Supplementary Note for the MAC-lecture November 17

November 14, 2003

Equations expressing that $[\mathcal{A} \equiv \mathcal{B}]^{\cdot}$ holds under some hypothesis

In the Mac derivation system there is a basic rule saying that:

If
the equation $[\mathcal{A} \equiv \mathcal{B}]^{\cdot}$ holds when $x \in \mathbf{T} \equiv T$
and
the equation $[\mathcal{A} \equiv \mathcal{B}]^{\cdot}$ holds when $x \in \mathbf{F} \equiv T$
then
the equation $[\mathcal{A} \equiv \mathcal{B}]^{\cdot}$ holds when $x \in \mathbf{B} \equiv T$

Until now we have not seen any axiom expressing an 'if-then-statement', so we may expect this rule to be an inference rule of the form:

[**Mac rule MacroName**: ($[x \in \mathbf{T} \equiv T]^{\cdot}$ implies that $[\mathcal{A} \equiv \mathcal{B}]^{\cdot}$ holds)
$\vdash$ ($[x \in \mathbf{F} \equiv T]^{\cdot}$ implies that $[\mathcal{A} \equiv \mathcal{B}]^{\cdot}$ holds)
$\vdash$ ($[x \in \mathbf{B} \equiv T]^{\cdot}$ implies that $[\mathcal{A} \equiv \mathcal{B}]^{\cdot}$ holds)$]^{o}$

We know that **premises** and **conclusions** are something that may *hold* or not hold, i.e. they **must be equations** (or terms interpreted as equations) and thus they cannot contain the infer-directive! But how can we then write premises and a conclusion expressing that $[\mathcal{A} \equiv \mathcal{B}]^{\cdot}$ holds under some hypothesis?

Well, consider the equation

$$\text{case}(x \in \mathbf{B}, \mathcal{A}, T) \equiv \text{case}(x \in \mathbf{B}, \mathcal{B}, T)$$

Since **hypotheses** are *true* or not, they **must be terms**, and thus they may appear as the first argument of the case-operator.
If the hypothesis $[x \in \mathbf{B}]^{\cdot}$ is true then this equation tells us (or rather Map) that the equation $[\mathcal{A} \equiv \mathcal{B}]^{\cdot}$ holds, *whereas* the equation does not say more than the rule Reflexivity, when $[x \in \mathbf{B}]^{\cdot}$ is $[\perp]^{\cdot}$ or a λ-function like $[F]^{\cdot}$.
Hence, we may write the rule in question as:

[**Mac rule Cases**: $\text{case}(x \in \mathbf{T}, \mathcal{A}, T) \equiv \text{case}(x \in \mathbf{T}, \mathcal{B}, T)$
$\vdash$ $\text{case}(x \in \mathbf{F}, \mathcal{A}, T) \equiv \text{case}(x \in \mathbf{F}, \mathcal{B}, T)$
$\vdash$ $\text{case}(x \in \mathbf{B}, \mathcal{A}, T) \equiv \text{case}(x \in \mathbf{B}, \mathcal{B}, T)]^{\cdot}$

In Chapter 12 the new 'deduce-directive' $[\rightarrow]^{o}$ is introduced by the *term reduction rule*:

$$[x \rightarrow y \equiv z \overset{o}{\rightarrow} \text{case}(x, y, T) \equiv \text{case}(x, z, T)]^{\cdot}$$

and due to the priorities $[x \equiv y \check{>} x \rightarrow y \check{>} x \vdash y]^{\cdot}$ the inference rule Cases may thus be written in a more compact way:

[**Mac rule Cases**: $x \in \mathbf{T} \rightarrow \mathcal{A} \equiv \mathcal{B} \vdash x \in \mathbf{F} \rightarrow \mathcal{A} \equiv \mathcal{B} \vdash x \in \mathbf{B} \rightarrow \mathcal{A} \equiv \mathcal{B}]^{\cdot}$

In Sections 12.4 to 12.11 it is shown how the so-called deduction proof technique may be used for proving lemmas containing the deduce-directive (*without introducing the 'case-equations'* !).

Since all the lemmas in Chapter 12 can be proved without using *blocks*, you may have a glance at the proof of Lemma L14.1.1, where blocks are really necessary!

Exercise 1

Consider the following lemma (proved in Chapter 12)

[**Mac lemma L12.5.1**: $x \in \mathbf{Z} \rightarrow y \in \mathbf{Z} \rightarrow x + y \cdot x \in \mathbf{Z}$]

L12.5.1 is a macro for the *implication* $[x \in \mathbf{Z} \rightarrow y \in \mathbf{Z} \rightarrow x + y \cdot x \in \mathbf{Z}]^{\cdot}$, which consists of two *hypothesis* and a *consequence* (the term $[x + y \cdot x \in \mathbf{Z}]^{\cdot}$ interpreted as the equation $[x + y \cdot x \in \mathbf{Z} \equiv T]^{\cdot}$). The new directive $[\rightarrow]^{o}$ is right-associative (like $[\vdash]^{o}$), but is L12.5.1 an axiom or an inference rule?

Exercise 2

Assume that x, y, z, u and v are terms. Which of the following expressions are then syntactically correct?

$[x \rightarrow y \rightarrow z]^{o}$ (the *implication* may be read as: x *implies* that y *implies* that z ≡ T)
$[x \vdash y \vdash z]^{o}$ (the *inference* may be read as: x ≡ T *infers* that y ≡ T *infers* that z ≡ T)
$[x \rightarrow y \rightarrow z \equiv u]^{o}$
$[x \rightarrow y \equiv u \rightarrow z \equiv v]^{o}$
$[(x \rightarrow y \equiv u) \rightarrow z \equiv v]^{o}$
$[(x \rightarrow y \equiv u) \vdash z \equiv v]^{o}$
$[x \equiv u \rightarrow y \rightarrow z]^{o}$
$[x \equiv u \vdash y \vdash z]^{o}$

Exercise 3

In Section E.2, Volume 3, you have a Mac-statement expressing the priorities of most of the operators and directives in the Mac System (e.g. *not* the substitution operator). Until now you have learned about all these constructs except the following four:

The directive $[\preceq]^{o}$ (i.e. 'weakly less information than') and
the three ternary operators $[\forall x{\in}y{:}z]^{\cdot}$, $[\exists x{\in}y{:}z]^{\cdot}$, $[\varepsilon x{\in}y{:}z]^{\cdot}$.

Try to recall what all the other constructs stand for!