On Local Non-Compactness in Recursive Mathematics

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A metric space is said to be locally non-compact if every neighborhood contains a sequence that is bounded away from every element of the space, hence contains no accumulation point. We show within recursive mathematics that a nonvoid complete metric space is locally non-compact iff it is without isolated points.

The result has an interesting consequence in computable analysis: If a complete metric space has a computable witness that it is without isolated points, then every neighborhood contains a computable sequence that is eventually computably bounded away from every computable element of the space.

1 introduction

Let (X, d) be a metric space. In classical mathematics, the following are equivalent for any subset $C \subseteq X$:

1. C is complete and totally bounded.
2. Any cover of C by open balls can be trimmed to a finite subcover.
3. Every sequence of elements of C has an accumulation point.

Any one of these can, classically, be taken as the definition of "C is a compact subset of X". However, in the recursive branch of constructive mathematics, any considered concept from classical mathematics is assumed to have the prefix: "There is a program computing ...". For, say, (2) above, this means that there exists a program which given an "effective representation" of a cover, itself generated by a program, must produce a finite subset of the cover that itself covers C. For (1) and (2) to be *effectively* equivalent in this setting, we must have a recursive procedure transforming a program witnessing (1) to a program witnessing (2), and vice versa.

For the special case of the computable real numbers, standard results [Spe49, TZ62] state that not only do the above fail to be effectively equivalent, but a much stronger property holds: We can construct computable *counterexamples* to the equivalences such that, for instance, (1) holds, but (2) and (3) do not. As an example, the interval $[0, 1]$ of computable reals between 0 and 1 satisfies (1), but not (2): There exists a, computably given, sequence of open intervals with rational endpoints covering $[0, 1]$ for which all subcovers are infinite. Furthermore, there exists a computable sequence of rationals from $[0, 1]$ (a "Specker sequence") that is computably bounded away from all computable reals, hence can have no accumulation point, showing that (3) cannot hold. The choice of $[0, 1]$ is inessential: We can reconstruct the counterexamples in every neighborhood of every computable real; the set of computable reals is therefore said to be *locally non-compact*.

In this paper, we show within recursive mathematics that every nonvoid, complete metric space without isolated points is locally non-compact. In this form of constructivism, also known as the

Russian or *Markov* school of constructivism, all topological properties are assumed to have recursive witnesses. This provides an answer to a question posed by Bauer and Simpson [BS04] (indeed, provides an answer to a more general question, as Bauer and Simpson posed their question in the context of effectively *separable* metric spaces). As, conversely, every computable nonvoid metric space that is locally non-compact is also without isolated points, we obtain a precise characterization of local non-compactness in nonvoid complete metric spaces. In addition, as the proofs are entirely within recursive mathematics, the characterization is effective: We can transform a recursive witness of being complete and without isolated points to a suitable recursive witness of being locally non-compact and vice versa.

The result also sheds some light on the situation in computable analysis, e.g. the Type 2 computable analysis of Weihrauch [Wei98]: If we have an *effectively complete* metric space, i.e. a space where every computable Cauchy sequence of computable elements of the space converges effectively to some computable element of the space and an effective witness of being without isolated points, then we can, in every neighborhood of every points, construct a computable sequence of computable elements bounded away from every *computable* element of the space (though not necessarily from non-computable elements).

2 Preliminaries

The standard makeup of recursive constructive mathematics is assumed to be known; we refer the interested reader to the monographs by Aberth [Abe80] and by Bridges and Richman [BR87] and to the relevant parts of Troelstra and van Dalen's treatise [TvD87]; Markov's principle is freely employed. As always, there is the problem of whether to write definitions and proofs *without* explicit mention of Turing machines or partial recursive functions (as in e.g. [BR87]), or whether to explicitly mention them. The former approach is highly readable, especially to classical mathematicians, but presupposes a fair amount of maturity on the part of the reader. The latter approach has the advantage that the connection to actual programs — recursive "witnesses"— implementing the various concepts is clear, but — alas — is cluttered due to the heavy use of the indices lamentably common in recursion theory. We choose the first approach but give recursion-theoretic formulations of the non-standard definitions in Remarks 2.5 and 2.8. Interested readers are referred to [BS04] to see other standard concepts expressed in standard recursion theory nomenclature, and to [RJ87] for a fairly comprehensive treatise on recursion theory itself.

