Facility Location

Bjørn Petersen
DIKU, University of Copenhagen

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Outline

• Introduction
• Uncapacitated Facility Location Problem (UFL)
• Capacitated Facility Location Problem (CFLP)
• Plants and multiple commodities
• Integrating with vehicle routing
• Integrating with inventory management
• Stochastic facility location
• Facility failures
• Exercises
Introduction

Facility Location:

- Select facilities to open
- Assign clients to facilities
- Minimize total cost
Uncapacitated Facility Location Problem (UFL)

Also known as Simple Plant Location Problem (SPLP)

\( y_i \) Indicator variable that tells if facility \( i \in F \) is used or not

\( x_{ij} \) Indicator variable that tells if client \( j \in C \) is serviced by facility \( i \in F \)

\( f_i \) Cost of opening facility \( i \in F \)

\( c_{ij} \) Cost of serving client \( j \in C \) by facility \( i \in F \)

\[
\min \sum_{i \in F} f_i y_i + \sum_{i \in F} \sum_{j \in C} c_{ij} x_{ij} \quad (1)
\]

\[
\text{s.t. } \sum_{i \in F} x_{ij} = 1 \quad \forall j \in C \quad (2)
\]

\[
x_{ij} \leq y_i \quad \forall i \in F, \forall j \in C \quad (3)
\]

\[
x_{ij} \in \{0, 1\} \quad \forall i \in F, \forall j \in C \quad (4)
\]

\[
y_i \in \{0, 1\} \quad \forall i \in F \quad (5)
\]

Complexity (UFL)

(● Se exercises ●)
Capacitated Facility Location Problem (CFLP)

Two versions:

1) Less-than-truckload (LTL) not allowed
2) LTL allowed

\( y_i \) Indicator variable that tells if facility \( i \in F \) is used or not
\( x_{ij} \) Indicator variable that tells if client \( j \in C \) is serviced by facility \( i \in F \)
\( f_i \) Cost of opening facility \( i \in F \)
\( c_{ij} \) Cost of serving client \( j \in C \) by facility \( i \in F \)
\( d_j \) The amount of demand from client \( j \in C \)
\( u_i \) Upper limit on the amount of demand serviced from facility \( i \in F \)
The model (CFLP)

Case 1:

\[
\min \sum_{i \in F} f_i y_i + \sum_{i \in F} \sum_{j \in C} c_{ij} d_j x_{ij} \tag{6}
\]

s.t. \[
\sum_{i \in F} x_{ij} = 1 \quad \forall j \in C \tag{7}
\]

\[
x_{ij} \leq y_i \quad \forall i \in F, \forall j \in C \tag{8}
\]

\[
\sum_{j \in C} d_j x_{ij} \leq u_i y_i \quad \forall i \in F \tag{9}
\]

\[
x_{ij} \in \{0, 1\} \quad \forall i \in F, \forall j \in C \tag{10}
\]

\[
y_i \in \{0, 1\} \quad \forall i \in F \tag{11}
\]

In case 2, (10) is replaced by

\[
0 \leq x_{ij} \leq 1 \quad \forall i \in F, \forall j \in C \tag{12}
\]
Complexity (CFLP)

Case 1) is easily seen to be \(\mathcal{NP}\)-hard, by reduction from Knapsack or Bin Packing:

- \(f_i\) is set to one
- \(u_i\) is capacity of Knapsack/Bin \(i\)
- \(d_j\) is size of item \(j\)
- \(c_{ij}\) is set to zero
- \(x_{ij}\) assigns item \(j\) to Knapsack/Bin \(i\)

Case 2) is easily reduced from Knapsack the same way

Solution methods (CFLP)

Have been solved with many different methods, e.g.

- Optimally
  - Branch and cut
  - Lagrange relaxation
- Heuristically
  - Simulated annealing
  - Ant Colony
Plants and multiple commodities (PMC)

In the real world facilities are dependent on plants (e.g. factories)

There are often more than one commodity handled at each facility

There are often an upper limit on how much commodity can be handled by a given facility

There might be a lower limit on if a facility is allowed to open
The model (PMC)

$x_{ij}$ Indicator variable that tells if client $j \in C$ is serviced by facility $i \in F$

$y_i$ Indicator variable that tells if facility $i \in F$ is used or not

$z_{lki}j$ Variable that tells how much of commodity $l \in L$ is shipped from plant $k \in K$ to facility $i \in F$ and further to client $j \in C$

$f_i$ Cost of opening facility $i \in F$

$c_{lki}j$ Unit cost of producing and shipping commodity $l \in L$ from plant $k \in K$ to facility $i \in F$ and further to client $j \in C$

$d_{l}j$ The amount of demand of commodity $l \in L$ from client $j \in C$

$\underline{u}_i$ Lower limit on the amount of demand serviced from facility $i \in F$

