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REPRESENTING SIGNALS BY THEIR TOPPOINTS IN SCALE SPACE

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Abstract

Witkin (1983) introduced the *fingerprint* of a function as a set **F** of points in *scale space*, where scale space is the plane. Fingerprints are calculated by convolving the function with a Gaussian with continuously varying standard deviation. He defined the toppoints of the signal as points in scale space where **F** has a horizontal tangent. This paper proves that periodic, bandlimited functions are defined up to a multiplicative constant by their toppoints, if this concept is properly generalized. The uniqueness theorem may be regarded as a sampling theorem for signals in the scale space.

Introduction.

During the past decades researchers have realized that conventional mathematical tools like series expansion and Fourier analysis are inadequate for image understanding and image analysis. At the same time the idea of multi-resolution or multiscale representation of images has received increasing attention. Multiresolution representation of image features has been used for edge detection [1,2], region extraction [3,4], image segmentation [5], motion analysis [6], stereo matching [7] and image encoding and compression [8].

A signal or image can be represented at a given *resolution* or *scale* by convolution (filtering) with a gaussian-shaped lowpass filter [9]. The result is an averaged or blurred version of the original image or signal. The purpose of filtering is to wipe out irrelevant details below a certain chosen scale before subsequent analysis. The blurring process can be controlled by the filter width: filtering with a filter of small scale retains more details than at larger scales. In the limit as the scale converges to zero the filter becomes a Dirac delta-function, which leaves the signal or image unchanged, whereas all features are removed at infinite scale. Multiscale analysis, i.e. tracing a feature (zero-crossing, extremum) of the signal or one of its derivatives over a continuous scale interval can provide information useful for signal processing or signal understanding [9]. It is conjectured [10,11] that the human eye contains optimum edge detection mechanisms comprised of Gaussian low-pass filters and spatial Laplacian operators.

The Scale Space Image.

Witkin [12] suggested to describe a signal g in the following way. Assume that g is locally L_1 and that there exists a polynomial $P:\mathbf{R}\to\mathbf{R}$ such that $|g(x)| \le P(x)$ for all $x\in \mathbf{R}$. The *scale space image* **g** of g is the function $\mathbf{g}:\mathbf{R}\times\mathbf{R}_+\to\mathbf{R}$ given by

$$\mathbf{g}(x,t) = \int_{\mathbf{R}} g(y)\, (4\pi t)^{-1/2} \exp(-(x-y)^2/(4t))\, dy$$

for $x\in\mathbf{R}$, $t>0$. The domain $\mathbf{S}_+=\mathbf{R}\times\mathbf{R}_+$ of the scale space image is called the *scale space.* The scale space image is defined and analytical in all of scale space.

The set $\mathbf{F}_+ = \{(x,t)\in \mathbf{S}_+ \mid \mathbf{g}(x,t)=0\}$ is called the *fingerprint* of g due to its appearence (Fig. 1). Let $\mathbf{g}^{(p,q)}(x,t)$ denote **g** differentiated p times with respect to x and q times with respect to t. The set $\mathbf{T}_+ = \{(x,t)\in \mathbf{S}_+ \mid \mathbf{g}(x,t)=\mathbf{g}^{(1,0)}(x,t)=0\}$ is called the set of *toppoints* of g. Due to the analyticity of **g** the set $\mathbf{T}_+$ is discrete unless g is the zero function.

In general g is not uniquely determined by its toppoints or its fingerprint. Consider the function

$$h(x) = a_1 \sin(x) + a_3 \sin(3x) + a_5 \sin(5x) + \ldots$$

which has the fingerprint $\{(p\pi,t) \mid p\in \mathbf{Z}, t\in \mathbf{R}_+\}$ regardless of the a_i's as long as $|a_1|\ge 3|a_3|+5|a_5|+\ldots$ Obviously many considerably different functions have identical fingerprints.

The purpose of this paper is to prove that if we also consider negative scales and toppoints occurrring at negative scales, and at $+\infty$ and if $g:\mathbf{R}\to\mathbf{R}$ is *periodic* and *bandlimited,* then g is determined up to a constant factor by its toppoints.