Definition 2.1 We set $\mathbf{2} \triangleq \{0, 1\}$, define $\mathbf{2}^*$ as the set of finite binary sequences, and let $\mathbb{N} = \{1, 2, 3, \ldots\}$, resp. $\mathbb{N}_0 \triangleq \{0\} \cup \mathbb{N}$ as usual. $\mathbf{2}^{\mathbb{N}}$ is the set of infinite binary sequences (represented by indices i of total recursive functions $\phi_i : \mathbb{N} \longrightarrow \mathbf{2}$).

Elements of $\mathbf{2}^*$ are ranged over by the Greek letters α, β, ξ. If $\gamma \in \mathbf{2}^{\mathbb{N}}$ and $j \in \mathbb{N}$, we denote by $\gamma(j)$ the jth bit of γ, and by $\gamma(1) \cdots \gamma(j)$ the initial prefix of γ of length j. The latter notation is used for elements of $\mathbf{2}^*$ *mutatis mutandis.*

Note that equality in $\mathbf{2}^*$ is decidable.

The definition of metric spaces and their completeness is assumed to be known. A set X is said to be *nonvoid* if there exists $x \in X$. If (X, d) is a metric space with X nonvoid, the ball with center x and radius 2^{-k} (for $k \in \mathbb{N}$) is denoted $B_k(x)$. We remind the reader of the concepts of total boundedness and locatedness:

Definition 2.2 Let (X, d) be a metric space and let $Y \subseteq X$. A 2^{-n}-*approximation* of Y is a subset $Z \subseteq Y$ such that for any $y \in Y$ there exists $z \in Z$ with $d(y, z) \leq 2^{-n}$. Y is said to be *totally bounded* if, for each $n \in \mathbb{N}$, there exists a *finite* 2^{-n}-approximation to Y.

Y is said to be *located* if, for all $x \in X$, the *distance* $d_Y(x) \triangleq \inf\{d(x, y) : y \in Y\}$ exists.

The following is a staple, classically valid, result of all major schools of constructivism:

Proposition 2.3 *Let (X, d) be a metric space and let $Y \subseteq X$ be totally bounded. Then Y is located.*

Proof. See e.g. [BR87, Cor 4.6]. □

Definition 2.4 A metric space (X, d) is said to be *without isolated points* if, for each $x \in X$ and each $n \in \mathbb{N}$, there exists $y \in X$ with $y \neq x$ and $d(x, y) < 2^{-n}$.

A sequence $(x_n)_{n\in\mathbb{N}}$ in X is said to be *without accumulation point* if, for all $y \in X$, there exists $j, l \in \mathbb{N}$ such that $d(x_n, y) > 2^{-j}$ for all $n \geq l$.

Observe that $y \neq x$ in the above asserts existence of $m \in \mathbb{N}$ such that $d(x, y) > 2^{-m}$.

Remark 2.5 In the lingo of recursion theory, X must be a *numbered space* (see e.g. [Erš73, Erš75, Erš77]): We assume a (classical, possibly non-computable) *partial* map $\nu_X : \mathbb{N} \longrightarrow X$ (intution: every element of X is represented by some natural number).

A sequence in X is then a total recursive function $\phi_i : \mathbb{N} \longrightarrow \mathbb{N}$ such that $\phi_i(j) \in dom(\nu_X)$ for all $j \in \mathbb{N}$. The sequence is without accumulation point if there exists a partial recursive function $\phi_p : \mathbb{N} \longrightarrow \mathbb{N} \times \mathbb{N}$ such that if $n \in dom(\nu_X)$, then $n \in dom(\phi_p)$ and $\phi_p(n) = (m, l)$ implies that $d(\nu_X(\phi_i(j)), \nu_X(n)) > 2^{-m}$ for all $j \geq l$. We say that ϕ_p *witnesses* that the sequence is without accumulation point.

Example 2.6 Standard examples of metric spaces without isolated points are $\mathbb{Q}$ and $\mathbb{R}$ with the usual metrics, any metric vector space over $\mathbb{Q}$ or one of its extension fields, and the set of uniformly continuous functions on totally bounded spaces equipped with the supremum norm (see e.g. [BS04]).

