Semidefinite programming
— an introduction

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Exercise 1

Consider the positive semidefinite matrix:

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \succeq 0$$

- show that $a_{ii} \geq 0$
- show that if $a_{ii} = 0$ then $a_{ij} = 0$ for all $j = 1, \dots, n$

Exercise 2

A *cone* $\mathcal{K}$ is an object satisfying:

$$\forall x \in \mathcal{K}, \lambda \geq 0 : \lambda x \in \mathcal{K}$$

- Prove that the set of positive semidefinite matrices is a cone.

For a given inner product $\langle \cdot, \cdot \rangle$ the *dual cone* of $\mathcal{K}$ is given by

$$\mathcal{K}^* = \{s \in \mathbb{R}^n : \langle s, x \rangle \geq 0, \forall x \in \mathcal{K}\}$$

A cone $\mathcal{K}$ is *self-dual* if an inner product can be chosen such that $\mathcal{K} = \mathcal{K}^*$

- Show that the set of positive semidefinite matrices is a self-dual cone

Exercise 3

The dense subgraph problem is defined as follows: Given a weighted graph $G = (V, E, c)$ and a positive integer k. Choose a subset $U \subseteq V$ of the nodes, with $|U| = k$ such that the sum of the edge weights in the subgraph spanned by U is maximized. The problem can formally be defined as follows:

$$z = \max \left\{ \sum_{i \in U} \sum_{j \in U} c_{ij} : U \subseteq V, |U| = k \right\} \tag{1}$$

Formulate the dense subgraph problem as a semidefinite optimization problem. Consider various improved formulations.