	Dual stage in forming $\hat{\Phi}$
(10.4)	11.3	$\check{Q}'(\alpha)$	The relativization of $\hat{Q}'(\alpha)$
	14.2	$\acute{Q}'(\alpha)$	The counterpart of $\check{Q}'(\alpha)$ in $\acute{M}$
	14.2	$Q'_{\partial}(\alpha)$	The boundary (i.e. set of minimal elements) of $Q'(\alpha)$

R

	10.1	$r(f)$	The root of the map f, i.e. the label of the root node of f
(10.7)	11.4	$\acute{r}(f)$	The root of the map denoted by the term f
	11.4	$\acute{r}'(v)$	Function which, given an approximation v to $\acute{r}$, produces a better approximation. The function $\acute{r}$ is the least fixed point of $\acute{r}'$
	11.4	$\acute{r}''(\alpha)$	The function $\acute{r}'$ iterated α times
	9.2	f^r	The range of the function f
	9.2	$\mathbf{R}$	The set of real numbers
	11.2	$R'_{\alpha,\beta}$	Predicate describing the domain and range of certain functions. Used for proving Corollary 11.2.2 by transfinite induction
	9.2	G^R	The union of ranges of functions in the set G of functions. We have $G^{*R} = G$ and $(G \rightarrow H)^R = H$ for $G \neq \emptyset$

S

(10.1)	15.1	$s(f)$	The set represented by the well-founded map f
(10.6)	15.2	$\hat{s}(f)$	The set represented by the map $f \in \hat{\Phi}$
(10.1)	11.9	S	A particular map
	9.9	$\dot{S}$	The Gödel number of S
	11.2	$S'_{\alpha,\beta}$	Predicate stating the surjectivity of certain functions. Used for proving Corollary 11.2.2 by transfinite induction
	9	SI	An axiom beyond ZFC which asserts the existence of a strongly inaccessible ordinal

T

(10.2)	14.2	$t_G(f)$	The observational class of the map f when using elements of the set G of maps for observations
	11.1	$\hat{t}_G(f)$	The observational class of $f \in \hat{\Phi}$ when using elements of the set $G \subseteq \hat{\Phi}$ for observations
	11.2	$\check{t}_G(f)$	The relativization of $\hat{t}_G(f)$
	14.2	$\acute{t}_G(f)$	The observational behavior of the map denoted by the term f when using elements of the set G of maps for observations
	14.2	$t^*_G(f)$	Coordinatewise application of $t_G(\cdot)$
	11.1	$\hat{t}^*_G(f)$	Coordinatewise application of $\hat{t}_G(\cdot)$
	14.2	$\acute{t}^*_G(f)$	Coordinatewise application of $\acute{t}_G(\cdot)$
	9.5	$tc(G)$	The transitive closure (w.r.t. $\in$) of the set G
	9.4	$tpl\,(x)$	Predicate stating that the set x is a tuple
(10.1)	11.9	T	A particular map
	9.9	$\dot{\mathsf{T}}$	The Gödel number of T
	11.10	$\hat{\mathsf{T}}$	The element of $\hat{\Phi}$ that corresponds to T
	11.10	$\check{\mathsf{T}}$	The relativization of $\hat{\mathsf{T}}$
	9.7	$\tilde{\mathsf{T}}$	The root of T, i.e. the label of the root node of T
	12.5	${}^{T}_{d}\mathcal{A}$	Predicate stating that the term $\mathcal{A}$ is true (i.e. has the value T) for the assignment $d \in M^V$ of values to free variables

U

	12.1	$\dot{u}$	The Gödel number of a particular variable of map theory
	9.1	$\bigcup G$	The union of all elements of the set G

V

	12.1	$\dot{v}$	The Gödel number of a particular variable of map theory
	9.9	$\dot{v}_i$	The Gödel number of the i'th variable of map theory
	9.9	$\ddot{v}_i$	The Gödel number of the i'th variable of ZFC
	12.1	$\dot{V}$	The set of Gödel numbers of variables