Examples of complete metric spaces without isolated points that are not separable abound in classical mathematics, but none of the standard examples give rise to a computable metric (which is necessary for the metric to be well-defined in recursive mathematics). Instead, an example—lamentably invalid in classical mathematics— can be constructed as follows: Select a subset $A \subseteq \mathbb{N}$ that is not recursively enumerable (hence not countable in the sense of recursive mathematics), e.g. $A = \{i : \phi_i \text{ is not total}\}$. Consider the set $A \times [0, 1]$ endowed with the metric defined by $d((m, x), (n, y)) = 1$ if $m \neq n$, and $d(m, x), (n, y)) = |y - x|$ if $m = n$. The metric is well-defined as equality on $\mathbb{N}$ is decidable, and the space is complete and without isolated points as $[0, 1]$ has both of these properties. Were the space separable, we could extract a (possibly non-injective) enumeration of A from any sequence witnessing separability, a contradiction.

Definition 2.7 A metric space (X, d) is said to be *locally non-compact* if, for each $y \in X$ and each $m \in \mathbb{N}$, there exists a sequence $(x_n)_{n\in\mathbb{N}}$ such that $d(y, x_n) < 2^{-m}$, but $(x_n)_{n\in\mathbb{N}}$ is without accumulation point.

Remark 2.8 In the lingo of recursion theory, X is locally non-compact if there is a partial recursive function $\phi_l : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N} \times \mathbb{N}$ such that if $n \in dom(\nu_X)$, then $(n, m) \in dom(\phi_l)$ for all $m \in \mathbb{N}$, and if $\phi_l(n, m) = (i, k)$, then $\phi_i : \mathbb{N} \longrightarrow \mathbb{N}$ and $\phi_k : \mathbb{N} \longrightarrow \mathbb{N}\times\mathbb{N}$ are both total recursive, $d(\nu_X(\phi_i(j)), \nu_X(n)) < 2^{-m}$ for all $j \in \mathbb{N}$, and ϕ_k witnesses that the sequence given by ϕ_i is without accumulation point.
Intuition: $\nu_X(p)$ is y and $(\nu_X(\phi_i(n)))_{n\in\mathbb{N}}$ is $(x_n)_{n\in\mathbb{N}}$ in Definition 2.7.

Proposition 2.9 *A nonvoid, locally non-compact space* (X, d) *is without isolated points.*

Proof. Let $y \in X$ and $B_k(y)$ be any ball with center y. By local non-compactness, there is a sequence $(x_n)_{n\in\mathbb{N}}$ in $B_k(y)$ such that $d(y, x_n) < 2^{-k}$, but $(x_n)_{n\in\mathbb{N}}$ is without accumulation point. Then, there are $N, l \in \mathbb{N}$ with $d(x_n, y) > 2^{-l}$ for $n \geq N$, i.e. $x_N \neq y$ and $x_N < 2^{-k}$; as $y \in X$ and $k \in \mathbb{N}$ were arbitrary, this shows that (X, d) is without isolated points. □

3 Embedding the Cantor Space

Bauer and Simpson prove the existence of a uniformly continuous embedding with closed image of $\mathbf{2}^{\mathbb{N}}$ in every nonvoid separable space without isolated points. For our main theorem, we need such an embedding in every neighborhood of every point.

While the construction of Bauer and Simpson may be easily adapted so that the image of $\mathbf{2}^{\mathbb{N}}$ is contained in any prescribed open ball, their proof uses separability of the metric space. We adopt a more pedestrian approach which first constructs an embedding of $\mathbf{2}^*$ in each neighborhood of each point in every space without isolated points and subsequently extends this to an embedding of $\mathbf{2}^{\mathbb{N}}$ when the considered space is complete. The advantage of this approach is that the considered space need not be assumed separable, but the price paid is use of the Axiom of Dependent Choice in Lemma 3.1 below (Bauer and Simpson used only the weaker $AC_{0,1}$, i.e. choice from $\mathbb{N}$ to $\mathbb{N}^{\mathbb{N}}$, but their proof only works for separable spaces).

The embedding of $\mathbf{2}^*$ is given implicitly by the infinite binary tree of the following lemma.