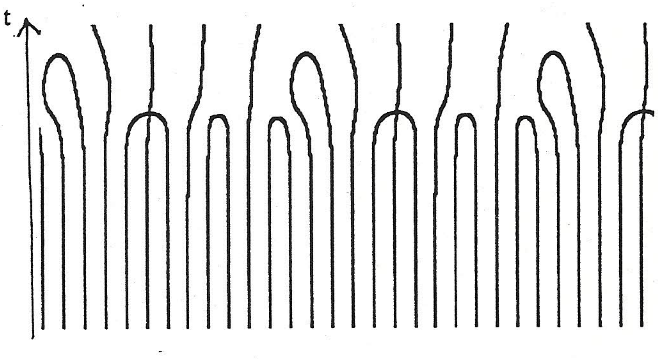

Figure 1. The figure shows the fingerprint of the periodic bandlimited function g(x)=cos(2x)-3cos(3x+3)+2cos(4x+2) -cos(5x+1.178)+cos(6x). The graph shows 2.5 periods. For small t the high frequency contents dominate, for large t the low frequency dominates.

The uniqueness is proved by considering global behavior (i.e. for $t = \infty$ and $-\infty$) and local behavior of the scale space representation of the periodic and bandlimited function g.

A function $g:\mathbf{R}\to\mathbf{R}$ is *bandlimited* if there exists a distribution G with compact support such that

$$g(x)=\int_{\mathbf{R}} G(\omega)\exp(j\omega x)d\omega.$$

For bandlimited g, only one G satisfies the equation, and this G is the Fourier Transform of g. A necessary and sufficient condition for a function to be bandlimited is stated in the Paley-Wiener theorem [13].

In general it does not make sense to consider $g(x,t)$ for $t<0$. If g is bandlimited, however, one may extend the scale space image $\mathbf{g}$ of g

$$\mathbf{g}(x,t)=\int_{\mathbf{R}} G(\omega)\exp(j\omega x)\exp(-t\omega^2)d\omega.$$

This definition extends analytically the scale space image of g to all of $\mathbf{S}=\mathbf{R}\times\mathbf{R}$. Likewise, one may extend fingerprints by

$$\mathbf{F} = \{(x,t)\in \mathbf{S} \mid \mathbf{g}(x,t)=0\}$$ and toppoints by

$$\mathbf{T} = \{(x,t)\in \mathbf{S} \mid \mathbf{g}(x,t)=\mathbf{g}^{(1,0)}(x,t)=0\}.$$

The set $\mathbf{T}$ is countable as is $\mathbf{T}_+$ (unless g is the zero function).

The Scale Space Image of Periodic and Bandlimited Functions

From now on we assume that g is periodic with period 2π, bandlimited, and not the zero-function. Then there exist integers L and H with $L\leq H$ and real numbers $a_L,...,a_H$, and $\phi_L,...,\phi_H$ such that $a_L\neq0$, $a_H\neq0$, and

$$g(x) =\Sigma(f=L,...,H)\ a_f \cos(fx-\phi_f).$$

We refer to L and H as the lower and higher bandlimit of g, respectivly. By straightforward calculation we find the scale space image $\mathbf{g}$ of g to be

$$\mathbf{g}(x,t)=\Sigma(f=L,...,H)\ a_f \cos(fx-\phi_f) \exp(-f^2t).$$

I. Global Behavior of the Signal.

a. The Fingerprint as $t\rightarrow+\infty$

Let $\mathbf{F}$ be the fingerprint of g. For large t we have

$$\mathbf{g}(x,t) \equiv a_L \cos(Lx-\phi_L) \exp(-L^2t),$$

so it is trivial to prove the following lemma:

Lemma 1. (The behavior of $\mathbf{F}$ in the neighbourhood of $+\infty$) There exists a real constant A and continuous real curves $d_1,...,d_{2L}$ such that in each 2π period interval $I= \{(x,t) \mid x\in [0,2\pi), t\in \mathbf{R}\cup\{-\infty,\infty\}\}$:

$$d_1(t)<d_2(t)<...<d_{2L}(t) \text{ for all } t>A \text{ and}$$

$$\mathbf{F}\cap(\mathbf{R}\times(A,+\infty)) = \{(d_i(t),t) \mid t>A, i=1,...,2L\}.$$

(For $L=0$ this is to be interpreted as $\mathbf{F}\cap(\mathbf{R}\times(A,+\infty)) = \emptyset$).