W

	12.1	$\dot{w}$	The Gödel number of a particular variable of map theory
(10.3)	14.2	$wf\,(G)$	The set of maps f that are well-founded w.r.t. the set G of maps, c.f. G°
	14.2	$\acute{wf}\,(G)$	The set of terms that denote maps that are well-founded w.r.t. the set G of maps
(10.1)	11.9	W	A particular map
	9.9	$\dot{W}$	The Gödel number of W

X

12.1	$\dot{x}$	The Gödel number of a particular variable of map theory
9.9	$\ddot{x}$	The Gödel number of a particular variable of *ZFC*

Y

12.1	$\dot{y}$	The Gödel number of a particular variable of map theory
9.9	$\ddot{y}$	The Gödel number of a particular variable of *ZFC*
13.5	Y	The Gödel number of Y (c.f. Section 3.3)

Z

12.1	$\dot{z}$	The Gödel number of a particular variable of map theory
15.4	Z	The set of well-formed formulas of *ZFC*
9	ZFC	An axiomatic theory of sets
9	ZFC^{+}	An extension of *ZFC*

Other constructs

9.1	$\neg\mathcal{A}$	Negation of $\mathcal{A}$
13.5	$\dot{\neg}\mathcal{A}$	The Gödel number of $\neg\mathcal{A}$ (c.f. Section 3.1)
9.9	$\ddot{\neg}\mathcal{A}$	The Gödel number of $\neg\mathcal{A}$ in *ZFC*
13.5	$\mathcal{A}\dot{\wedge}\mathcal{B}$	The Gödel number of $\mathcal{A}\wedge\mathcal{B}$ (c.f. Section 3.1)
13.5	$\mathcal{A}\dot{\vee}\mathcal{B}$	The Gödel number of $\mathcal{A}\vee\mathcal{B}$ (c.f. Section 3.1)
9.1	$\mathcal{A} \Rightarrow \mathcal{B}$	Implication, $\mathcal{A}_1 \Rightarrow \mathcal{A}_2 \Rightarrow \cdots \Rightarrow \mathcal{A}_n$ means $(\mathcal{A}_1 \Rightarrow \mathcal{A}_2) \wedge (\mathcal{A}_2 \Rightarrow \mathcal{A}_3) \wedge \ldots \wedge (\mathcal{A}_{n-1} \Rightarrow \mathcal{A}_n)$
13.5	$\mathcal{A}\dot{\Rightarrow}\mathcal{B}$	The Gödel number of $\mathcal{A}\Rightarrow\mathcal{B}$ (c.f. Section 3.1)
9.9	$\mathcal{A}\ddot{\Rightarrow}\mathcal{B}$	The Gödel number of $\mathcal{A} \Rightarrow \mathcal{B}$ in *ZFC*
9.1	$\mathcal{A} \Leftrightarrow \mathcal{B}$	Biimplication, $\mathcal{A}_1 \Leftrightarrow \mathcal{A}_2 \Leftrightarrow \cdots \Leftrightarrow \mathcal{A}_n$ means $(\mathcal{A}_1 \Leftrightarrow \mathcal{A}_2) \wedge (\mathcal{A}_2 \Leftrightarrow \mathcal{A}_3) \wedge \ldots \wedge (\mathcal{A}_{n-1} \Leftrightarrow \mathcal{A}_n)$
13.5	$\mathcal{A}\dot{\Leftrightarrow}\mathcal{B}$	The Gödel number of $\mathcal{A}\Leftrightarrow\mathcal{B}$ (c.f. Section 3.1)
13.5	$\dot{\forall}x.\mathcal{A}$	The Gödel number of $\forall x.\mathcal{A}$ (c.f. Section 3.2)
9.9	$\ddot{\forall}x : \mathcal{A}$	The Gödel number of $\forall x : \mathcal{A}$ in *ZFC*
13.5	$\dot{\exists}x.\mathcal{A}$	The Gödel number of $\exists x.\mathcal{A}$ (c.f. Section 3.2)
15.1	$x\dot{\in}y$	The Gödel number of the map $x \in y$. In contrast, $x\ddot{\in}y$ is the Gödel number of the term $x \in y$ in *ZFC*. Note that Part I and II in general and Section 3.6 in particular use $\dot{\in}$ for the map itself rather than the Gödel number
9.9	$x\ddot{\in}y$	The Gödel number of $x \in y$ in *ZFC*
15.3	$x\dot{=}y$	The Gödel number of the map set equality predicate expressed in map theory. See also $\dot{\in}$. Note that Part I and II in general and Section 3.6 in particular use $\dot{\in}$ for the map itself rather than the Gödel number