Lemma 3.1 *Let (X, d) be a nonvoid metric space without isolated points and let $B_k(x)$ be a ball of radius 2^{-k} with center $x \in X$. For each $n \in \mathbb{N}_0$ and each $\alpha \in \mathbf{2}^*$ with $|\alpha| = n$, there is an $x_\alpha \in X$ and a natural number $k_n > n$ such that:*

1. $d(x_\alpha, x_{\alpha\cdot b}) + 2^{-k_{n+1}} < 2^{-k_n}$ *for both* $b \in \mathbf{2}$.
2. $d(x_\alpha, x_{\alpha\cdot b}) > 3 \cdot 2^{-k_{n+1}}$ *for both* $b \in \mathbf{2}$.
3. $d(x_{\alpha\cdot 0}, x_{\alpha\cdot 1}) > 3 \cdot 2^{-k_{n+1}}$.
4. *For any $\alpha, \beta \in \mathbf{2}^*$ with $\alpha \neq \beta$, let $j \in \mathbb{N}_0$ be the least non-negative integer such that α and β disagree in the jth bit (if $|\alpha| > |\beta|$ and α and β agree on $|\beta|$ bits, then j is $|\beta| + 1$). We then have $d(x_\alpha, x_\beta) > 2^{-k_j}$.*

P r o o f. We construct x_α by induction on n. If $n = 0$, we have $\alpha = \lambda$ in which case we set $x_\lambda \triangleq x$, and $k_0 = k$, respectively. For the inductive case, assume that the x_α and k_n have been constructed up to length n, and consider any α with $|\alpha| = n$. As (X, d) is without isolated points, there is an $x'_\alpha \in X$ and a $k'_\alpha \in \mathbb{N}$ such that $2^{-k'_\alpha} < d(x_\alpha, x'_\alpha) < 2^{-k_n}$ and $d(x_\alpha, x'_\alpha) + 2^{-k'_\alpha} < 2^{-k_n}$. Define $x_{\alpha\cdot 0} \triangleq x'_\alpha$ and $k'_n \triangleq 2 + \max\{k'_\alpha : |\alpha| = n\}$.

Again, as (X, d) is without isolated points, there is a $k''_\alpha \in \mathbb{N}$ and an x''_α such that $2^{-k''_\alpha} < d(x_\alpha, x''_\alpha) < 2^{-k'_n}$. Define $x_{\alpha\cdot 1} \triangleq x''_\alpha$ and $k_{n+1} \triangleq 2 + \max\{k'_n, \max\{k''_\alpha : |\alpha| = n\}\}$.

We now prove each of the lemma's claims in turn:

1. The choice of $x_{\alpha\cdot 0}$ and $x_{\alpha\cdot 1}$ clearly ensures that $d(x_\alpha, x_{\alpha\cdot b}) + 2^{-k_{n+1}} < 2^{-k_n}$ for both $b \in \mathbf{2}$.
2. This is immediate by the construction of $x_{\alpha\cdot 0}$, $x_{\alpha\cdot 1}$ and k_{n+1}.
3. This is immediate by the construction of $x_{\alpha\cdot 0}$, $x_{\alpha\cdot 1}$ and k_{n+1}.
4. Let $\alpha, \beta \in \mathbf{2}^*$. If $j = n-1$ or $j = n$, the result follows by the second and third claims of the lemma. If $j \leq n-2$, let $\xi \in \mathbf{2}^{j-1}$ be such that (wlog.) $\alpha = \xi \cdot 0 \cdot \delta$ and $\beta = \xi \cdot 1 \cdot \delta'$ for suitable (and unique) $\delta, \delta' \in \mathbf{2}^*$. A straightforward induction using the first claim of the lemma shows that for any string $\delta \in \mathbf{2}^*$, we have $d(x_{\xi\cdot 0}, x_{\xi\cdot 0\cdot\delta}) < 2^{-k_j}$ (the corresponding fact holds for $\xi \cdot 1$). Thus:

$$\begin{array}{rcll} d(x_{\xi\cdot 0\cdot\delta}, x_{\xi\cdot 1\cdot\delta'}) & \geq & d(x_{\xi\cdot 0}, x_{\xi\cdot 1\cdot\delta'}) - d(x_{\xi\cdot 0}, x_{\xi\cdot 0\cdot\delta}) & \text{Triangle Inequality} \\ & \geq & d(x_{\xi\cdot 0}, x_{\xi\cdot 1}) - d(x_{\xi\cdot 1\cdot\delta'}, x_{\xi\cdot 1}) - d(x_{\xi\cdot 0}, x_{\xi\cdot 0\cdot\delta}) & \text{Triangle Inequality} \\ & > & 3 \cdot 2^{-k_j} - 2^{-k_j} - 2^{-k_j} & \text{Third claim} \\ & \geq & 2^{-k_j} & \end{array}$$

concluding the proof.

