# Semidefinite Programming
# — an introduction

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**Integer programming**

$$\begin{array}{ll} \text{minimize} & cx \\ \text{subject to} & Ax = b \\ & x \geq 0 \\ & x \in \mathbb{Z} \end{array}$$

Solved through branch-and-bound, using lower bounds

- LP-relaxation can be solved efficiently
- Natural way of formulating ILP models

Many problems cannot be formulated as linear models, or the formulation is inefficient.

## Interior point methods

Interior point methods can be extended to a number of cones (*self-dual homogeneous cones*)

- $\mathbb{R}^n$ (linear programming)
- vectorized symmetric matrices over real numbers (semidefinite programming)
- vectorized Hermitian matrices over complex numbers
- vectorized Hermitian matrices over quaternions
- vectorized Hermitian $3 \times 3$ matrices over octonions

Grötschel, Lovász and Schrijver [3]:
semidefinite programming solved in polynomial time

- *semidefinite relaxations* attractive tool
- Modeling part not well-developed

Concrete semidefinite optimizers: Sturm [6], Toh, Tutuncu and Todd [7], Fujisawa, Kojima and Nakata [2].

## Semidefinite programming, definitions

- *vector* $a = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$ $\qquad a^T = (a_1, \ldots, a_n)$
- *inner product between matrices* $A, B \in \mathbb{R}^{m,n}$

$$\langle A, B \rangle := \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} b_{ij}.$$

- *vector product* of two vectors $a, b \in \mathbb{R}^n$ is

$$a^T b := \sum_{j=1}^{n} a_j b_j$$

$$ab^T := \begin{pmatrix} a_1 b_1 & \cdots & a_1 b_n \\ \vdots & & \vdots \\ a_n b_1 & \cdots & a_n b_n \end{pmatrix}$$

- $\text{diag}(A)$ *vector of diagonal elements* of $A \in \mathbb{R}^{n \times n}$

$$\text{diag}(A) := \text{diag} \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} = \begin{pmatrix} a_{11} \\ \vdots \\ a_{nn} \end{pmatrix}$$

- $\text{Diag}(a)$ *diagonal matrix* of vector $a \in \mathbb{R}^n$

$$\text{Diag}(a) = \text{Diag}(a_1, \ldots, a_n) = \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & a_n \end{pmatrix}$$

## Semidefinite programming

- Matrix $A \in \mathbb{R}^{n\times n}$ is *positive semidefinite*

$$\forall y \in \mathbb{R}^n : y^T A y \geq 0 \tag{1}$$

- $\boxed{A \succeq 0}$ $\Leftrightarrow$ $A$ pos. semidef. and symmetric

## Example

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \succeq 0$$

symmetric, $y^T A y \geq 0$ since

$$\begin{aligned} y^T A y &= \begin{pmatrix} y_1 & y_2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \\ &= \begin{pmatrix} y_1 & y_2 \end{pmatrix} \begin{pmatrix} y_1 + y_2 \\ y_1 + y_2 \end{pmatrix} \\ &= (y_1^2 + y_1 y_2) + (y_1 y_2 + y_2^2) \\ &= (y_1 + y_2)^2 \geq 0 \end{aligned}$$

**Example**

$$A = aa^T = \begin{pmatrix} a_1 a_1 & \cdots & a_1 a_n \\ \vdots & & \vdots \\ a_n a_1 & \cdots & a_n a_n \end{pmatrix}$$

is positive semidefinite (and symmetric) since

$$y^T A y = y^T (aa^T) y = (y^T a)(a^T y) = (a^T y)^2 \geq 0$$

**Example**

$$A = \mathrm{Diag}(a_1, \ldots, a_n) = \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & a_n \end{pmatrix}$$

is positive semidefinite if and only if $a_i \geq 0$.

$$y^T A y = \sum_{i=1}^{n} y_i^2 a_i$$

to prove "only if" choose $y = (0, \ldots, 1, \ldots, 0)$

**Cone**

The set of semidefinite matrices is a cone (see exercise)

## Properties of semidefinite matrices

Observations from linear programming:

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \succeq 0$$

All submatrices are again positive semidefinite

Diagonal elements $a_{ii} \geq 0$ (see exercise)

$$A_1 \succeq 0, A_2 \succeq 0, \ldots, A_k \succeq 0 \quad \Leftrightarrow \quad \begin{pmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & A_k \end{pmatrix} \succeq 0$$

## Characterisation of positive semiefinite matrices

**Proposition 1** The following are equivalent

1 $A \in \mathbb{R}^{n\times n}$ is positive semidefinite

2 Eigenvalues $\lambda_i \geq 0$ for $i = 1,\ldots,n$

3 $\exists C \in \mathbb{R}^{n\times n}$ such that $A = C^TC$ and $\text{rank}(C) = \text{rank}(A)$

$A \in \mathbb{R}^{n\times n}$: eigenvector $v \neq 0$, eigenvalue $\lambda$ $\boxed{Av = \lambda v}$
$A = P\Lambda P^T$ eigenvalue decomposition of $A$

**Proof**

- **1 $\Rightarrow$ 2** Let $v \in \mathbb{R}^n$ be an eigenvector with $|v| = 1$ corresponding to $\lambda$. Since $A$ is pos. semidef. $v^TAv \geq 0$ and $v^T(Av) = v^T\lambda v = \lambda v^Tv = \lambda|v|^2 = \lambda$
- **2 $\Rightarrow$ 3** Let $A = P\Lambda P^T$ eigenvalue decomp. of $A$

$$\Lambda^{\frac{1}{2}} := \begin{pmatrix} \sqrt{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & \sqrt{\lambda_n} \end{pmatrix}$$

$$C := \Lambda^{\frac{1}{2}}P^T$$

$$C^TC = (P\Lambda^{\frac{1}{2}})(\Lambda^{\frac{1}{2}}P^T) = P\Lambda P^T = A$$

- **3 $\Rightarrow$ 1** Show: $\forall y \in \mathbb{R}^n : y^TAy \geq 0$
  Given $y \in \mathbb{R}^n$ let $w = Cy$

$$y^TAy = y^T(C^TC)y = (y^TC^T)(Cy) = w^Tw \geq 0$$

## Semidefinite programming

Linear optimization problem in standard form

$$\begin{array}{rl} \text{maximize} & cx \\ \text{subject to} & a_1 x = b_1, \\ & \vdots \\ & a_m x = b_m, \\ & x = (x_1, \ldots, x_n) \geq 0 \end{array}$$

*Semidefinite optimization problem* in standard form

$$\begin{array}{rl} \text{maximize} & \langle C, X \rangle \\ \text{subject to} & \langle A_1, X \rangle = b_1, \\ & \vdots \\ & \langle A_m, X \rangle = b_m, \\ & X \succeq 0 \end{array}$$

where $X = (X_{ij}) \in \mathbb{R}^{n\times n}$, $C \in \mathbb{R}^{n\times n}$, $A_i \in \mathbb{R}^{n\times n}$, and $b_i \in \mathbb{R}$.
Grötschel, Lovász and Schrijver [3] polynomial algorithm

## Several semidefinite matrices

Problem defined in several matrices

$$
\begin{array}{ll}
\text{maximize} & \sum_{j=1}^{k} \langle C_j, X_j \rangle \\
\text{subject to} & \sum_{j=1}^{k} \langle A_{1j}, X_j \rangle = b_1, \\
& \vdots \\
& \sum_{j=1}^{k} \langle A_{mj}, X_j \rangle = b_m, \\
& X_1 \succeq 0, \ldots, X_k \succeq 0
\end{array}
$$

