Mathematics and Computation
Exam like questions, October 24, 2004

Exams consist of three questions, but here you have four exam like questions.
I define

$$\left[H(\mathsf{x}) \doteq \mathsf{x} < 0 \left\{ \begin{array}{c} 0 \\ 1 \end{array} \right. \right]$$

Exercise 1. (Too small for an exam question) Prove one of the following:
[**Mac lemma L04.0.4A:** $\mathsf{x} \in \mathbf{B} \rightarrow \mathrm{if}(\mathsf{x}, \mathsf{u}, \mathrm{if}(\mathsf{x}, \mathsf{v}, \mathsf{w})) \equiv \mathrm{if}(\mathsf{x}, \mathsf{u}, \mathsf{w})$]
[**Mac antilemma L04.0.4B:** $\mathsf{x} \in \mathbf{B} \rightarrow \mathrm{if}(\mathsf{x}, \mathsf{u}, \mathrm{if}(\mathsf{x}, \mathsf{v}, \mathsf{w})) \equiv \mathrm{if}(\mathsf{x}, \mathsf{u}, \mathsf{w})$]

Exercise 2. Prove one of the following:
[**Mac lemma L04.0.5A:** $\mathsf{x} \in \mathbf{D} \rightarrow H(\mathsf{x}) \in \mathbf{N}$]
[**Mac antilemma L04.0.5B:** $\mathsf{x} \in \mathbf{D} \rightarrow H(\mathsf{x}) \in \mathbf{N}$]

Exercise 3. Prove one of the following:
[**Mac lemma L04.0.6A:** $\mathsf{x} \in \mathbf{B} \rightarrow \mathsf{x} \wedge \mathsf{x} \equiv \mathsf{x}$]
[**Mac antilemma L04.0.6B:** $\mathsf{x} \in \mathbf{B} \rightarrow \mathsf{x} \wedge \mathsf{x} \equiv \mathsf{x}$]

Exercise 4. (Too large for an exam question) Prove one of the following:
[**Mac lemma L04.0.7A:** $\mathsf{Y}\,'\,\lambda \mathsf{f}.\lambda \mathsf{x}.\mathrm{case}(\mathsf{x}, \mathsf{T}, \mathsf{f}\,'\,\mathsf{x}) \equiv \lambda \mathsf{x}.\mathrm{case}(\mathsf{x}, \mathsf{T}, \bot)$]
[**Mac antilemma L04.0.7B:** $\mathsf{Y}\,'\,\lambda \mathsf{f}.\lambda \mathsf{x}.\mathrm{case}(\mathsf{x}, \mathsf{T}, \mathsf{f}\,'\,\mathsf{x}) \equiv \lambda \mathsf{x}.\mathrm{case}(\mathsf{x}, \mathsf{T}, \bot)$]

The following rules may be used:
[**Mac rule Case1:** $\mathrm{case}(\mathsf{x}, \mathrm{case}(\mathsf{x}, \mathsf{u}, \mathsf{v}), \mathsf{w}) \equiv \mathrm{case}(\mathsf{x}, \mathsf{u}, \mathsf{w})$]
[**Mac rule Case2:** $\mathrm{case}(\mathsf{x}, \mathsf{u}, \mathrm{case}(\mathsf{x}, \mathsf{v}, \mathsf{w})) \equiv \mathrm{case}(\mathsf{x}, \mathsf{u}, \mathsf{w})$]

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A proof of

[**Mac lemma L04.0.4A** : $\mathsf{x} \in \mathbf{B} \rightarrow \mathrm{if}(\mathsf{x}, \mathsf{u}, \mathrm{if}(\mathsf{x}, \mathsf{v}, \mathsf{w}) \equiv \mathrm{if}(\mathsf{x}, \mathsf{u}, \mathsf{w}))$]

could be as follows:

[**Mac proof of** L04.0.4A**:**

L01 :	Block ▷	Begin	;
L02 :	Hypothesis ▷	$\mathsf{x} \in \mathbf{T}$	;
L03 :	IfT ▷ L2 ▷	$\mathrm{if}(\mathsf{x}, \mathsf{u}, \mathrm{if}(\mathsf{x}, \mathsf{v}, \mathsf{w})) \equiv \mathsf{u}$	;
L04 :	IfT ▷ L2 ▷	$\mathrm{if}(\mathsf{x}, \mathsf{u}, \mathsf{w}) \equiv \mathsf{u}$	;
L05 :	Commutativity ▷ L4 ▷	$\mathsf{u} \equiv \mathrm{if}(\mathsf{x}, \mathsf{u}, \mathsf{w})$	;
L06 :	Transitivity ▷ L3 ▷ L5 ▷	$\mathrm{if}(\mathsf{x}, \mathsf{u}, \mathrm{if}(\mathsf{x}, \mathsf{v}, \mathsf{w})) \equiv \mathrm{if}(\mathsf{x}, \mathsf{u}, \mathsf{w})$	;
L07 :	Block ▷	End	;
L08 :	Block ▷	Begin	;
L09 :	Hypothesis ▷	$\mathsf{x} \in \mathbf{F}$	;
L10 :	IfF ▷ L9 ▷	$\mathrm{if}(\mathsf{x}, \mathsf{v}, \mathsf{w}) \equiv \mathsf{w}$	;
L11 :	Replace ▷ L10 ▷	$\mathrm{if}(\mathsf{x}, \mathsf{u}, \mathrm{if}(\mathsf{x}, \mathsf{v}, \mathsf{w})) \equiv \mathrm{if}(\mathsf{x}, \mathsf{u}, \mathsf{w})$	;
L12 :	Block ▷	End	;
L13 :	Cases ▷ L6 ▷ L11 ▷	$\mathsf{x}{\in}\mathbf{B} \rightarrow \mathrm{if}(\mathsf{x}, \mathsf{u}, \mathrm{if}(\mathsf{x}, \mathsf{v}, \mathsf{w})) \equiv \mathrm{if}(\mathsf{x}, \mathsf{u}, \mathsf{w})$	]

Line 6 could also have argumentation [Reverse' ▷ L4 ▷ L3]˙ in which case Line 5 can be omitted.

[L04.0.4A]˙ rule L04.0.4A

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A proof of

[**Mac lemma L04.0.5A** : $\mathsf{x} \in \mathbf{D} \to H(\mathsf{x}) \in \mathbf{N}$]

could be as follows:

[**Mac proof of** L04.0.5A**:**

L01 :	Block $\rhd$	Begin	;
L02 :	Hypothesis $\rhd$	$(\mathsf{x} < 0) \in \mathbf{T}$	;
L03 :	IfT $\rhd$ L2 $\rhd$	$\text{if}(\mathsf{x} < 0, 0, 1) \equiv 0$	;
L04 :	TypeNumeralInN $\rhd$	$0 \in \mathbf{N}$	;
L05 :	Reverse' $\rhd$ L3 $\rhd$ L4 $\rhd$	$\text{if}(\mathsf{x} < 0, 0, 1) \in \mathbf{N}$	;
L06 :	Block $\rhd$	End	;
L07 :	Block $\rhd$	Begin	;
L08 :	Hypothesis $\rhd$	$(\mathsf{x} < 0) \in \mathbf{F}$	;
L09 :	IfF $\rhd$ L8 $\rhd$	$\text{if}(\mathsf{x} < 0, 0, 1) \equiv 1$	;
L10 :	TypeNumeralInN $\rhd$	$1 \in \mathbf{N}$	;
L11 :	Reverse' $\rhd$ L9 $\rhd$ L10 $\rhd$	$\text{if}(\mathsf{x} < 0, 0, 1) \in \mathbf{N}$	;
L12 :	Block $\rhd$	End	;
L13 :	Cases $\rhd$ L5 $\rhd$ L11 $\rhd$	$(\mathsf{x} < 0) \in \mathbf{B} \to \text{if}(\mathsf{x} < 0, 0, 1) \in \mathbf{N}$	;
L14 :	Hypothesis $\rhd$	$\mathsf{x} \in \mathbf{D}$	;
L15 :	TypeNumeralInD $\rhd$	$0 \in \mathbf{D}$	;
L16 :	Type**D**<**D** $\rhd$ L14 $\rhd$ L15 $\rhd$	$(\mathsf{x} < 0) \in \mathbf{B}$	;
L17 :	L13 $\trianglerighteq$ L16 $\rhd$	$\text{if}(\mathsf{x} < 0, 0, 1) \in \mathbf{N}$	;
L18 :	Definition $\rhd$	$H(\mathsf{x}) \equiv \text{if}(\mathsf{x} < 0, 0, 1)$	;
L19 :	Reverse' $\rhd$ L18 $\rhd$ L17 $\rhd$	$H(\mathsf{x}) \in \mathbf{N}$	]

[$H(\mathsf{x})$] was used by Oliver Heaviside for investigations of electronic circuits.