	13.5	$\dot{\approx} x$	The Gödel number of $\approx x$ (c.f. Section 3.1)
	13.5	$\dot{!}x$	The Gödel number of $!x$ (c.f. Section 3.1)
	13.5	$\dot{¡}x$	The Gödel number of $¡x$ (c.f. Section 3.1)
	13.5	$x\dot{:}y$	The Gödel number of $x{:}y$ (c.f. Section 5.3)
	13.5	$x_1, \ldots, x_n \dot{\rightarrow} y$	The Gödel number of the equation $x_1, \ldots, x_n \rightarrow y$ (c.f. Section 5.3)
(10.1)	11.9	$\bot$	A particular map
	9.9	$\dot{\bot}$	The Gödel number of $\bot$
	9.7	$\tilde{\bot}$	The root of $\bot$, i.e. the label of the root node of $\bot$
	9.2	$G \times H$	The Cartesian product of the sets G and H
	9.2	$G \rightarrow H$	The set of functions from G into H
	9.2	$x \in G \mapsto \mathcal{A}$	The function with domain G that maps x to $\mathcal{A}$ where the term $\mathcal{A}$ may contain x free
	9.2	$f \circ g$	Functional composition
	9.2	$f \vert G$	The function f restricted to the domain G
	9.3	0	The ordinal *zero*, $0 = \emptyset$
	9.3	α^+	The successor of the ordinal α
	9.4	G^*	The set of tuples of elements of the set G
	9.4	$x \cdot y$	Tuple concatenation
	9.7	$\sqcup G$	The least upper bound of a chain G of labels
	9.8	G°	The set of functions f that are well-founded on the set G. The elements of G° are set theoretic functions (i.e. sets of pairs) as opposed to maps. See also $wf\,(G)$
(10.4)	14.2	∂G	The set of minimal elements of the set G of maps (the 'boundary' of G)
	14.2	∇G	The set of maps with the same or more information contents than elements of the set G of maps
	14.2	$\dot{\nabla} G$	A version of ∇ that operates on terms rather than maps
	15.4	$\downarrow\!\mathcal{A}$	The well-formed formula $\mathcal{A}$ of *ZFC* translated to map theory
	12.5	${}_d\mathcal{A}$	The interpretation of the term $\mathcal{A}$ for the assignment $d \in M^V$

The grammar of map theory

variable	::=	$x \mid y \mid z \mid \ldots$
term	::=	variable | λvariable.term | (term term) | T | $\perp$ |
		(if term term term) | ϕterm | εterm
wff	::=	term = term