Remind that

$$
X_1 \succeq 0, X_2 \succeq 0, \ldots, X_k \succeq 0 \quad \Leftrightarrow \quad \begin{pmatrix} X_1 & 0 & \cdots & 0 \\ 0 & X_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & X_k \end{pmatrix} \succeq 0
$$

Defining

$$
X = \begin{pmatrix} X_1 & 0 & \cdots & 0 \\ 0 & X_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & X_k \end{pmatrix} \qquad A_i = \begin{pmatrix} A_{i1} & 0 & \cdots & 0 \\ 0 & A_{i2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & A_{ik} \end{pmatrix}
$$

and noting that

$$
\langle A_i, X \rangle = \left\langle \begin{pmatrix} A_{i1} & 0 & \cdots & 0 \\ 0 & A_{i2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & A_{ik} \end{pmatrix}, \begin{pmatrix} X_1 & 0 & \cdots & 0 \\ 0 & X_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & X_k \end{pmatrix} \right\rangle = \sum_{j=1}^{k} \langle A_{ij} X_j \rangle
$$

Semidefinite optimization problem in standard form

## Embedding LP in semidefinite programming

Linear optimization problem

$$\begin{array}{ll} \text{minimize} & \sum_{j=1}^{k} c_j x_j \\ \text{subject to} & \sum_{j=1}^{k} a_{1j} x_j = b_1 \\ & \vdots \\ & \sum_{j=1}^{k} a_{mj} x_j = b_m \\ & x_1, \ldots, x_m \geq 0 \end{array}$$

Define matrices

- $A_{ij} = (a_{ij})$
- $X_i = (x_i)$

Nonnegativity constraint

$$X_i \succeq 0 \Leftrightarrow y X_i y \geq 0 \Leftrightarrow x_i y^2 \geq 0 \Leftrightarrow x_i \geq 0$$

We may use all LP-constraints as usual

## Rank of matrices

rank$(A)$ $\Leftrightarrow$ number of linearly independent columns of $A$

**Proposition 2** matrix $X \in \mathbb{R}^{n\times n}$:

1. $X \succeq 0$ and $\text{rank}(X) = 1$

$\Leftrightarrow$

2. $X = xx^T$ for some vector $x \in \mathbb{R}^n$

$$X = \begin{pmatrix} X_{11} & \cdots & X_{1n} \\ \vdots & & \vdots \\ X_{n1} & \cdots & X_{nn} \end{pmatrix} \qquad xx^T = \begin{pmatrix} x_1x_1 & \cdots & x_1x_n \\ \vdots & & \vdots \\ x_nx_1 & \cdots & x_nx_n \end{pmatrix}$$

**Proof**

- **1 $\Rightarrow$ 2** From Prop. 1: $X = C^TC$ where $C = \Lambda^{\frac{1}{2}}P^T$
  Since $\text{rank}(X) = 1$ only one eigenvalue $\lambda_i \neq 0$

$$C = \Lambda^{\frac{1}{2}}P^T = \begin{pmatrix} 0 & & \\ & \sqrt{\lambda_i} & \\ & & 0 \end{pmatrix} \begin{pmatrix} p_{11} & \cdots & p_{n1} \\ \vdots & & \vdots \\ p_{1n} & \cdots & p_{nn} \end{pmatrix} = \begin{pmatrix} 0 & \cdots & 0 \\ c_1 & \cdots & c_n \\ 0 & \cdots & 0 \end{pmatrix}$$

  set $x = (c_1, \ldots, c_n)$. Then $X = C^TC = xx^T$.

- **2 $\Rightarrow$ 1** Construct matrix

$$C := \begin{pmatrix} x_1 & \cdots & x_n \\ 0 & \cdots & 0 \\ & & \\ 0 & \cdots & 0 \end{pmatrix} \qquad C^TC = \begin{pmatrix} x_1 & 0 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ x_n & 0 & \ldots & 0 \end{pmatrix} \begin{pmatrix} x_1 & \cdots & x_n \\ 0 & \cdots & 0 \\ & & \\ 0 & \cdots & 0 \end{pmatrix} = xx^T$$

  Since $\text{rank}(C) = 1$ from Prop. 1: $X \succeq 0 \wedge \text{rank}(X) = 1$

**Quadratic functions**

If objective function or constraint

$$\sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}x_ix_j = x^TCx$$

$\exists$ matrix $X = xx^T$ where $X \succeq 0$ and $\mathrm{rank}(X) = 1$ such that

$$\langle C, X\rangle = \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}X_{ij}$$

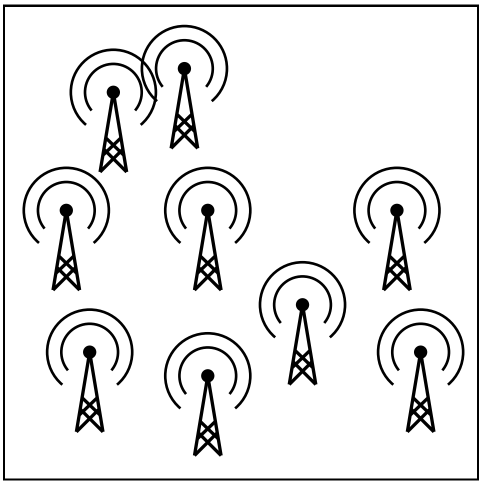
since $X_{ij} = x_ix_j$ we get

$$\sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}x_ix_j = \langle C, X\rangle$$

$$X \succeq 1$$

$$\mathrm{rank}(X) = 1$$

## The Quadratic Knapsack Problem



- $N = \{1, \ldots, n\}$ items
- item $j \in N$ has weight $w_j$
- knapsack capacity $c$
- profit matrix $P = (p_{ij})$
- $p_{ij} + p_{ji}$ profit achieved if items $i$ and $j$ are selected

Binary variable $x_j = 1 \Leftrightarrow$ item $j$ is selected

$$
\begin{aligned}
\text{maximize} \quad & \sum_{i \in N} \sum_{j \in N} p_{ij} x_i x_j \\
\text{subject to} \quad & \sum_{j \in N} w_j x_j \leq c, \\
& x_j \in \{0, 1\}, \quad j \in N.
\end{aligned}
\tag{2}
$$

## Upper Bounds

LP-relaxation

$$\begin{aligned} \text{maximize} \quad & \sum_{i \in N} \sum_{j \in N} p_{ij} x_i x_j \\ \text{subject to} \quad & \sum_{j \in N} w_j x_j \le c, \\ & 0 \le x_j \le 1, \quad j \in N. \end{aligned}$$

objective function is not linear

## Bounds from Linearisation

Variables $y_{ij} \Leftrightarrow x_i = 1$ and $x_j = 1$

$$y_{ij} \le x_i, \qquad y_{ij} \le x_j, \qquad x_i + x_j \le 1 + y_{ij},$$

ILP model

$$\begin{array}{lll} \text{maximize} & \sum_{i\in N}\sum_{j\in N} p_{ij} y_{ij} & \\ \text{subject to} & \sum_{j\in N} w_j x_j \le c, & \\ & y_{ij} \le x_i, & i,j \in N, \\ & y_{ij} \le x_j, & i,j \in N, \\ & x_i + x_j \le 1 + y_{ij}, & i,j \in N, \\ & x_j, y_{ij} \in \{0,1\}, & i,j \in N. \end{array}$$