□

Corollary 3.2 *Let $j \in \mathbb{N}$; then $\alpha, \beta \in \mathbf{2}^*$ have a common prefix of length j iff $d(x_\alpha, x_\beta) < 2^{-k_j}$.*

P r o o f. "If": Assume that α and β agree on a prefix of length $0 \leq j' < j$. By the fourth claim of the lemma, we then have $d(x_\alpha, x_\beta) > 2^{-k_{j'}} > 2^{-k_j}$. "Only if": If α, β have a common prefix of length j, consider $\alpha(1) \cdots \alpha(j)$. It is an easy induction on $n \in \mathbb{N}_0$, using the first part of the lemma, to show that if $\xi \in \mathbf{2}^n$ with, then $d(x_{\alpha(1)\cdots\alpha(j)}, x_{\alpha(1)\cdots\alpha(j)\cdot\xi}) < 2^{-k_j}$, whence the result. □

Corollary 3.3 *Let $(x_{\alpha_n})_{n\in\mathbb{N}}$ be convergent with limit a and assume that $d(x_{\alpha_m}, x_{\alpha_{m'}}) < 2^{-k_m}$ for all $m' \geq m$. Then, $d(x_{\alpha_n}, a) \leq 2^{-k_n}$ for all $n \in \mathbb{N}$.*

Proof. By the previous corollary, α_m and $\alpha_{m'}$ agree on a prefix of length m for all $m \in \mathbb{N}$. By repeated application of Triangle Inequality, we obtain $d(x_{\alpha_n}, a) \leq d(x_{\alpha_{n+k}}, a) + \sum_{j=0}^{k-1} d(x_{n+j}, x_{n+j+1})$ for all positive integers k, n. By the first claim of the lemma, we have $d(x_{n+j}, x_{n+j+1}) < 2^{-k_{n+j}} - 2^{-k_{n+j+1}}$, and we thus have $d(x_{\alpha_n}, a) \leq d(x_{\alpha_{n+k}}, a) + 2^{-k_{n+1}} - 2^{-k_{n+k}}$ where the right-hand side of the inequality is bounded above by $d(x_{\alpha_{n+k}}, a) + 2^{-k_{n+1}}$ (for all $k \in \mathbb{N}$). As $(x_{\alpha_n})_{n\in\mathbb{N}}$ is convergent, there is an $l \in \mathbb{N}$ such that $d(x_{\alpha_{n+l}}, a) < 2^{-k_{n+1}}$, whence $d(x_{\alpha_n}, a) \leq 2 \cdot 2^{-k_{n+1}} \leq 2^{-k_n}$, as desired. □

Definition 3.4 Let (X, d) be nonvoid, complete, and without isolated points, and let $B_k(x)$ and x_α be as in Lemma 3.1. Then we define $A(k, x)$ as the closure of $\{x_\alpha : \alpha \in \mathbf{2}^*\}$, i.e. $A(k, x)$ is the set of elements $y \in X$ such that y is the limit of a Cauchy sequence of elements of $\{x_\alpha : \alpha \in \mathbf{2}^*\}$.

Lemma 3.5 *Let (X, d) be nonvoid, complete and without isolated points. For any $k \in \mathbb{N}$ and $x \in X$, the set $A(k, x)$ is complete and totally bounded (hence closed and located).*

Proof. For each $m \in \mathbb{N}$, the set $K_m = \{x_\alpha : |\alpha| = m\}$ is finite, and we claim that it is a 2^{-m}-approximation of $A(k, x)$, entailing that $A(k, x)$ is totally bounded. To prove the claim, let $a \in A(k, x)$, whence there is a Cauchy sequence $(x_{\alpha_n})_{n\in\mathbb{N}}$ converging to a which we wlog. may assume satisfies $d(x_{\alpha_m}, x_{\alpha_{m'}}) < 2^{-k_m}$ for all $m' \geq m$. By Corollary 3.2, we have that the initial prefix of length m of $\alpha_{m'}$ are identical for all $m' \geq m$; write this prefix as $\gamma(1) \cdots \gamma(m)$. By Corollary 3.3, we have $d(a, x_{\gamma(1)\cdots\gamma(m)}) \leq 2^{-k_m}$, and as $k_m > m$, we are done.