The curves $d_i(t)$ all converge for $t\rightarrow+\infty$. Define D_i such that $d_i(t)\rightarrow D_i$ for $t\rightarrow+\infty$. We have $D_1<D_2<...D_{2L}$ and the D_i's are equally spaced over each interval of length 2π. For large t, the fingerprint $\mathbf{F}$ looks almost like $\{(D_i,t) \mid t \text{ large}, i=1,...,2L\}$. We shall refer to $\{(D_i,+\infty) \mid i=1,...,2L\}$ as the infinite toppoint of multiplicity L of $\mathbf{F}$.

b. The fingerprint as $t\rightarrow-\infty$

By analogywith lemma 1 we have:

Lemma 2. (The behavior of $\mathbf{F}$ in the neighbourhood of $-\infty$) There exists a real constant B and continuous real curves $s_1,...,s_{2H}$ such that in each 2π period interval I

$$s_1(t)<s_2(t)<...<s_{2H}(t) \text{ for all } t<B \text{ and}$$

$$\mathbf{F}\cap(\mathbf{R}\times(-\infty,B)) = \{(s_i(t),t) \mid t<B, i=1,...,2H\}.$$

The curves $s_i(t)$ all converge for $t\rightarrow-\infty$. Define S_i such that $s_i(t)\rightarrow S_i$ for $t\rightarrow-\infty$. We have $S_1<S_2<...S_{2H}$, and the S_i's are equally spaced over each interval of length 2π. For small t (i.e. for numerically large, negative t), the fingerprint $\mathbf{F}$ looks almost like $\{(S_i,t) \mid t \text{ small}, i=1,...,2H\}$. We shall refer to $\{(S_i,-\infty) \mid i=1,...,2H\}$ as the as the source of multiplicity H of $\mathbf{F}$.

II. Local Behavior of the Signal.

For all fixed $(x_0,t_0)\in \mathbf{R}^2$ define the *order* of g at (x_0,t_0) as the least number m for which $\mathbf{g}^{(m,0)}(x_0,t_0)\neq0$. If $\mathbf{g}(x_0,t_0)\neq0$ then g has order zero at (x_0,t_0). If $\mathbf{g}(x_0,t_0)=0$ and $\mathbf{g}^{(1,0)}(x_0,t_0)\neq0$ then g has order one at (x_0,t_0). If g has a toppoint at (x_0,t_0) then g has order at least two at (x_0,t_0). In appendix A we prove:

Lemma 3. (The local behavior of $\mathbf{F}$ where g has even order). Suppose the order m of g at (x_0,t_0) is even. Then there exists a neighbourhood Ω of (x_0,t_0) and m real curves $d_1,...,d_m$ such that

$$d_1(t)<d_2(t)<...<d_m(t) \text{ for all } t<t_0,$$

$$d_1(t_0)=...=d_m(t_0)=x_0, \text{ and}$$

$$\mathbf{F}\cap\Omega = \Omega\cap\{(d_i(t),t) \mid t\leq t_0\}.$$

Figure 2 displays $\mathbf{F}$ in the neighbourhood of an (x_0,t_0) where g has order four. If the order m of g at (x_0,t_0) is even, then we may describe $\mathbf{F}$ in the neighbourhood of (x_0,t_0) as m curves coming from below and disappearing in the point (x_0,t_0). In this case we say that g has a toppoint of multiplicity $m/2$ in (x_0,t_0).

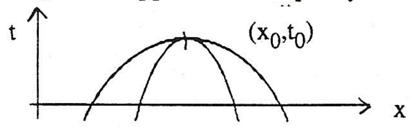

Figure 2. The neighbourhood of a point (x_0,t_0) where g has the order four, and, accordingly, a toppoint of multipicity 2.