LP-relaxation

$$\begin{array}{lll} \text{maximize} & \sum_{i\in N}\sum_{j\in N} p_{ij} y_{ij} & \\ \text{subject to} & \sum_{j\in N} w_j x_j \le c, & \\ & y_{ij} \le x_i, & j \in N, \\ & y_{ij} \le x_j, & i \in N, \\ & x_i + x_j \le 1 + y_{ij}, & i,j \in N, \\ & 0 \le x_j \le 1, & j \in N, \\ & y_{ij} \ge 0, & i,j \in N. \end{array}$$

Upper bound $U^1_{\text{BC}}$

**Bounds from Semidefinite Relaxation**

Objective function of QKP

$$x^T P x = \sum_{i \in N} \sum_{j \in N} p_{ij} x_i x_j = \langle P, X \rangle$$

Formulation of QKP

$$\begin{array}{rl} \text{maximize} & \langle P, X \rangle \\ \text{subject to} & \langle \mathrm{Diag}(w), X \rangle \leq c, \\ & X \succeq 0, \\ & \mathrm{rank}(X) = 1, \\ & X_{ii} \in \{0, 1\}. \end{array}$$

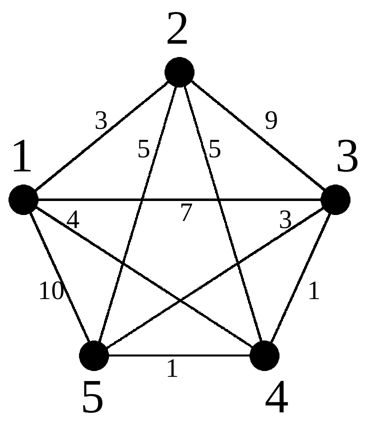
Dropping $\mathrm{rank}(X) = 1$ constraint, relax last constraint

$$\begin{array}{rl} \text{maximize} & \langle P, X \rangle \\ \text{subject to} & \langle \mathrm{Diag}(w), X \rangle \leq c, \\ & 0 \leq X_{ii} \leq 1 \\ & X \succeq 0, \end{array}$$

Upper bound $U^0_{\text{HRW}}$

## Max-cut problem

Graph $G=(V,E,a)$, split $V=\{U,V\setminus U\}$




$$x_i = \begin{cases} 1 & \text{if } i \in U \\ -1 & \text{if } i \in V\setminus U \end{cases} \qquad \frac{1-x_i x_j}{2} = \begin{cases} 1 & \text{if } x_i \neq x_j \\ 0 & \text{if } x_i = x_j \end{cases}$$

Model

$$\begin{array}{ll} \text{maximize} & \displaystyle\sum_{i<j} a_{ij}\frac{1-x_i x_j}{2} \\ \text{subject to} & x \in \{-1,1\}^n \end{array}$$

Rewriting objective

$$\begin{aligned} \frac{1}{2}\sum_{i<j} a_{ij}(1-x_i x_j) &= \frac{1}{4}\sum_{i,j} a_{ij}(1-x_i x_j) \\ &= \frac{1}{4}\sum_{i=1}^{n}\left(\sum_{j=1}^{n} a_{ij}x_i x_i - \sum_{j=1}^{n} a_{ij}x_i x_j\right) \\ &= \frac{1}{4}x^T(\mathrm{Diag}(Ae)-A)x \end{aligned}$$

## Max-cut problem

Setting $C = \frac{1}{4}(\mathrm{Diag}(Ae) - A)$ we get objective

$$\frac{1}{2}\sum_{i<j} a_{ij}(1 - x_i x_j) = x^T C x = \langle C, xx^T \rangle$$

Semidefinite program

$$\begin{array}{rl} \text{maximize} & \langle C, X \rangle \\ \text{subject to} & \mathrm{diag}(X) = e \\ & X \succeq 0, \\ & \mathrm{rank}(X) = 1, \end{array}$$

Since $\mathrm{diag}(X) = e$ we have

$$X_{ii} = 1 = x_i x_i$$

Drop $\mathrm{rank}(X) = 1$, semidefinite relaxation

$$\begin{array}{rl} \text{maximize} & \langle C, X \rangle \\ \text{subject to} & \mathrm{diag}(X) = e \\ & X \succeq 0 \end{array}$$

## Quadratic 0-1 programming

Problem

$$\max_{x\in\{0,1\}^n} x^T C x$$

Semidefinite program

$$\begin{array}{rl} \text{maximize} & \langle C, X\rangle \\ \text{subject to} & X \succeq 0, \\ & \operatorname{rank}(X) = 1, \\ & X_{ii} \in \{0,1\} \end{array}$$

Drop $\operatorname{rank}(X) = 1$, LP-relax $X_{ii} \in \{0,1\}$

Semidefinite relaxation

$$\begin{array}{rl} \text{maximize} & \langle P, X\rangle \\ \text{subject to} & 0 \leq X_{ii} \leq 1 \\ & X \succeq 0 \end{array}$$

## Tighter bounds for QKP

**Proposition 3** If $X \succeq 0$ and $\mathrm{rank}(X) = 1$ and $X_{ii} \in \{0,1\}$ then also

$$X - \mathrm{diag}(X)\,\mathrm{diag}(X)^T \succeq 0. \tag{3}$$

## Proof

- $X = xx^T$ due to Proposition 2

$$X = \begin{pmatrix} X_{11} & \cdots & X_{1n} \\ \vdots & & \vdots \\ X_{n1} & \cdots & X_{nn} \end{pmatrix} \qquad xx^T = \begin{pmatrix} x_1x_1 & \cdots & x_1x_n \\ \vdots & & \vdots \\ x_nx_1 & \cdots & x_nx_n \end{pmatrix}$$

- $\mathrm{diag}(X) = \mathrm{diag}(xx^T) = x$ when $x \in \{0,1\}^n$
- $\forall v \in \mathbb{R}^n$: $(x+v)(x+v)^T \succeq 0$
  choose $v = -\mathrm{diag}(X)$

$$\begin{aligned}
&(x+v)(x+v)^T = \\
&xx^T + vx^T + xv^T + vv^T = \\
&X + v\,\mathrm{diag}(X)^T + \mathrm{diag}(X)v^T + vv^T = \\
&X + (v+\mathrm{diag}(X))(v+\mathrm{diag}(X))^T - \mathrm{diag}(X)\,\mathrm{diag}(X)^T = \\
&X - \mathrm{diag}(X)\,\mathrm{diag}(X)^T \succeq 0
\end{aligned}$$

## Tighter bounds for QKP

### Proposition 4

$$X - \operatorname{diag}(X)\operatorname{diag}(X)^T \succeq 0 \qquad \Leftrightarrow \qquad \overline{X} \succeq 0$$

where

$$\overline{X} := \begin{pmatrix} 1 & \operatorname{diag}(X)^T \\ \operatorname{diag}(X) & X \end{pmatrix}.$$

### Proof

Define the regular matrix $B$ as

$$B = \begin{pmatrix} 1 & -\operatorname{diag}(X)^T \\ 0 & I \end{pmatrix},$$

and observe that

$$\begin{aligned} B^T\overline{X}B &= \begin{pmatrix} 1 & 0 \\ -\operatorname{diag}(X) & I \end{pmatrix}\begin{pmatrix} 1 & \operatorname{diag}(X)^T \\ \operatorname{diag}(X) & X \end{pmatrix}\begin{pmatrix} 1 & -\operatorname{diag}(X)^T \\ 0 & I \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & X - \operatorname{diag}(X)\operatorname{diag}(X)^T \end{pmatrix}. \end{aligned}$$