To show completeness of $A(k, x)$, let $(a_n)_{n\in\mathbb{N}}$ be a Cauchy sequence of elements of $A(k, x)$. Wlog. we may assume that $d(a_k, a_j) < 2^{-k_m}$ for all $k, j \geq m$. Corollary 3.2 then entails that a_k and a_j agree on m bits for all $k, j \geq m$, and $(a_n)_{n\in\mathbb{N}}$ thus induces a sequence $(\alpha_n)_{n\in\mathbb{N}}$ of elements of $\mathbf{2}^*$ such that $|\alpha_n| = n$ for all $n \in \mathbb{N}$ and α_n is a (proper) prefix of α_k for all $k \geq n$. The sequence $(x_{\alpha_n})_{n\in\mathbb{N}}$ is Cauchy by Corollary 3.2, hence converges to some limit $a \in A(k, x)$; as $d(x_{\alpha_m}, a_m) < 2^{-k_m}$, we have $d(a, a_m) \leq d(a, x_{\alpha_m}) + d(x_{\alpha_m}, a_m) < 2 \cdot 2^{-k_m}$ for all $m \in \mathbb{N}$, i.e. $(a_n)_{n\in\mathbb{N}}$ converges to a, concluding the proof. □

To see that every $\gamma \in \mathbf{2}^\mathbb{N}$ induces a corresponding element x_γ, we have the following proposition:

Proposition 3.6 *Let $\gamma \in \mathbf{2}^\mathbb{N}$ and let $(\alpha_n)_{n\in\mathbb{N}}$ and $(\beta_n)_{n\in\mathbb{N}}$ be sequences in $\mathbf{2}^*$ both converging to γ. Then the sequences $(x_{\alpha_n})_{n\in\mathbb{N}}$ and $(x_{\beta_n})_{n\in\mathbb{N}}$ both converge in $A(k, x)$ and have the same limit.*

Proof. As $(\alpha_n)_{n\in\mathbb{N}}$ converges in $\mathbf{2}^\mathbb{N}$, we may assume wlog. that, for any $n \in \mathbb{N}$, α_m and $\alpha_{m'}$ agree on the first n bits for all $m, m' > n$. Then Corollary 3.2 yields that $(x_{\alpha_n})_{n\in\mathbb{N}}$ is Cauchy, hence convergent by completeness of (X, d). As $A(k, x)$ is closed by Lemma 3.5, $(x_{\alpha_n})_{n\in\mathbb{N}}$ converges to some element of $A(k, x)$. By symmetry, $(x_{\beta_n})_{n\in\mathbb{N}}$ converges as well. As $(\alpha_n)_{n\in\mathbb{N}}$ and $(\beta_n)_{n\in\mathbb{N}}$ were convergent in $\mathbf{2}^\mathbb{N}$, for each $j \in \mathbb{N}$, there is some N such that $m > N$ implies $d(\alpha_m, \beta_m) < 2^{-j}$, i.e. α_m and β_m agree on at least j bits, whence Corollary 3.2 yields that $d(x_{\alpha_m}, x_{\beta_m}) < 2^{-k_j}$. As $j \in \mathbb{N}$ was arbitrary and $k_j > j$, $(x_{\alpha_n})_{n\in\mathbb{N}}$ and $(x_{\beta_n})_{n\in\mathbb{N}}$ converge to the same limit, as desired. □

Thus, x_γ in the following does not depend on the concrete sequence $(\alpha_n)_{n\in\mathbb{N}}$:

Definition 3.7 For any $\gamma \in \mathbf{2}^\mathbb{N}$, let $(\alpha_n)_{n\in\mathbb{N}}$ converge to γ. Define x_γ to be the limit of $(x_{\alpha_n})_{n\in\mathbb{N}}$ in $A(k, x)$.

Thus, the stipulation $\gamma \mapsto x_\gamma$ yields a well-defined map from $\mathbf{2}^\mathbb{N}$ to the complete metric space (X, d), and said map can easily be proved to be injective and satisfy $f(\mathbf{2}^\mathbb{N}) = A(k, x)$. However, these properties are not essential for our exposition; upcoming proofs need only refer to x_γ and $A(k, x)$.