In appendix A we also prove:

Lemma 4. (The local behavior of $\mathbf{F}$ where g has odd order). Suppose the order m of g at (x_0,t_0) is odd. Then there exist a neighbourhood Ω of (x_0,t_0), a real curve c_1 and $m-1$ real curves $d_2,...,d_m$ such that

$$c_1(t)<d_2(t)<...<d_m(t) \text{ for all } t<t_0,$$

$$c_1(t_0)=d_2(t_0)=...=d_m(t_0)=x_0, \text{ and}$$

$$\mathbf{F}\cap\Omega = \Omega \cap (\{(c_1(t),t) \mid t\in \mathbf{R}\}\cup\{(d_i(t),t) \mid t\leq t_0\}).$$

Figure 3 displays $\mathbf{F}$ in the neighbourhood of an (x_0,t_0) where g has order five. If the order m of g at (x_0,t_0) is odd, then we may describe $\mathbf{F}$ in the neighbourhood of (x_0,t_0) as one curve which goes through (x_0,t_0) together with $m-1$ curves coming from below and disappearing in the point (x_0,t_0). In this case we say that g has a toppoint of multiplicity $(m-1)/2$ in (x_0,t_0).

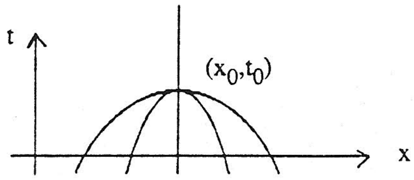

Figure 3. The neighbourhood of a point (x_0,t_0) where g has the order five and a toppoint of multiplicity 2.

The Uniqueness Theorem

Lemma 5. The upper bandlimit equals the sum of toppoints counted with multiplicity.

Proof. We prove that $H = N(t_1) + T(t_1)$, where $N(t_1)$ is half the number of points of intersection between $\mathbf{F}$ and the line $t = t_1$, and where $T(t_1)$ is the number of finite toppoints (counted with multiplicity) below the line. From lemma 2 the statement follows for $t_1<B$. Let the total multiplicity of the toppoints at $t = t_0$ be k. If $H = N(t_1) + T(t_1)$ for $t_1 = t_0 - \varepsilon$, it remains true for $t_1 = t_0 + \varepsilon$ since by lemma 3 or lemma 4, T is increased by k and N is decreased by k. From lemma 1 $N(t_1) = L$ for $t_1 > A$ and lemma 5 follows.

Theorem. If g and h are both periodic and bandlimited, if neither is the zero function, and if g and h have the same toppoints counted with multiplicity, then $g=ah$ for some real non-zero constant a.

Proof. If g is periodic with period A and h is periodic with period B, then the ratio between A and B must be a rational number, for otherwise g and h could not have the same toppoints. Hence g and h have a common period C, i.e. there exists a C such that both g and h are periodic with period C. Without loss of generality we may assume $C=2\pi$.

The functions g and h have a toppoint of the same multiplicity at infinity in **I**. Hence, their lower bandlimits are equal. Let L and H denote the lower and higher bandlimit, respectively.

Define $a_L,\dots,a_H$, $b_L,\dots,b_H$, $\phi_L,\dots,\phi_H$, $\eta_L,\dots,\eta_H$ such that

$$g(x)=\Sigma(f=L,\dots,H)\ a_f \cos(fx-\phi_f),$$
$$h(x)=\Sigma(f=L,\dots,H)\ b_f \cos(fx-\eta_f), \text{ and}$$
$$\phi_L=\eta_L.$$

(It is always posible to satisfy $\phi_L=\eta_L$ because the infinite toppoint of g and h coincide). From the definition of L we have $a_L \neq 0$ and $b_L \neq 0$. Define $a=a_L/b_L$ and $v=g-ah$.

We now prove indirectly that v is the zero function. Assume that v is not the zero function.

If g has a finite toppoint of multiplicity k at some (x_o,t_o), then h also has a finite toppoint of multiplicity k at (x_o,t_o), and then v has a finite toppoint at (x_o,t_o) of multiplicity at least k. Hence, as g has H-L finite toppoints counted with multiplicity, v has at least H-L finite toppoints counted with multiplicity. We have

$$v(x) = \Sigma\ (f=L,\dots,H)\ (a_f \cos(fx-\phi_f) - a\ b_f \cos(fx-\eta_f))$$
$$= \Sigma\ (f=L+1,\dots,H)\ (a_f \cos(fx-\phi_f) - a\ b_f \cos(fx-\eta_f))$$

Hence, the lower bandlimit of v is at least L+1 so v has a toppoint at infinity of multiplicity at least L+1.