Using $\bar{y} = B^{-1}y$ we have

$$y^T\overline{X}y = y^T(B^{-1})^T B^T\overline{X}BB^{-1}y = \bar{y}^T B^T\overline{X}B\bar{y},$$

hence: $\overline{X} \succeq 0 \Leftrightarrow B^T\overline{X}B \succeq 0$

$$B^T\overline{X}B \succeq 0 \;\Leftrightarrow\; 1 \succeq 0 \text{ and } X - \operatorname{diag}(X)\operatorname{diag}(X)^T \succeq 0$$

## Tighter bounds for QKP (bound 1)

Relax QKP to QKP’

$$\begin{array}{ll} \text{maximize} & \langle P, X\rangle \\ \text{subject to} & \langle \mathrm{Diag}(w), X\rangle \le c, \\ & X - \mathrm{diag}(X)\,\mathrm{diag}(X)^T \succeq 0, \\ & \mathrm{rank}(X) = 1, \\ & X_{ii} \in \{0,1\} \end{array}$$

Drop $\mathrm{rank}(X) = 1$ and relax last constraint to $0 \le X_{ii} \le 1$

$$\begin{array}{ll} \text{maximize} & \langle P, X\rangle \\ \text{subject to} & \langle \mathrm{Diag}(w), X\rangle \le c, \\ & X - \mathrm{diag}(X)\,\mathrm{diag}(X)^T \succeq 0, \\ & X_{ii} \le 1 \end{array}$$

upper bound $U^1_{\text{HRW}}$

## Tighter bounds for QKP (bound 2)

- $w^T x = x^T w$
- $w^T x \leq c \Rightarrow w^T x x^T w \leq c^2$
- $w^T x x^T w = \langle w w^T, x x^T \rangle$
- relax $x x^T$ to $X$

relaxation

$$\begin{array}{ll} \text{maximize} & \langle P, X \rangle \\ \text{subject to} & \langle w w^T, X \rangle \leq c^2, \\ & X - \operatorname{diag}(X) \operatorname{diag}(X)^T \succeq 0 \end{array}$$

upper bound $U^2_{\text{HRW}}$

## Tighter bounds for QKP (bound 3)

The third semidefinite relaxation is based on the observation that $w^T x \leq c$ can be multiplied by the real number $w^T x$ on both sides gives the constraint $(w^T x)^2 \leq w^T x c$, leading to the inequality

$$\begin{aligned} 0 \leq w^T x(c - x^T w) &= w^T x \begin{pmatrix} 1, x^T \end{pmatrix} \begin{pmatrix} c \\ -w \end{pmatrix} \\ &= \begin{pmatrix} 0, w^T \end{pmatrix} \begin{pmatrix} 1 \\ x \end{pmatrix} \begin{pmatrix} 1, x^T \end{pmatrix} \begin{pmatrix} c \\ -w \end{pmatrix}. \end{aligned} \tag{4}$$

Setting $X' = \begin{pmatrix} 1 \\ x \end{pmatrix} \begin{pmatrix} 1, x^T \end{pmatrix} = x' x'^T$ the right hand side expression in (4) can be written

$$\left\langle \begin{pmatrix} c \\ -w \end{pmatrix} \begin{pmatrix} 0, w^T \end{pmatrix}, X' \right\rangle. \tag{5}$$

This leads to the following relaxation:

$$\begin{aligned} \text{maximize} \quad & \langle P, X \rangle \\ \text{subject to} \quad & \left\langle \begin{pmatrix} c \\ -w \end{pmatrix} \begin{pmatrix} 0, w^T \end{pmatrix}, X' \right\rangle \geq 0, \\ & X - \operatorname{diag}(X) \operatorname{diag}(X)^T \succeq 0. \end{aligned} \tag{6}$$

Solving the above problem gives bound $U^3_{\text{HRW}}$.

## Tighter bounds for QKP (bound 4)

The last relaxation is obtained by multiplying the capacity constraint with each of the variables $x_i$ for $i \in N$ getting $x_i w^T x \leq x_i c$. By writing the vector product explicitly we get the sum

$$\sum_{j \in N} w_j x_i x_j \leq x_i c. \tag{7}$$

By introducing in (7) $X_{ij}$ for $x_i x_j$ and $X_{ii}$ for $x_i$ we get

$$\sum_{j \in N} w_j X_{ij} \leq X_{ii} c, \tag{8}$$

which leads to the following relaxation:

$$\begin{array}{ll} \text{maximize} & \langle P, X \rangle \\ \text{subject to} & \sum_{j \in N} w_j X_{ij} - X_{ii} c \leq 0, \quad \text{for } i \in N, \\ & X - \operatorname{diag}(X) \operatorname{diag}(X)^T \succeq 0, \end{array} \tag{9}$$

giving the bound $U^4_{\text{HRW}}$.

## Strength of bounds for QKP

Helmberg, Rendl and Weismantel [4] together with Bauvin and Goemans [1] prove that

$$U^1_{\text{HRW}} \geq U^2_{\text{HRW}} \geq U^3_{\text{HRW}} \geq U^4_{\text{HRW}}$$

## Computational Experiments

Randomly generated instances by Gallo, Hammer and Simeone
Let $\Delta$ be the *density* of an instance

- weight $w_j$ is randomly distributed in $[1,50]$ while
- profits $p_{ij} = p_{ji}$ are nonzero with probability $\Delta$, and in this case randomly distributed in $[1,100]$.
- capacity $c$ is randomly distributed in $[50, \sum_{j=1}^{n} w_j]$

Implementation and experimental study was carried out by Rasmussen and Sandvik [5]