$\bar{u}_i$ Upper limit on the amount of demand serviced from facility $i \in F$

$s_{lk}$ Supply of commodity $l \in L$ at plant $k \in K$

$v_i$ Unit cost of throughput at facility $i \in F$
The model (PMC)

\[
\begin{align*}
\min & \sum_{i \in F} f_i y_i + \sum_{i \in F} v_i \sum_{l \in L} \sum_{j \in C} d_{lj} x_{ij} + \sum_{l \in L} \sum_{k \in K} \sum_{i \in F} \sum_{j \in C} c_{lki} z_{lki}j \\
\text{s.t.} & \sum_{i \in F} \sum_{j \in C} z_{lki}j \leq s_{lk} \quad \forall l \in L, \forall k \in K \quad (14) \\
& \sum_{k \in K} z_{lki}j = d_{lj} x_{ij} \quad \forall l \in L, i \in F, j \in C \quad (15) \\
& \sum_{i \in F} x_{ij} = 1 \quad \forall j \in C \quad (16) \\
& u_i y_i \leq \sum_{l \in L} \sum_{j \in C} d_{lj} x_{ij} \leq u_i y_i \quad \forall i \in F \quad (17) \\
& x_{ij} \in \{0, 1\} \quad \forall i \in F, \forall j \in C \quad (18) \\
& y_i \in \{0, 1\} \quad \forall i \in F \quad (19) \\
& z_{lki}j \geq 0 \quad \forall l, k, i, j \quad (20)
\end{align*}
\]

(14) secures plant production-limit is satisfied

(15) flow balance between consumption at client/facility and supply by plant

(16) single source constraint (map client to facility)

(17) secures that lower and upper limit is satisfied at facilities
Integrating with vehicle routing

Full truckload ⇒ costs of deliveries constants

LTL ⇒ costs of deliveries a Vehicle Routing Problem (VPR)

Better ‘approximation’ by actually solving the VRP

Can be combined with multiple levels (e.g. incorporating plants)

In the simple case, three aspect at the same time:

- Locate facilities
- Assign clients to facilities
- Vehicle routing

A picture of a truck:
The model (vehicle routing)

Column generation based model:

- $\lambda_p$ Indicator variable that tells if path $p \in P$ is used or not
- $y_i$ Indicator variable that tells if facility $i \in F$ is used or not
- $c_p$ Cost of path $p \in P$
- $P_i$ Set of paths representing all paths from facility $i \in F$ that are in the set of feasible paths $P_i \subseteq P$
- $a_{pj}$ Tells if path $p$ is serving client $j \in C$

\[
\begin{align*}
\min & \quad \sum_{i \in F} f_i y_i + \sum_{i \in F} \sum_{p \in P_i} c_p \lambda_p \\
\text{s.t.} & \quad \sum_{i \in F} \sum_{p \in P_i} a_{pj} \lambda_p = 1 \quad \forall j \in C \\
& \quad \lambda_p \leq y_i \quad \forall i \in F, \forall p \in P_i \\
& \quad y_i \in \{0, 1\} \quad \forall i \in F \\
& \quad \lambda_p \in \{0, 1\} \quad \forall i \in F, \forall p \in P_i
\end{align*}
\]

Solution methods

Delayed column generation with a resource constrained shortest path problem (SPPRC) as pricing problem for each possible facility $i \in F$
Integrating with inventory management

In fixed charge facility location the cost is a tradeoff between:

- Travel cost from facilities to clients
- Cost of opening facilities

In the real world the cost of running a facility is also dependent on the amount of inventory.

Cost of the working stock increases ‘approximately’ as the square root of the amount of commodity going through the facility.

Cost for the safety stock increases ‘approximately’ as the square root of the variance.
The model (inventory management)

$y_i$ Indicator variable that tells if facility $i \in F$ is used or not

$x_{ij}$ Indicator variable that tells if client $j \in C$ is serviced by facility $i \in F$

$c_{ij}$ Cost of serving client $j \in C$ by facility $i \in F$

$\mu_j$ Mean of the demand at client $j \in C$ (mu)

$\sigma^2_j$ Variance of the demand at client $j \in C$ (sigma)

$\rho_i$ Cost of working stock at facility $i \in F$ (rho)

$\omega_i$ Cost of safety stock at facility $i \in F$ (omega)

$$\min \sum_{i \in F} f_i y_i + \sum_{i \in F} \sum_{j \in C} c_{ij} \mu_j x_{ij} + \sum_{i \in F} \rho_i \sqrt{\sum_{j \in C} \mu_j x_{ij}} + \sum_{i \in F} \omega_i \sqrt{\sum_{j \in C} \sigma^2_j x_{ij}}$$ (26)

s.t. $\sum_{i \in F} x_{ij} = 1 \quad \forall j \in C$ (27)

$x_{ij} \leq y_i \quad \forall i \in F, \forall j \in C$ (28)

$x_{ij} \geq 0 \quad \forall i \in F, \forall j \in C$ (29)

$y_i \in \{0, 1\} \quad \forall i \in F$ (30)
Solving the model (inventory management)

Two strategies have been proposed:

- Column generation
- Lagrange relaxation

In both cases $O(|F|)$ pricing problems need to be solved to prove optimality.