The following result is valid in recursive mathematics (but not classically valid, nor derivable in Bishop's constructivism):

Proposition 3.8 *There exists a sequence in $\mathbf{2}^*$ without accumulation point in $\mathbf{2}^\mathbb{N}$.*

Proof. See e.g. [Ric83] or alternatively [BR87, Prop. 3.1]. The result is occasionally phrased as follows: There exists a detachable[1] *bar* (subset B of $\mathbf{2}^*$ such that every element of $\mathbf{2}^\mathbb{N}$ has a prefix in

[1] Constructive lingo for the recursion-theoretic "decidable".

B) such that there is a sequence $(\alpha_n)_{n\in\mathbb{N}}$ in $\mathbf{2}^*$ with $|\alpha_n| \geq n$, such that no prefix of any α_n is in the bar. Then $(\alpha_n)_{n\in\mathbb{N}}$ has no accumulation point in $\mathbf{2}^{\mathbb{N}}$: For every $\gamma \in \mathbf{2}^{\mathbb{N}}$, some prefix $\gamma(1)\cdots\gamma(k)$ is in B, but then $d(\alpha_n, \gamma) > 2^{-(k+1)}$ for all $n > k$. □

Proposition 3.9 *Let $B_k(x)$ be a ball in the complete metric space (X, d). If $(\alpha_n)_{n\in\mathbb{N}}$ is a sequence in $\mathbf{2}^*$ without accumulation point in $\mathbf{2}^{\mathbb{N}}$, then $(x_{\alpha_n})_{n\in\mathbb{N}}$ is without accumulation point in $A(k, x)$.*

Proof. Let $x_\gamma \in A(k, x)$. Then x_γ is the limit of a sequence $(x_{\beta_n})_{n\in\mathbb{N}}$ where $\beta_n \in \mathbf{2}^*$, and we may wlog. assume that $d(x_{\beta_n}, x_\gamma) < 2^{-n}$ for all $n \in \mathbb{N}$, i.e. the first n bits of β_m and $\beta_{m'}$ are identical for $m, m' \geq n$. As $(\alpha_n)_{n\in\mathbb{N}}$ is without accumulation point in $\mathbf{2}^{\mathbb{N}}$, there exists $N \in \mathbb{N}$ such that $d(\alpha_n, x_\gamma) > 2^{-(j+1)}$ for all $n > N$, in which case the jth bit of α_n and x_{β_n} differ for all $n > N$. By Corollary 3.2, $d(x_{\alpha_n}, x_{\beta_n}) \geq 2^{-k_j}$ for all $n > N$, whence $d(x_{\alpha_n}, x_\gamma) \geq 2^{k_j}$ for all $n > N$; as $x_\gamma \in A(k, x)$ was arbitrary, $(x_{\alpha_n})_{n\in\mathbb{N}}$ is thus without accumulation point. □

We shall also need the following result, colloquially known as Bishop's Lemma:

Lemma 3.10 *Let L be a closed, located subset of a complete metric space (X, d) and let $x \in X$. There exists $y \in L$ such that $d(x, L) > 0$ iff $d(x, y) > 0$.*

Proof. See e.g. [BR87, Lem. 3.3]. □

Corollary 3.11 *Let L be a closed, located subset of a complete metric space (X, d) and let $(a_n)_{n\in\mathbb{N}}$ be a sequence in L without accumulation point in L. Then $(a_n)_{n\in\mathbb{N}}$ is also without accumulation point in X.*

Proof. Let $x \in X$. As $(a_n)_{n\in\mathbb{N}}$ has no accumulation point in L, there is $\epsilon > 0$ such that $d(a_n, y) > \epsilon$ for all but finitely many $n \in \mathbb{N}$. By Bishop's Lemma there is $y \in L$ such that $d(x, L) > 0$ iff $d(x, y) > 0$. Let, hence, $N \in \mathbb{N}$ be such that $d(x, y) \geq \epsilon/2$ implies that $d(x, L) > 2^{-N}$.