| | | $U^1_{\mathrm{GHS}}$ | | $U^2_{\mathrm{GHS}}$ | | $U^3_{\mathrm{GHS}}$ | | $U^4_{\mathrm{GHS}}$ | | $\overline{U}^4_{\mathrm{GHS}}$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\Delta$ | $n$ | `time` | `dev` | `time` | `dev` | `time` | `dev` | `time` | `dev` | `time` | `dev` |
| 5 | 40 | 0.000 | 6.38 | 0.002 | 6.38 | 0.001 | 6.38 | 0.000 | 6.38 | 0.000 | 6.38 |
| | 60 | 0.000 | 17.61 | 0.000 | 17.61 | 0.001 | 17.61 | 0.002 | 17.61 | 0.001 | 17.49 |
| | 80 | 0.000 | 22.70 | 0.000 | 22.70 | 0.000 | 22.70 | 0.001 | 22.70 | 0.001 | 22.70 |
| | 100 | 0.000 | 13.84 | 0.002 | 13.84 | 0.003 | 13.84 | 0.001 | 13.84 | 0.000 | 13.84 |
| | 120 | 0.001 | 25.04 | 0.001 | 25.04 | 0.002 | 25.03 | 0.001 | 25.02 | 0.001 | 25.02 |
| | 140 | 0.000 | 40.17 | 0.000 | 40.17 | 0.006 | 39.22 | 0.002 | 39.09 | 0.003 | 38.99 |
| | 160 | 0.001 | 21.53 | 0.001 | 21.53 | 0.003 | 21.32 | 0.005 | 21.27 | 0.005 | 21.24 |
| | 180 | 0.002 | 21.69 | 0.002 | 21.69 | 0.008 | 21.69 | 0.002 | 21.69 | 0.002 | 21.69 |
| | 200 | 0.002 | 18.06 | 0.004 | 18.06 | 0.008 | 18.06 | 0.002 | 18.06 | 0.004 | 18.06 |
| | avrg | 0.001 | 20.78 | 0.001 | 20.78 | 0.004 | 20.65 | 0.002 | 20.63 | 0.002 | 20.60 |
| 25 | 40 | 0.000 | 35.22 | 0.001 | 34.00 | 0.002 | 27.82 | 0.001 | 27.33 | 0.000 | 26.45 |
| | 60 | 0.000 | 21.47 | 0.000 | 20.99 | 0.000 | 18.08 | 0.001 | 17.88 | 0.001 | 17.76 |
| | 80 | 0.000 | 16.83 | 0.001 | 16.78 | 0.001 | 14.89 | 0.003 | 14.81 | 0.001 | 14.74 |
| | 100 | 0.000 | 61.73 | 0.003 | 61.36 | 0.002 | 57.57 | 0.000 | 57.41 | 0.001 | 57.17 |
| | 120 | 0.000 | 31.08 | 0.001 | 30.64 | 0.002 | 27.30 | 0.001 | 27.24 | 0.004 | 27.18 |
| | 140 | 0.002 | 36.89 | 0.002 | 36.89 | 0.004 | 35.66 | 0.001 | 35.62 | 0.005 | 35.57 |
| | 160 | 0.000 | 18.54 | 0.001 | 18.54 | 0.006 | 17.96 | 0.002 | 17.94 | 0.006 | 17.91 |
| | 180 | 0.000 | 31.80 | 0.002 | 31.44 | 0.008 | 29.96 | 0.003 | 29.95 | 0.005 | 29.92 |
| | 200 | 0.003 | 49.02 | 0.003 | 49.02 | 0.009 | 42.59 | 0.010 | 42.50 | 0.006 | 42.31 |
| | avrg | 0.001 | 33.62 | 0.002 | 33.29 | 0.004 | 30.20 | 0.002 | 30.08 | 0.003 | 29.89 |
| 50 | 40 | 0.000 | 35.76 | 0.000 | 34.91 | 0.001 | 26.01 | 0.001 | 25.48 | 0.000 | 24.95 |
| | 60 | 0.000 | 36.11 | 0.000 | 34.17 | 0.000 | 27.61 | 0.001 | 27.35 | 0.002 | 26.98 |
| | 80 | 0.000 | 18.48 | 0.000 | 18.39 | 0.003 | 14.86 | 0.002 | 14.76 | 0.001 | 14.64 |
| | 100 | 0.001 | 52.13 | 0.001 | 46.00 | 0.003 | 34.87 | 0.001 | 34.70 | 0.003 | 34.42 |
| | 120 | 0.001 | 31.47 | 0.005 | 29.80 | 0.003 | 22.59 | 0.002 | 22.51 | 0.003 | 22.36 |
| | 140 | 0.000 | 43.17 | 0.002 | 43.14 | 0.003 | 38.05 | 0.006 | 37.97 | 0.003 | 37.84 |
| | 160 | 0.000 | 23.59 | 0.002 | 22.41 | 0.003 | 16.80 | 0.004 | 16.76 | 0.006 | 16.67 |
| | 180 | 0.003 | 30.88 | 0.007 | 28.91 | 0.005 | 21.90 | 0.005 | 21.87 | 0.007 | 21.79 |
| | 200 | 0.000 | 29.32 | 0.005 | 28.60 | 0.010 | 23.23 | 0.007 | 23.20 | 0.007 | 23.15 |
| | avrg | 0.001 | 33.43 | 0.002 | 31.81 | 0.003 | 25.10 | 0.003 | 24.96 | 0.004 | 24.75 |
| 75 | 40 | 0.000 | 35.10 | 0.000 | 28.80 | 0.000 | 18.48 | 0.000 | 18.00 | 0.000 | 17.55 |
| | 60 | 0.001 | 37.38 | 0.000 | 29.12 | 0.000 | 17.36 | 0.000 | 17.06 | 0.002 | 16.73 |
| | 80 | 0.000 | 26.77 | 0.001 | 23.04 | 0.002 | 15.37 | 0.003 | 15.27 | 0.000 | 15.12 |
| | 100 | 0.000 | 31.20 | 0.001 | 22.89 | 0.004 | 13.85 | 0.002 | 13.77 | 0.002 | 13.63 |
| | 120 | 0.000 | 33.20 | 0.002 | 21.57 | 0.002 | 11.59 | 0.003 | 11.53 | 0.003 | 11.39 |
| | 140 | 0.000 | 36.53 | 0.002 | 27.27 | 0.005 | 16.80 | 0.005 | 16.75 | 0.004 | 16.66 |
| | 160 | 0.000 | 25.79 | 0.002 | 22.07 | 0.004 | 16.13 | 0.007 | 16.11 | 0.007 | 16.05 |
| | 180 | 0.002 | 52.66 | 0.006 | 34.91 | 0.006 | 20.86 | 0.009 | 20.82 | 0.006 | 20.71 |
| | 200 | 0.001 | 51.06 | 0.004 | 26.00 | 0.007 | 13.30 | 0.009 | 13.26 | 0.009 | 13.19 |
| | avrg | 0.000 | 36.63 | 0.002 | 26.19 | 0.003 | 15.97 | 0.004 | 15.84 | 0.004 | 15.67 |
| 95 | 40 | 0.000 | 63.84 | 0.001 | 35.23 | 0.003 | 17.33 | 0.000 | 16.72 | 0.000 | 16.15 |
| | 60 | 0.000 | 61.20 | 0.000 | 35.03 | 0.002 | 16.16 | 0.001 | 15.86 | 0.003 | 15.51 |
| | 80 | 0.000 | 39.89 | 0.001 | 24.19 | 0.000 | 12.88 | 0.002 | 12.74 | 0.002 | 12.59 |
| | 100 | 0.002 | 49.27 | 0.002 | 27.00 | 0.001 | 13.50 | 0.006 | 13.39 | 0.000 | 13.28 |
| | 120 | 0.002 | 27.16 | 0.002 | 18.52 | 0.002 | 10.64 | 0.004 | 10.59 | 0.004 | 10.52 |
| | 140 | 0.002 | 28.02 | 0.004 | 19.79 | 0.006 | 11.69 | 0.003 | 11.65 | 0.005 | 11.58 |
| | 160 | 0.000 | 34.82 | 0.002 | 23.34 | 0.004 | 12.11 | 0.008 | 12.08 | 0.008 | 12.01 |
| | 180 | 0.002 | 31.71 | 0.004 | 20.00 | 0.006 | 11.27 | 0.008 | 11.24 | 0.003 | 11.18 |
| | 200 | 0.002 | 53.23 | 0.003 | 31.67 | 0.009 | 14.33 | 0.007 | 14.30 | 0.015 | 14.21 |
| | avrg | 0.001 | 43.24 | 0.002 | 26.08 | 0.004 | 13.32 | 0.004 | 13.17 | 0.004 | 13.00 |
| 100 | 40 | 0.001 | 23.67 | 0.001 | 15.24 | 0.001 | 9.64 | 0.001 | 9.32 | 0.000 | 9.15 |
| | 60 | 0.000 | 19.93 | 0.000 | 11.09 | 0.001 | 6.09 | 0.001 | 5.96 | 0.001 | 5.88 |
| | 80 | 0.000 | 40.13 | 0.000 | 21.15 | 0.001 | 10.72 | 0.003 | 10.58 | 0.001 | 10.45 |
| | 100 | 0.001 | 33.11 | 0.001 | 19.42 | 0.003 | 10.46 | 0.003 | 10.38 | 0.002 | 10.29 |
| | 120 | 0.001 | 37.67 | 0.004 | 22.86 | 0.002 | 11.33 | 0.003 | 11.27 | 0.005 | 11.19 |
| | 140 | 0.000 | 26.21 | 0.002 | 17.50 | 0.006 | 10.35 | 0.004 | 10.31 | 0.004 | 10.25 |
| | 160 | 0.000 | 38.14 | 0.001 | 19.63 | 0.007 | 9.68 | 0.006 | 9.64 | 0.005 | 9.59 |
| | 180 | 0.002 | 46.30 | 0.005 | 24.20 | 0.005 | 9.86 | 0.007 | 9.83 | 0.009 | 9.77 |
| | 200 | 0.003 | 40.76 | 0.004 | 22.01 | 0.009 | 10.09 | 0.010 | 10.07 | 0.008 | 10.01 |
| | avrg | 0.001 | 33.99 | 0.002 | 19.23 | 0.004 | 9.80 | 0.004 | 9.71 | 0.004 | 9.62 |
| total avrg | | 0.001 | 33.62 | 0.002 | 26.23 | 0.004 | 19.18 | 0.003 | 19.06 | 0.003 | 18.92 |