Various definitions in map theory

See the index of Part I and II for explanations of constructs

$$
\begin{array}{lcl}
\mathsf{F} & = & \lambda x.\mathsf{T} \\
\dot\neg x & = & (\mathsf{if}\ x\ \mathsf{F}\ \mathsf{T}) \\
\approx x & = & (\mathsf{if}\ x\ \mathsf{T}\ \mathsf{F}) \\
!x & = & (\mathsf{if}\ x\ \mathsf{T}\ \mathsf{T}) \\
¡x & = & (\mathsf{if}\ x\ \mathsf{F}\ \mathsf{F}) \\
x \dot\wedge y & = & (\mathsf{if}\ x\ (\mathsf{if}\ y\ \mathsf{F}\ \mathsf{T})\ (\mathsf{if}\ y\ \mathsf{F}\ \mathsf{F})) \\
x \dot\vee y & = & (\mathsf{if}\ x\ (\mathsf{if}\ y\ \mathsf{T}\ \mathsf{T})\ (\mathsf{if}\ y\ \mathsf{T}\ \mathsf{F})) \\
x \dot\Rightarrow y & = & (\mathsf{if}\ x\ (\mathsf{if}\ y\ \mathsf{T}\ \mathsf{F})\ (\mathsf{if}\ y\ \mathsf{T}\ \mathsf{T})) \\
x \dot\Leftrightarrow y & = & (\mathsf{if}\ x\ (\mathsf{if}\ y\ \mathsf{T}\ \mathsf{F})\ (\mathsf{if}\ y\ \mathsf{F}\ \mathsf{T})) \\
\dot\exists \mathcal{A} & = & \approx(\mathcal{A}\, \varepsilon \mathcal{A}) \\
\dot\exists x.\mathcal{A} & = & \dot\exists(\lambda x.\mathcal{A}) \\
\dot\forall x.\mathcal{A} & = & \dot\neg\dot\exists x.\dot\neg\mathcal{A} \\
x \dot= y & = & (\mathsf{if}\ x\ (\mathsf{if}\ y\ \mathsf{T}\ \mathsf{F})\ (\mathsf{if}\ y\ \mathsf{F}\ (\dot\forall u \dot\exists v.(x\,u) \dot= (y\,v)) \dot\wedge (\dot\forall v \dot\exists u.(x\,u) \dot= (y\,v)))) \\
a \dot\in b & = & (\mathsf{if}\ b\ \mathsf{F}\ \dot\exists v.a \dot= (b\,v)) \\
\mathsf{Y} & = & \lambda f.((\lambda x.(f\,(x\,x)))\,(\lambda x.(f\,(x\,x)))) \\
\mathsf{Y} f.\mathcal{A} & = & (\mathsf{Y}\,\lambda f.\mathcal{A}) \\
P & = & \lambda a.\lambda b.\lambda x.(\mathsf{if}\ x\ a\ b) \\
\mathit{Curry} & = & \lambda a.\lambda x.\lambda y.(a\,(P\,x\,y)) \\
\mathit{Prim} & = & \lambda f.\lambda a.\lambda b.\mathsf{Y} g.\lambda x.(\mathsf{if}\ x\ a\ (f\,\lambda u.(g\,(x\,(b\,u))))) \\
\mathsf{F}' & = & \lambda f.\lambda x.(f\,x) \\
\phi x.\mathcal{A} & = & \phi\lambda x.\mathcal{A} \\
x{:}y & = & (\mathsf{if}\ x\ y\ \mathsf{T})
\end{array}
$$

$\mathcal{A}_1, \ldots, \mathcal{A}_n \rightarrow (\mathcal{C} = \mathcal{D})$ is shorthand for the equation $\mathcal{A}_1{:}\ldots{:}\mathcal{A}_n{:}\mathcal{C} = \mathcal{A}_1{:}\ldots{:}\mathcal{A}_n{:}\mathcal{D}$
$\mathcal{A}_1, \ldots, \mathcal{A}_n \rightarrow \mathcal{C}$ is shorthand for the equation $\mathcal{A}_1{:}\ldots{:}\mathcal{A}_n{:}\mathcal{C} = \mathcal{A}_1{:}\ldots{:}\mathcal{A}_n{:}\mathsf{T}$

Axioms/inference rules of Section 4 (λ-calculus)