We have $d(x, L) > 2^{-(N+1)}$ or $d(x, L) < 2^{-N}$. In the former case, $d(a_n, x) > 2^{-(N+1)}$ for all $n \in \mathbb{N}$. In the latter case, we have $d(x, y) \geq \epsilon/2$ would imply $d(x, L) > 2^{-N}$, and must thus have $d(x, y) < \epsilon/2$. Hence, $d(x, a_n) \geq d(y, a_n) - d(x, y) > \epsilon/2$ for all but finitely many $n \in \mathbb{N}$. □

4 The Correspondence Theorem

The stage is now set for the main theorem of the paper:

Theorem 4.1 *A nonvoid complete metric space (X, d) is locally non-compact iff it is without isolated points.*

Proof. "Only if" is Proposition 2.9. "If" is proved as follows: Given any open ball in $B_k(x)$ in X, construct an embedding of the set $\mathbf{2}^*$ in $B_{k/2}(x)$ as in Lemma 3.1. Now, $A(k/2, x)$ is a subset of $B_k(x)$ and by Lemma 3.5 is complete and totally bounded, hence closed and located. By Proposition 3.8, let $(\alpha_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbf{2}^*$ without accumulation point in $\mathbf{2}^{\mathbb{N}}$. Proposition 3.9 furnishes that $(x_{\alpha_n})_{n\in\mathbb{N}}$ is without accumulation point in $A(k/2, x)$. Corollary 3.11 now yields the result.

□

Remark 4.2 Let $D \subseteq X$ be a dense subset of a complete metric space (X, d) without isolated points. By refining the 'if' part of the proof, we can show that the sequence witnessing local non-compactness in a given neighborhood may be taken to consist of elements of D.

Note that since the proof is valid in recursive mathematics, the 'iff' statement amounts to the claim that we can effectively transform a witness of being without isolated points to a witness of being locally non-compact and vice versa.

4.1 Discussion

We close our exposition with a small observation on conservation of the property of being without accumulation point:

Lemma 4.3 *Let (X, d) and (Y, e) be metric spaces, and let $f : X \longrightarrow Y$ have a pointwise continuous left-inverse on its image, i.e. there is $g : f(X) \longrightarrow X$ with $g(f(\gamma)) = \gamma$ for all $\gamma \in X$. If $(\gamma_n)_{n\in\mathbb{N}}$ is without accumulation point in X, then $(f(\gamma_n))_{n\in\mathbb{N}}$ is without accumulation point in $f(Y)$.*

P r o o f. Let $x \in X$; as $(\gamma_n)_{n\in\mathbb{N}}$ was without accumulation point, there is $\epsilon > 0$ and $N \in \mathbb{N}$ such that $d(\gamma_n, x) > \epsilon$ for all $n > N$. As g is continuous, there is $\delta > 0$ such that $d(f(x), f(z)) < \delta$ implies that $d(g(f(x), g(f(z))) = d(x, z) < \epsilon$ for all $z \in X$. But then $d(f(x), f(\gamma_n)) < \delta$ implies that $d(x, \gamma_n) < \epsilon$, an impossibility, whence $d(f(x), f(\gamma_n)) > \delta$ for all $n > N$. As $x \in X$ (and hence $f(x) \in f(X)$ was arbitrary, $(f(\gamma_n))_{n\in\mathbb{N}}$ is thus without isolation point in $f(X)$. □

In particular, if we have constructed a homeomorphic embedding of $\mathbf{2}^{\mathbb{N}}$ into *any* metric space (X, d), we obtain a sequence without accumulation point in that space by Proposition 3.8. One can then proceed to prove that the image of $\mathbf{2}^{\mathbb{N}}$ was closed and located; Bauer and Simpson construct a uniformly continuous embedding $f : \mathbf{2}^{\mathbb{N}} \longrightarrow X$ where (X, d) is complete, separable and without isolated points and a pointwise continuous $g : X \longrightarrow \mathbb{R}^{\mathbb{N}}$ such that the restriction of g to $f(\mathbf{2}^{\mathbb{N}})$ maps into $\mathbf{2}^{\mathbb{N}}$ and is a left-inverse of f on its image. While they prove the image of f closed, they do not prove it located.; it would be interesting to ascertain whether the image is also located.

We could have proven that the conditions of the above lemma hold for the particular construction in out paper: Essentially, $A(k, x)$ is the image of a map $f_{k,x} : \mathbf{2}^{\mathbb{N}} \longrightarrow X$ defined by $\gamma \mapsto x_\gamma$. However, we would then be faced with the task of avoiding dubious choice axioms (in particular $AC_{1,0}$) when constructing the corresponding map $g : A(k, x) \longrightarrow \mathbf{2}^{\mathbb{N}}$. We have thus opted for the more straightforward solution presented in Section 3.

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