Table 1: Bounds from upper planes (Intel Pentium III, 933 MHz).

| Δ | $n$ | $U_{\mathrm{CHM}}$ time | dev | $\hat{U}^2_{\mathrm{MV}}$ time | dev | $\hat{U}^2_{\mathrm{BFS}}$ time | dev |
|---|---|---|---|---|---|---|---|
| 5 | 40 | 0.0 | 0.80 | 0.3 | 0.27 | 3.6 | 0.74 |
| | 60 | 0.0 | 0.65 | 0.6 | 0.37 | 9.6 | 0.49 |
| | 80 | 0.0 | 0.45 | 1.4 | 0.29 | 15.8 | 0.43 |
| | 100 | 0.0 | 0.32 | 2.0 | 0.16 | 24.8 | 0.31 |
| | 120 | 0.0 | 0.28 | 3.2 | 0.17 | 40.4 | 0.30 |
| | 140 | 0.0 | 0.56 | 2.2 | 0.53 | 75.6 | 0.70 |
| | 160 | 0.0 | 0.13 | 12.5 | 0.08 | 76.5 | 0.14 |
| | 180 | 0.0 | 0.14 | 7.3 | 0.11 | 95.0 | 0.17 |
| | 200 | 0.1 | 0.08 | 22.0 | 0.05 | 111.0 | 0.09 |
| | avrg | 0.0 | 0.38 | 5.7 | 0.22 | 50.2 | 0.37 |
| 25 | 40 | 0.0 | 3.07 | 0.6 | 2.38 | 3.7 | 1.65 |
| | 60 | 0.0 | 0.58 | 1.8 | 0.37 | 8.5 | 0.35 |
| | 80 | 0.0 | 1.02 | 6.5 | 0.77 | 16.1 | 0.65 |
| | 100 | 0.0 | 2.48 | 3.0 | 2.44 | 30.0 | 0.68 |
| | 120 | 0.1 | 0.57 | 4.9 | 0.53 | 38.6 | 0.26 |
| | 140 | 0.1 | 1.46 | 2.4 | 1.46 | 59.4 | 0.61 |
| | 160 | 0.2 | 0.50 | 64.7 | 0.47 | 62.2 | 0.16 |
| | 180 | 0.2 | 0.74 | 34.8 | 0.73 | 93.5 | 0.27 |
| | 200 | 0.3 | 1.47 | 20.3 | 1.47 | 122.3 | 0.36 |
| | avrg | 0.1 | 1.32 | 15.5 | 1.18 | 48.3 | 0.55 |
| 50 | 40 | 0.0 | 3.70 | 1.1 | 3.33 | 3.7 | 1.33 |
| | 60 | 0.0 | 2.85 | 3.1 | 2.70 | 8.4 | 0.67 |
| | 80 | 0.0 | 0.82 | 9.4 | 0.69 | 14.7 | 0.38 |
| | 100 | 0.1 | 3.61 | 15.4 | 3.57 | 24.7 | 0.50 |
| | 120 | 0.2 | 1.40 | 48.5 | 1.35 | 36.1 | 0.35 |
| | 140 | 0.3 | 1.33 | 28.4 | 1.32 | 47.3 | 0.19 |
| | 160 | 0.5 | 1.16 | 164.9 | 1.12 | 66.1 | 0.20 |
| | 180 | 0.6 | 1.57 | 27.9 | 1.57 | 84.8 | 0.21 |
| | 200 | 0.8 | 0.83 | 43.8 | 0.83 | 107.0 | 0.16 |
| | avrg | 0.3 | 1.92 | 38.1 | 1.83 | 43.7 | 0.44 |
| 75 | 40 | 0.0 | 3.92 | 3.3 | 3.36 | 3.7 | 1.58 |
| | 60 | 0.0 | 2.15 | 6.4 | 1.88 | 8.1 | 0.81 |
| | 80 | 0.1 | 1.52 | 2.0 | 1.52 | 13.5 | 0.23 |
| | 100 | 0.2 | 1.96 | 15.1 | 1.93 | 23.3 | 0.41 |
| | 120 | 0.3 | 1.89 | 195.5 | 1.80 | 31.2 | 0.49 |
| | 140 | 0.5 | 1.94 | 30.6 | 1.94 | 47.5 | 0.19 |
| | 160 | 0.7 | 1.24 | 506.8 | 1.21 | 63.1 | 0.18 |
| | 180 | 0.8 | 1.80 | 14.4 | 1.80 | 77.8 | 0.12 |
| | 200 | 3.7 | 2.03 | 17.9 | 2.03 | 102.5 | 0.17 |
| | avrg | 0.7 | 2.05 | 88.0 | 1.94 | 41.2 | 0.46 |
| 95 | 40 | 0.0 | 9.30 | 0.4 | 9.28 | 3.8 | 1.60 |
| | 60 | 0.0 | 4.21 | 0.6 | 4.21 | 8.4 | 0.68 |
| | 80 | 0.1 | 2.59 | 11.2 | 2.54 | 14.6 | 0.71 |
| | 100 | 0.2 | 1.63 | 13.1 | 1.62 | 21.0 | 0.46 |
| | 120 | 0.4 | 1.71 | 89.4 | 1.69 | 32.7 | 0.28 |
| | 140 | 0.5 | 1.82 | 113.2 | 1.81 | 41.5 | 0.22 |
| | 160 | 0.8 | 2.18 | 221.6 | 2.18 | 62.9 | 0.33 |
| | 180 | 1.0 | 2.06 | 15.1 | 2.06 | 80.8 | 0.32 |
| | 200 | 1.2 | 3.38 | 563.3 | 3.37 | 103.7 | 0.15 |
| | avrg | 0.5 | 3.21 | 114.2 | 3.20 | 41.0 | 0.53 |
| 100 | 40 | 0.0 | 4.22 | 2.0 | 4.08 | 3.6 | 1.55 |
| | 60 | 0.0 | 2.14 | 11.2 | 2.05 | 8.2 | 0.99 |
| | 80 | 0.1 | 2.22 | 20.4 | 2.20 | 14.0 | 0.53 |
| | 100 | 0.2 | 2.90 | 4.4 | 2.90 | 22.0 | 0.27 |
| | 120 | 0.4 | 2.23 | 20.0 | 2.22 | 33.1 | 0.38 |
| | 140 | 0.6 | 1.93 | 122.7 | 1.92 | 47.9 | 0.26 |
| | 160 | 0.8 | 2.25 | 978.7 | 2.21 | 60.0 | 0.31 |
| | 180 | 1.0 | 2.54 | 21.1 | 2.54 | 78.3 | 0.24 |
| | 200 | 1.4 | 1.66 | 79.8 | 1.66 | 103.1 | 0.50 |
| | avrg | 0.5 | 2.45 | 140.0 | 2.42 | 41.1 | 0.56 |
| total | avrg | 0.3 | 1.89 | 66.9 | 1.80 | 44.3 | 0.49 |