Trans	$\mathcal{A}=\mathcal{B};\mathcal{A}=\mathcal{C}\vdash\mathcal{B}=\mathcal{C}$
Sub1	$\mathcal{A}=\mathcal{B};\mathcal{C}=\mathcal{D}\vdash(\mathcal{A}\,\mathcal{C})=(\mathcal{B}\,\mathcal{D})$
Sub2	$\mathcal{A}=\mathcal{B}\vdash\lambda x.\mathcal{A}=\lambda x.\mathcal{B}$
Apply 1	$(\mathsf{T}\,\mathcal{B})=\mathsf{T}$
Apply 2	$((\lambda x.\mathcal{A})\,\mathcal{B})=[\mathcal{A}/x{:=}\mathcal{B}]$ if $\mathcal{B}$ is free for x in $\mathcal{A}$
Apply 3	$(\bot\,\mathcal{B})=\bot$
Select 1	$(\mathsf{if}\,\mathsf{T}\,\mathcal{B}\,\mathcal{C})=\mathcal{B}$
Select 2	$(\mathsf{if}\,(\lambda x.\mathcal{A})\,\mathcal{B}\,\mathcal{C})=\mathcal{C}$
Select 3	$(\mathsf{if}\,\bot\,\mathcal{B}\,\mathcal{C})=\bot$
Rename	$\lambda x.[\mathcal{A}/y{:=}x]=\lambda y.[\mathcal{A}/x{:=}y]$ if x is free for y in $\mathcal{A}$ and vice versa

Axioms/inference rules of Section 5 (propositional calculus)

QND'	$[\mathcal{A}/x{:=}\mathsf{T}]=[\mathcal{B}/x{:=}\mathsf{T}]$; $[\mathcal{A}/x{:=}(\mathsf{F}'\,x)]=[\mathcal{B}/x{:=}(\mathsf{F}'\,x)]$; $[\mathcal{A}/x{:=}\bot]=[\mathcal{B}/x{:=}\bot]$ $\vdash\mathcal{A}=\mathcal{B}$

Axioms/inference rules of Section 6 (first order predicate calculus)

Quantify 1	$\phi\mathcal{A},\dot\forall x.\mathcal{B}\rightarrow((\lambda x.\mathcal{B})\,\mathcal{A})$
Quantify 2	$\varepsilon x.\mathcal{A}=\varepsilon x.(\phi x\dot\wedge\mathcal{A})$
Quantify 3	$\phi\varepsilon x.\mathcal{A}=\dot\forall x.!\mathcal{A}$
Quantify 4	$\dot\exists x.\mathcal{A}\rightarrow\phi\varepsilon x.\mathcal{A}$
Quantify 5	$\dot\forall x.\mathcal{A}=\dot\forall x.(\phi x\dot\wedge\mathcal{A})$

Axioms/inference rules of Section 7 (set theory)

Well 1	$\phi\mathsf{T}$
Well 2	$\phi\lambda x.\mathcal{A}=\phi\lambda x.\phi\mathcal{A}$
Well 3	$\phi\bot=\bot$
C-A	$\phi a,\phi b\rightarrow\phi(a\,b)$
C-K'	$\phi x.\mathsf{T}$
C-P'	$\phi x.(\mathsf{if}\,x\,\mathsf{T}\,\mathsf{T})$
C-Curry	$\phi a\rightarrow\phi(\mathit{Curry}\,a)$
C-Prim	$\dot\forall x.\phi(f\,x),\phi a,\phi b\rightarrow\phi(\mathit{Prim}\,f\,a\,b)$
C-M1	$\dot\forall u.\phi x.\mathcal{A}\rightarrow\dot\forall u.\phi x.((\lambda u.\mathcal{A})\,(u\,x))$
C-M2	$\dot\forall u.\phi x.\mathcal{A}\rightarrow\dot\forall u.\phi x.((\lambda x.\mathcal{A})\,(x\,u))$
Induction	If x does not occur free in $\mathcal{A}$ and y does not occur (free or bound) in $\mathcal{B}$, then $\mathcal{A},x\rightarrow\mathcal{B};\mathcal{A},\dot\neg x,\phi x,\dot\forall y.[\mathcal{B}/x{:=}(x\,y)]\rightarrow\mathcal{B}\vdash\mathcal{A}\rightarrow\mathcal{B}$

See the addendum in this volume for an explanation of the correspondence between Sections 4 to 7 and λ-calculus, propositional calculus, first order predicate calculus and set theory.