Prize collecting traveling salesman problem with time windows

Approximation algorithms
DIKU, January 2007
Order of the day

- Defining the prize collecting traveling salesman problem with time windows (TW-TSP).
- Factor 8 approximation for a very restricted problem in one dimension.
- Factor $12 \log(n)$ for a less restricted problem in one dimension.
- An exact algorithm for a very restricted problem in any dimension.
- Factor $\lfloor \sigma(\pi) \rfloor + 1$ for a nearly unrestricted problem in any dimension.
TW-TSP: Definition

- In the real world it is similar to a repairman visiting a number of different houses (jobs).
- Each job $v$ must be carried out within a specified time window, $[r(v), d(v)]$.
- Each job $v$ has a profit $p(v)$ that can be collected if it is carried out within the time window.
TW-TSP: Graphical representation
TW-TSP: Not TSP

- Not a classic traveling salesman problem.
- Object is to maximize the profit, not minimize the tour.
- It is not required that the tour is Hamiltonian.
- Points may be visited several times (no extra profit)
TW-TSP: Formal definition

- \((V,l)\) is a space, where \(V\) is a point set and \(l(u,v)\) is a distance function.
- The set of jobs \(S\), subset of \(V\).
- Any path must start at \(v_0\).
- \(p(v_0) = r(v_0) = d(v_0) = 0\).
TW-TSP on a line: Definition

- Also referred to as metric TW-TSP
- \( V = \mathbb{R} \), jobs are points on the real line
- Distance function \( l(u, v) = |u-v| \)
- Can be reduced to MAX-MONOTONE-TOUR (MMT)
MAX-MONOTONE-TOUR: Definition

- Object is to find a path in the plane intersecting the maximum number of line segments.
- The path must be weakly monotonely increasing, with regard to $x$ and $y$. 
MAX-MONOTONE-TOUR: Graphical representation
TW-TSP on a line: Graphical description

- Describe TW-TSP as vertical segments in the plane, where axes are location \((x)\) and time \((y)\).
- Length of the segment equals the length of the time interval.
- Problem can be solved by finding a maximum intersecting path with slopes in \([45, 135]\) degrees.
TW-TSP on a line:
Graphical representation

Path can only move in this cone

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TW-TSP on a line: Reduction to MMT

Path can move in this cone
Factor 8 algorithm: Overview

- Works for unit-sized time windows.
- Imposes a grid of squares with diagonal 1.
- Shifts the grid.
- Creates a DAG from the grid.
- Assigns weights to the edges in the dag.
- Finds the heaviest weight path in the DAG.
Factor 8 algorithm: Graphical

- $w(e) = 0$
- $w(e) = 2$
- $w(e) = 1$
Factor 8 algorithm: Proof (short)

- Proof in two parts
  - Prove factor 2 bound w.r.t. optimal tour in grid
  - Prove factor 4 bound for optimal tour in grid w.r.t. optimal tour.

- A few definitions are needed:
  - \( p \) is the algorithm's path through the grid.
  - \( p' \) is the optimal path through the grid.
  - \( p^* \) is the optimal path.
  - \( k(q) \) is the number of slanted segments intersected by path \( q \).
Factor 8 algorithm: Proof (part 1)

- Factor 2 bound achieved by combining:
  - $k(q) \geq \frac{1}{2} w(q)$
  - $w(p) \geq w(p')$
  - $w(q) \geq k(q)$

- This results in $k(p) \geq \frac{1}{2}k(p')$. 
Factor 8 algorithm: Proof (part 2)

- Factor 4 bound is achieved by analyzing the optimal path.
- The optimal path $p^*$ traverses a set of grid cells. These are divided into horizontal and vertical blocks.
- At least half the segments on $p^*$ intersect the horizontal (or vertical) blocks.
Factor 8 algorithm: Proof (part 2)
Factor 8 algorithm - proof

- We can now construct a grid-path $p''$ that intersects at least half the segments of the horizontal blocks.
- Since $p'$ is the optimal grid-path then $k(p') \geq k(p'') \geq \frac{1}{2} \times \frac{1}{2} \times k(p^*)$
Factor 8 algorithm - proof
Factor 12 log $n$ algorithm

- Arbitrary time windows.
- Unit profits.
- Partitions slanted segments into $\log n$ disjoint sets.
- Factor 12 algorithm for these sets.
Factor 12 algorithm: Combs

- Each set satisfies the ”comb” property.
- A set is a comb if
  - There exists a set of vertical lines, s.t. each slanted segment is intersected by exactly one vertical line.
- Partitioning any set into combs with an interval tree.
Factor 12 log $n$
Factor 12 algorithm: combs

- Each level of the interval tree, corresponds to a comb. Comb is an instance of TW-TSP.
- Grid constructed:
  - $O(n)$ vertical lines: Bisectors.
  - $O(2n)$ horizontal lines: Through endpoints.
- Dag derived from grid like before.
- Path intersects segments max 3 times.
Factor 12 $\log n$
Additional results

- Generalization to non-unit profits
- $4+\varepsilon$ algorithm (unit sized time windows)

Arbitrary time windows:
- $16 \log L$ algorithm
- $4+\varepsilon \log L$ algorithm
- $4+\varepsilon \log n$ algorithm
And now…

…for something completely different.
Asymmetric TW-TSP

- \( l(u,v) \neq l(v,u) \).
- Impose triangle inequality by metric completion.
- Utilizes the concept of density \( \sigma \).
  - Density is the maximum number of round trips that can be made within a jobs time window.
Asymmetric TW-TSP: Density

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Asymmetric TW-TSP: Algorithm

- Uses dynamic programming
- Builds layers $L_i$ containing states $(v,t)$.
- $L_i$ contains states that visit $i$ points. Max $n$ states in any layer.
Asymmetric TW-TSP: Example
Asymmetric TW-TSP: Analysis

Exact algorithm

- If density < 1
- Unit profits

Factor $\sigma(\pi) + 1$ algorithm

- Arbitrary density
- Unit profits
Asymmetric TW-TSP: Additional results

- $(1+\varepsilon) \left( \frac{1}{\sigma(T)} \right) + 1$ algorithm based on reduction from KP.
- Solves general problem with arbitrary density and arbitrary profits.
- Can be expanded to handle processing time.
Conclusion

- The approximation factors are generally very high.
- TW-TSP on a line generally has poor approximation factors considering how restricted the problem is.