As v has at least H-L finite toppoints counted with multiplicity and an infinite toppoint of multiplicity L+1, the upper bandlimit of v must be at least H+1. However, the higher bandlimit of v is at most H, which gives the contradiction. Hence v=g-ah is the zero function, so g=ah as stated.

APPENDIX A.

Lemma 6. The Fourier transform G of $g(x)=a_0x^0+\dots+a_nx^n$ reads $G=a_0j^{-0}\delta^{(0)}+\dots+a_nj^{-n}\delta^{(n)}$ where $\delta^{(i)}$ denotes i'th derivative of the delta-distribution.

Proof. $\int_R G(\omega)\exp(j\omega x)d\omega = a_0x^0+\dots+a_nx^n$.

Lemma 7. Any polynomial is bandlimited with upper bandlimit zero.

Proof. Follows from Lemma 6.

Define $v_i(x)=x^i$, $V_i(\omega)=j^{-i}\delta^{(i)}$, $W(\omega,x,t) = \exp(j\omega x-t\omega^2)$, and $v_i(x,t) = \int_R V_i(\omega)\ W(\omega,x,t)\ d\omega$. V_i is the Fourier transform and v_i is the scale space image of v_i. Let $W_i(\omega,x,t)$ denote the i'th derivative of $W(\omega,x,t)$ with respect to ω. We have $v_i(x,t)=\int_R j^{-i}\delta^{(i)}W(\omega,x,t)d\omega = j^{-i}W_i(0,x,t)$. The functions v_i are the *Heat Polynomials* defined by Widder [14].

Lemma 8.

$W_{i+2}(\omega,x,t)=(jx-2t\omega)W_{i+1}(\omega,x,t)-2(i+1)tW_i(\omega,x,t)$.

Proof. By induction in i.

Lemma 9. $v_o(x,t)=1$.

$v_1(x,t)=x$. $v_{i+2}(x,t)=xv_{i+1}(x,t)+2(i+1)tv_i(x,t)$.

Proof. From lemma 8 and the definitions.

Lemma 10. $v_i(x,-1)$ is a polynomial in x of degree i which has i distinct, real roots.

Proof. By lemma 9 and induction in i we prove that $v_i(x,-1)$ is a polynomial of degree i. Define $q_o=1$, $q_1=i$, $q_p=2(i-p+1)q_{p-2}$, $p=2,\dots,i$. We have $q_p>0$ for $p=0,\dots,i$. Define $X_p(x) = q_pv_{i-p}(x,-1)$. From Widder [14] we get:

$$X_1(x) = X_o'(x).$$

Using lemma 9 we can prove that the sequence $X_0, X_1, \dots, X_i$ is the Sturm sequence of the polynomial $X_0 = v_i(x,-1)$

Hence, by Sturm's Theorem [15] we have that $X_o = v_i(x,-1)$ has i distinct, real roots.

Lemma 11. $v_i(\lambda x,\lambda^2t)=\lambda v_i(x,t)$.

Proof. Follows from the definition (the result is also stated in Widder [14]).

Lemma 12. $v_i(x,1)$ is a polynomial in x of degree i which has i distinct, imaginary roots.

Proof. $v_i(jx,1)=jv_i(x,-1)$ by lemma 11. Hence, lemma 12 restates lemma 10.

Lemma 13. For i even, none of the roots of $v_i(x,1)$ are real. For i odd, exactly one root is real (and this root has order one).

Proof. This is a direct consequence of lemma 12.

Lemma 14. The fingerprint function g of any bandlimited g satisfies $g(x,t)=\Sigma(i=0..+\infty)\ g^{(i)}(0)\ (i!)^{-1}\ v_i(x,t)$.

Proof. Se Widder [14].

Lemma 15. $g(\lambda\tau,-\lambda^2)= \Sigma(i=0..+\infty)\ g^{(i)}(0)\ (i!)^{-1}\ \lambda^i\ v_i(\tau,-1)$.

Proof. Follows from lemma 11 and 14.

Lemma 16. $g(\lambda\tau,\lambda^2)= \Sigma(i=0..+\infty)\ g^{(i)}(0)\ (i!)^{-1}\ \lambda^i\ v_i(\tau,1)$.

Proof. As 15.

Lemma 3 and 4 follows from lemma 10, 13, 15, and 16.

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