| Δ | $n$ | $\hat{U}^2_{\mathrm{CPT}}$ time | dev | $U^2_{\mathrm{BC}}$ time | dev |
|---|---|---|---|---|---|
| 5 | 40 | 0.0 | 1.23 | 25.2 | 0.70 |
| | 60 | 0.1 | 1.41 | 160.0 | 0.45 |
| | 80 | 0.2 | 1.56 | | |
| | 100 | 0.4 | 1.32 | | |
| | 120 | 0.7 | 1.37 | | |
| | 140 | 1.5 | 2.15 | | |
| | 160 | 2.1 | 1.49 | | |
| | 180 | 3.7 | 1.64 | | |
| | 200 | 4.8 | 1.44 | | |
| | avrg | 1.5 | 1.51 | 92.6 | 0.57 |
| 25 | 40 | 0.0 | 2.91 | 35.5 | 2.21 |
| | 60 | 0.1 | 0.92 | 326.4 | 0.38 |
| | 80 | 0.3 | 1.34 | | |
| | 100 | 0.7 | 2.79 | | |
| | 120 | 1.0 | 0.93 | | |
| | 140 | 2.1 | 1.96 | | |
| | 160 | 2.5 | 1.01 | | |
| | 180 | 4.5 | 1.18 | | |
| | 200 | 7.7 | 1.91 | | |
| | avrg | 2.1 | 1.66 | 181.0 | 1.30 |
| 50 | 40 | 0.1 | 3.29 | 55.9 | 1.70 |
| | 60 | 0.2 | 2.11 | 699.3 | 0.84 |
| | 80 | 0.3 | 0.85 | | |
| | 100 | 0.7 | 2.40 | | |
| | 120 | 1.2 | 1.31 | | |
| | 140 | 2.2 | 1.45 | | |
| | 160 | 2.9 | 0.89 | | |
| | 180 | 5.2 | 1.57 | | |
| | 200 | 6.0 | 0.84 | | |
| | avrg | 2.1 | 1.64 | 377.6 | 1.27 |
| 75 | 40 | 0.1 | 2.45 | 29.4 | 1.59 |
| | 60 | 0.1 | 1.39 | 116.4 | 0.80 |
| | 80 | 0.3 | 1.03 | 978.8 | 0.22 |
| | 100 | 0.5 | 1.21 | 2110.9 | 0.41 |
| | 120 | 0.8 | 0.73 | 2982.4 | 0.47 |
| | 140 | 1.7 | 1.11 | | |
| | 160 | 2.5 | 0.84 | | |
| | 180 | 3.6 | 0.78 | | |
| | 200 | 4.2 | 0.78 | | |
| | avrg | 1.5 | 1.15 | 1243.6 | 0.70 |
| 95 | 40 | 0.1 | 2.69 | 17.7 | 1.59 |
| | 60 | 0.1 | 1.25 | 102.2 | 0.68 |
| | 80 | 0.3 | 1.12 | 445.8 | 0.70 |
| | 100 | 0.4 | 0.91 | 1346.5 | 0.46 |
| | 120 | 0.7 | 0.66 | | |
| | 140 | 1.3 | 0.65 | | |
| | 160 | 2.1 | 0.63 | | |
| | 180 | 3.2 | 0.71 | | |
| | 200 | 4.7 | 0.54 | | |
| | avrg | 1.4 | 1.02 | 478.0 | 0.86 |
| 100 | 40 | 0.1 | 1.89 | 24.8 | 1.55 |
| | 60 | 0.1 | 1.17 | 151.1 | 0.98 |
| | 80 | 0.3 | 0.74 | 397.7 | 0.51 |
| | 100 | 0.5 | 0.61 | 1990.1 | 0.27 |
| | 120 | 0.9 | 0.74 | | |
| | 140 | 1.3 | 0.53 | | |
| | 160 | 1.9 | 0.50 | | |
| | 180 | 2.9 | 0.46 | | |
| | 200 | 4.2 | 0.66 | | |
| | avrg | 1.3 | 0.81 | 640.9 | 0.83 |
| total | avrg | 1.7 | 1.30 | 631.4 | 0.87 |

Table 2: Bounds based on Lagrangian relaxation (left table) and bounds from Linearisation (right table). (Intel Pentium III, 933 MHz).