If demands are Poisson distributed the running time of each pricing problem is $O(|C| \log |C|)$.

If demands are not Poisson distributed running time of each pricing problem is $O(|C|^2 \log |C|)$.

Quick results

The number of facilities in an optimal solution is significantly smaller than when the inventory costs are not considered.

Literature reports solving instances with $|C| = 600$ and $|F| = 600$. 
Stochastic facility location

Cost and demand may change

A set of scenarios $S$ are given

The clients are known in advance, but the scenario/orders are not

Minimize expected cost

The costs for serving clients are dependent on the scenario
The model (stochastic)

$y_i$ Indicator variable that tells if facility $i \in F$ is used or not

$x_{ijs}$ Indicator variable that tells if client $j \in C$ is serviced by facility $i \in F$ in scenario $s \in S$

$f_i$ Cost of opening facility $i \in F$

$c_{ijs}$ Cost of serving client $j \in C$ by facility $i \in F$ in scenario $s \in S$

$d_{js}$ The amount of demand from client $j \in C$ in scenario $s \in S$

$q_s$ The probability for scenario $s \in S$ happening

\[
\min \sum_{i \in F} f_i y_i + \sum_{s \in S} \sum_{i \in F} \sum_{j \in C} c_{ijs} q_s d_{js} x_{ijs} \tag{31}
\]

s.t. \[
\sum_{i \in F} x_{ijs} = 1 \quad \forall j \in C, \forall s \in S \tag{32}
\]

\[
x_{ijs} \leq y_i \quad \forall i \in F, \forall j \in C, \forall s \in S \tag{33}
\]

\[
x_{ijs} \geq 0 \quad \forall i \in F, \forall j \in C, \forall s \in S \tag{34}
\]

\[
y_i \in \{0, 1\} \quad \forall i \in F \tag{35}
\]
Facility failures

A single facility can fail at a given time (fire, strikes, nuclear wars, etc.)

Once a facility is failing it’s clients must be served by another facility

A failure must not cost more than a given amount $u$

Each client is assigned a primary and a backup facility (the assignment is primary and backup, not the facility)

Minimize primary cost
The model (facility failures)

$y_i$ Indicator variable that tells if facility $i \in F$ is used or not

$x_{ijk}$ Indicator variable that tells if client $j \in C$ is serviced as primary by facility $i \in F$ and as backup by facility $k \in F$

$f_i$ Cost of opening facility $i \in F$

$c_{ij}$ Cost of serving client $j \in C$ by facility $i \in F$

$d_j$ The amount of demand from client $j \in C$

$u$ The upper bound on the cost of a failure
The model (facility failures)

\[
\min \sum_{i \in F} f_i y_i + \sum_{i \in F} \sum_{j \in C} \sum_{k \in F} c_{i,j} d_{j} x_{i,j,k} 
\]  \hspace{1cm} (36)

s.t. \[ \sum_{i \in F} \sum_{k \in F} x_{i,j,k} = 1 \hspace{1cm} \forall j \in C \]  \hspace{1cm} (37)
\[ \sum_{k \in F} x_{i,j,k} \leq y_i \hspace{1cm} \forall i \in F, \forall j \in C \]  \hspace{1cm} (38)
\[ x_{i,j,k} \leq y_k \hspace{1cm} \forall i, k \in F, \forall j \in C \]  \hspace{1cm} (39)
\[ \sum_{k \neq i} \sum_{j \in C} \sum_{l \in F} d_{j} c_{k,j} x_{k,l} + \sum_{j \in C} \sum_{k \in F} d_{j} c_{k,j} x_{i,j,k} \leq u \hspace{1cm} \forall i \in F \]  \hspace{1cm} (40)
\[ x_{i,i} = 0 \hspace{1cm} \forall i \in F, \forall j \in C \]  \hspace{1cm} (41)
\[ x_{i,j,k} \in \{0, 1\} \hspace{1cm} \forall i, k \in F, \forall j \in C \]  \hspace{1cm} (42)
\[ y_i \in \{0, 1\} \hspace{1cm} \forall i \in F \]  \hspace{1cm} (43)

(36) cost of primary facility solution
(37) single source constraint (map client to facilities)
(38) secures that facilities are open if used as primary facility
(39) secures that facilities are open if used as backup facility
(40) reliability constraint, secures that failures does not cost more than \(u\)
(41) secures a clients primary facility is not also used as backup
Exercises

1) Prove that Uncapacitated Facility Location is \( \mathcal{NP} \)-hard or give a polynomial algorithm (or both ;-) ). (Hint: Set-Partition)

2) In the model for ‘Plants and multiple commodities’ replace (18) with:

\[
0 \leq x \leq 1, \quad \forall i \in F, \forall j \in C
\]

and remove \( v \)’s, \( d \)’s, \( x \)’s, and redundant constraints and change the costs \( c \) accordingly to obtain a smaller model.

3) What is the problem with constraint (23)? Replace (23) with constraints that strengthens the LP relaxation. (Hint: see article)

4) In the model for facility failures: What can be done to (39) to strengthen the model?