[L04.0.5A] rule L04.0.5A

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A proof of

[**Mac antilemma L04.0.6B** : $\mathsf{x} \in \mathbf{B} \rightarrow \mathsf{x} \wedge \mathsf{x} \equiv \mathsf{x}$]

could use a non-standard representation [$F \equiv \mathsf{N} \therefore (\mathsf{N} \therefore \mathsf{N})$]˙ of falsehood. One may prove [$F \in \mathbf{B}$]˙ using the type function [$\overline{\mathbf{B}}$]˙ from page 336 in the text book and one may then combine [$F \in \mathbf{B}$]˙ with the antilemma to get [$F \wedge F \equiv F$]˙. On the other hand, we have [$F \wedge F \equiv \mathsf{F}$]˙ (by Computation) so we get [$\mathsf{F} \equiv F$]˙. Hence, by the definitions of [F]˙ and [F]˙ we get [$\mathsf{N} \therefore \mathsf{N} \equiv \mathsf{N} \therefore (\mathsf{N} \therefore \mathsf{N})$]˙. Taking the Tail of both sides gives [$\mathsf{N} \equiv \mathsf{N} \therefore \mathsf{N}$]˙ which, by the definitions of [T]˙ and [F]˙ is the same as [$\mathsf{T} \equiv \mathsf{F}$]˙. CounterTF then yields the contradiction. The proof may look thus:

[**Mac proof of** L04.0.6B**:**

L01 :	Local ▷	$F \equiv \mathsf{N} \therefore (\mathsf{N} :: \mathsf{N})$	;
L02 :	Computation ▷	$\overline{\mathbf{B}}\,'\,F$	;
L03 :	Block ▷	Begin	;
L04 :	Algebra ▷	$\ell\,'\,F$	;
L05 :	Replace ▷ ClassicalPair ▷	$\ell\,'\,\mathsf{N} \wedge \ell\,'(\mathsf{N} \therefore \mathsf{N})$	;
L06 :	Replace ▷ ClassicalPair ▷	$\ell\,'\,\mathsf{N} \wedge (\ell\,'\,\mathsf{N} \wedge \ell\,'\,\mathsf{N})$	;
L07 :	Replace ▷ ClassicalNil ▷	$\mathsf{T} \wedge (\mathsf{T} \wedge \mathsf{T})$	;
L08 :	Computation ▷	T	;
L09 :	Block ▷	End	;
L10 :	IntroB ▷ L8 ▷ L2 ▷	$F \in \mathbf{B}$	;
L11 :	Antilemma ▷	$F \in \mathbf{B} \rightarrow F \wedge F \equiv F$	;
L12 :	L11 ⊵ L10 ▷	$F \wedge F \equiv F$	;
L13 :	Block ▷	Begin	;
L14 :	Algebra ▷	T	;
L15 :	Computation ▷	$(F \wedge F)$ Tail	;
L16 :	Replace ▷ L12 ▷	F Tail	;
L17 :	Computation ▷	F	;
L18 :	Block ▷	End	;
L19 :	CounterTF ▷ L17 ▷	$\perp$	]

[L04.0.6B]˙ rule L04.0.6B

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A proof of

[**Mac lemma L04.0.7A** : $\mathsf{Y}\,'\,\lambda\mathsf{f}.\lambda\mathsf{x}.\mathrm{case}(\mathsf{x},\mathsf{T},\mathsf{f}\,'\,\mathsf{x}) \equiv \lambda\mathsf{x}.\mathrm{case}(\mathsf{x},\mathsf{T},\bot)$]

could be constructed as follows: First, define [A] to be shorthand for [$\mathsf{Y}\,'\,\lambda\mathsf{f}.\lambda\mathsf{x}.\mathrm{case}(\mathsf{x},\mathsf{T},\mathsf{f}\,'\,\mathsf{x})$] (Line 1 below). Then compute [A] using Lemma L8.18.2 (Line 2–6 below). After Line 6 we know that [$A \equiv \lambda\mathsf{x}.\mathrm{case}(\mathsf{x},\mathsf{T},A\,'\,\mathsf{x})$] which makes it easy to prove [$\lambda\mathsf{x}.\mathrm{case}(\mathsf{x},\mathsf{T},\bot) \preceq A$] (Line 7–10 below).

This proves half of the Lemma L04.0.7A: InfoAntiSymmetry says that if [$\lambda\mathsf{x}.\mathrm{case}(\mathsf{x},\mathsf{T},\bot) \preceq A$] and [$A \preceq \lambda\mathsf{x}.\mathrm{case}(\mathsf{x},\mathsf{T},\bot)$] then the lemma holds (Line 18 below).