| | | $U^0_{\mathrm{HRW}}$ | | $U^1_{\mathrm{HRW}}$ | | $U^2_{\mathrm{HRW}}$ | | $U^3_{\mathrm{HRW}}$ | | $U^4_{\mathrm{HRW}}$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Δ | $n$ | time | dev | time | dev | time | dev | time | dev | time | dev |
| 5 | 40 | 3.9 | 3.40 | 5.6 | 1.29 | 5.3 | 1.22 | 5.9 | 1.22 | 14.2 | 1.21 |
| | 60 | 9.7 | 7.64 | 12.4 | 2.70 | 13.5 | 1.84 | 14.2 | 1.83 | 75.8 | 1.83 |
| | 80 | 25.5 | 10.23 | 29.1 | 3.60 | 34.2 | 1.95 | 36.8 | 1.95 | 222.1 | 1.95 |
| | 100 | 54.3 | 6.82 | 68.1 | 2.24 | 83.5 | 1.12 | 90.6 | 1.12 | 579.5 | 1.12 |
| | 120 | 125.4 | 11.02 | 143.0 | 4.19 | 170.4 | 1.58 | 180.1 | 1.58 | 1053.8 | 1.58 |
| | 140 | 243.1 | 17.97 | 273.7 | 8.24 | 309.4 | 2.47 | 327.9 | 2.47 | 1810.4 | 2.47 |
| | 160 | 408.0 | 10.42 | 509.7 | 3.95 | 566.3 | 1.03 | 619.8 | 1.03 | 2410.6 | 1.03 |
| | 180 | 644.5 | 11.45 | 799.2 | 5.01 | 909.3 | 1.19 | 1014.2 | 1.19 | 3928.0 | 1.19 |
| | 200 | 904.3 | 9.76 | 1107.9 | 3.43 | 1345.5 | 0.69 | 1564.6 | 0.69 | | |
| | avrg | 268.8 | 9.86 | 327.6 | 3.85 | 381.9 | 1.46 | 428.2 | 1.45 | 1261.8 | 1.55 |
| 25 | 40 | 3.8 | 17.36 | 4.2 | 10.44 | 4.7 | 2.99 | 5.4 | 2.97 | 17.8 | 2.95 |
| | 60 | 10.0 | 11.79 | 11.5 | 5.07 | 14.8 | 0.71 | 15.3 | 0.70 | 69.6 | 0.70 |
| | 80 | 24.4 | 9.78 | 28.7 | 4.96 | 38.2 | 0.87 | 42.1 | 0.86 | 255.5 | 0.85 |
| | 100 | 55.2 | 31.42 | 61.1 | 18.31 | 72.0 | 1.23 | 80.6 | 1.22 | 537.0 | 1.21 |
| | 120 | 120.3 | 16.75 | 138.6 | 8.77 | 185.1 | 0.49 | 204.2 | 0.48 | 1090.5 | 0.47 |
| | 140 | 240.7 | 21.13 | 274.3 | 11.59 | 340.3 | 0.49 | 350.7 | 0.49 | 1968.3 | 0.49 |
| | 160 | 373.4 | 11.62 | 457.7 | 5.78 | 626.3 | 0.25 | 679.3 | 0.25 | 3146.9 | 0.25 |
| | 180 | 583.3 | 18.92 | 664.0 | 10.00 | 888.5 | 0.31 | 929.8 | 0.30 | 4885.8 | 0.30 |
| | 200 | 916.2 | 24.76 | 1019.1 | 14.33 | 1306.5 | 0.34 | 1320.2 | 0.34 | | |
| | avrg | 258.6 | 18.17 | 295.5 | 9.92 | 386.3 | 0.85 | 403.1 | 0.85 | 1496.4 | 0.90 |
| 50 | 40 | 3.8 | 20.46 | 4.1 | 12.23 | 5.1 | 1.85 | 5.7 | 1.83 | 17.3 | 1.82 |
| | 60 | 10.1 | 21.03 | 11.0 | 12.16 | 13.9 | 1.03 | 15.7 | 1.01 | 76.0 | 0.99 |
| | 80 | 23.0 | 11.22 | 27.5 | 5.86 | 39.6 | 0.48 | 39.4 | 0.48 | 247.7 | 0.48 |
| | 100 | 55.1 | 27.69 | 61.0 | 16.58 | 79.2 | 0.66 | 87.7 | 0.63 | 576.6 | 0.61 |
| | 120 | 121.9 | 17.60 | 136.7 | 10.02 | 193.2 | 0.40 | 206.4 | 0.39 | 1185.8 | 0.38 |
| | 140 | 240.2 | 25.51 | 265.6 | 14.00 | 396.7 | 0.23 | 397.2 | 0.23 | 2370.4 | 0.23 |
| | 160 | 397.0 | 13.32 | 449.8 | 7.54 | 679.1 | 0.23 | 750.9 | 0.21 | 3333.8 | 0.21 |
| | 180 | 627.1 | 17.62 | 704.2 | 10.11 | 1022.0 | 0.21 | 1113.6 | 0.20 | 4668.1 | 0.20 |
| | 200 | 855.6 | 17.21 | 947.6 | 9.39 | 1399.9 | 0.13 | 1576.1 | 0.13 | | |
| | avrg | 259.3 | 19.07 | 289.7 | 10.88 | 425.4 | 0.58 | 465.9 | 0.57 | 1559.5 | 0.61 |
| 75 | 40 | 4.0 | 21.46 | 4.0 | 12.94 | 5.1 | 1.88 | 5.4 | 1.87 | 19.8 | 1.87 |
| | 60 | 9.5 | 20.72 | 10.8 | 11.82 | 14.3 | 1.01 | 16.4 | 0.98 | 72.8 | 0.97 |
| | 80 | 22.5 | 16.06 | 25.6 | 9.05 | 41.1 | 0.28 | 42.3 | 0.27 | 241.4 | 0.27 |
| | 100 | 53.9 | 16.96 | 61.0 | 10.15 | 89.8 | 0.47 | 94.0 | 0.47 | 628.0 | 0.46 |
| | 120 | 114.6 | 16.19 | 126.7 | 9.88 | 185.7 | 0.68 | 210.3 | 0.65 | 1254.4 | 0.64 |
| | 140 | 236.0 | 19.73 | 262.4 | 11.88 | 406.1 | 0.24 | 447.8 | 0.22 | 2168.5 | 0.21 |
| | 160 | 382.5 | 15.76 | 453.0 | 8.68 | 713.6 | 0.21 | 749.2 | 0.20 | 3191.2 | 0.20 |
| | 180 | 651.2 | 25.19 | 697.3 | 15.25 | 1064.5 | 0.35 | 1358.0 | 0.31 | | |
| | 200 | 913.6 | 20.60 | 963.6 | 13.32 | 1382.2 | 0.44 | 2036.5 | 0.36 | | |
| | avrg | 265.3 | 19.19 | 289.4 | 11.44 | 433.6 | 0.61 | 551.1 | 0.59 | 1082.3 | 0.66 |
| 95 | 40 | 3.7 | 38.28 | 3.9 | 25.05 | 4.9 | 1.80 | 5.3 | 1.78 | 17.3 | 1.78 |
| | 60 | 9.4 | 34.16 | 10.7 | 20.90 | 14.9 | 0.82 | 16.6 | 0.79 | 75.6 | 0.78 |
| | 80 | 23.2 | 22.26 | 26.2 | 13.48 | 40.8 | 0.78 | 43.8 | 0.76 | 256.5 | 0.75 |
| | 100 | 54.6 | 24.82 | 59.6 | 14.78 | 96.8 | 0.65 | 102.5 | 0.61 | 655.5 | 0.59 |
| | 120 | 117.1 | 16.32 | 131.7 | 9.32 | 228.7 | 0.34 | 233.0 | 0.32 | 1248.6 | 0.31 |
| | 140 | 236.5 | 17.29 | 259.9 | 9.82 | 428.0 | 0.27 | 464.6 | 0.27 | 2165.8 | 0.27 |
| | 160 | 369.1 | 21.00 | 419.3 | 12.44 | 729.5 | 0.34 | 732.6 | 0.34 | 3426.4 | 0.34 |
| | 180 | 619.4 | 18.34 | 662.9 | 11.02 | 1182.8 | 0.37 | 1227.4 | 0.36 | 5121.6 | 0.35 |
| | 200 | 906.3 | 29.32 | 967.7 | 18.20 | 1678.9 | 0.16 | 1762.3 | 0.16 | | |
| | avrg | 259.9 | 24.64 | 282.4 | 15.00 | 489.5 | 0.61 | 509.8 | 0.60 | 1620.9 | 0.65 |
| 100 | 40 | 3.7 | 17.04 | 4.1 | 10.59 | 5.8 | 1.66 | 5.9 | 1.66 | 21.1 | 1.66 |
| | 60 | 9.3 | 12.07 | 10.5 | 7.22 | 16.2 | 1.05 | 15.9 | 1.05 | 83.8 | 1.04 |
| | 80 | 22.3 | 22.30 | 25.8 | 13.17 | 38.1 | 0.59 | 41.5 | 0.58 | 258.0 | 0.57 |
| | 100 | 50.5 | 19.85 | 57.3 | 12.25 | 93.8 | 0.30 | 100.5 | 0.29 | 639.7 | 0.29 |
| | 120 | 119.5 | 22.26 | 130.4 | 13.04 | 238.6 | 0.40 | 235.2 | 0.40 | 1257.3 | 0.40 |
| | 140 | 238.9 | 16.86 | 260.2 | 9.61 | 459.0 | 0.27 | 457.8 | 0.26 | 2203.8 | 0.26 |
| | 160 | 405.8 | 20.41 | 442.0 | 12.41 | 717.2 | 0.39 | 803.8 | 0.38 | 3503.7 | 0.37 |
| | 180 | 620.9 | 24.25 | 678.2 | 14.85 | 1133.7 | 0.27 | 1210.9 | 0.26 | 4833.8 | 0.25 |
| | 200 | 893.2 | 21.91 | 970.0 | 13.16 | 1620.9 | 0.54 | 1757.8 | 0.53 | | |
| | avrg | 262.7 | 19.66 | 286.5 | 11.81 | 480.4 | 0.61 | 514.4 | 0.60 | 1600.1 | 0.60 |
| total avrg | | 262.4 | 18.43 | 295.2 | 10.48 | 432.8 | 0.79 | 478.7 | 0.78 | 1444.4 | 0.83 |

Table 3: Bounds from Semidefinite Programming. (Intel Pentium III, 933 MHz).

## Approximation algorithms

- Should run in polynomial time
- Bounds from semidefinite programming
- Semidefinite optimization can be solved in polynomial time

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