Hence, to prove the lemma, we have to prove [$A \preceq \lambda\mathsf{x}.\mathrm{case}(\mathsf{x},\mathsf{T},\bot)$] (Line 17 below). In other words, we have to prove that [A] is less that a term that contains [$\bot$]. One of the very few rules that allows that is MinimalY on page 427 in the text book. That rule looks promissing since [A] contains the fixed point operator [Y].

Applying MinimalY backwards to Line 17 gives

$$[\ \lambda\mathsf{f}.\lambda\mathsf{x}.\mathrm{case}(\mathsf{x},\mathsf{T},\mathsf{f}'\mathsf{x})\,'\,\lambda\mathsf{x}.\mathrm{case}(\mathsf{x},\mathsf{T},\bot) \equiv \lambda\mathsf{x}.\mathrm{case}(\mathsf{x},\mathsf{T},\bot)\ ]$$

which turns out to be easy to prove (Line 11–16 below). Line 15 uses Case2 which was provided together with the exercise.

[**Mac proof of** L04.0.7A**:**

L01 :	Local ▷	$A \equiv \mathsf{Y}\,'\,\lambda\mathsf{f}.\lambda\mathsf{x}.\mathrm{case}(\mathsf{x},\mathsf{T},\mathsf{f}\,'\,\mathsf{x})$	;
L02 :	Block ▷	Begin	;
L03 :	Algebra ▷	A	;
L04 :	Replace ▷ L8.18.2 ▷	$(\lambda\mathsf{f}.\lambda\mathsf{x}.\mathrm{case}(\mathsf{x},\mathsf{T},\mathsf{f}\,'\,\mathsf{x}))\,'\,A$	;
L05 :	Replace▷ApplyLambda ▷	$\lambda\mathsf{x}.\mathrm{case}(\mathsf{x},\mathsf{T},A\,'\,\mathsf{x})$	;
L06 :	Block ▷	End	;
L07 :	InfoBottom ▷	$\bot \preceq A\,'\,\mathsf{x}$	;
L08 :	Monotonicity' ▷ L7 ▷	$\mathrm{case}(\mathsf{x},\mathsf{T},\bot) \preceq \mathrm{case}(\mathsf{x},\mathsf{T},A\,'\,\mathsf{x})$	;
L09 :	InfoLambda ▷ L8 ▷	$\lambda\mathsf{x}.\mathrm{case}(\mathsf{x},\mathsf{T},\bot) \preceq \lambda\mathsf{x}.\mathrm{case}(\mathsf{x},\mathsf{T},A\,'\,\mathsf{x})$	;
L10 :	Reverse' ▷ L5 ▷ L9 ▷	$\lambda\mathsf{x}.\mathrm{case}(\mathsf{x},\mathsf{T},\bot) \preceq A$	;
L11 :	Block ▷	Begin	;
L12 :	Algebra ▷	$\lambda\mathsf{f}.\lambda\mathsf{x}.\mathrm{case}(\mathsf{x},\mathsf{T},\mathsf{f}'\mathsf{x})'\lambda\mathsf{x}.\mathrm{case}(\mathsf{x},\mathsf{T},\bot)$	;
L13 :	Replace▷ApplyLambda ▷	$\lambda\mathsf{x}.\mathrm{case}(\mathsf{x},\mathsf{T},(\lambda\mathsf{x}.\mathrm{case}(\mathsf{x},\mathsf{T},\bot))\,'\,\mathsf{x})$	;
L14 :	Replace▷ApplyLambda ▷	$\lambda\mathsf{x}.\mathrm{case}(\mathsf{x},\mathsf{T},\mathrm{case}(\mathsf{x},\mathsf{T},\bot))$	;
L15 :	Replace ▷ Case2 ▷	$\lambda\mathsf{x}.\mathrm{case}(\mathsf{x},\mathsf{T},\bot)$	;
L16 :	Block ▷	End	;
L17 :	MinimalY ▷ L15 ▷	$A \preceq \lambda\mathsf{x}.\mathrm{case}(\mathsf{x},\mathsf{T},\bot)$	;
L18 :	InfoAntiSymmetry ▷ L17 ▷ L10 ▷	$A \equiv \lambda\mathsf{x}.\mathrm{case}(\mathsf{x},\mathsf{T},\bot)$	]

[L04.0.7A] rule L04.0.7A