Interior point methods
— an introduction

David Pisinger

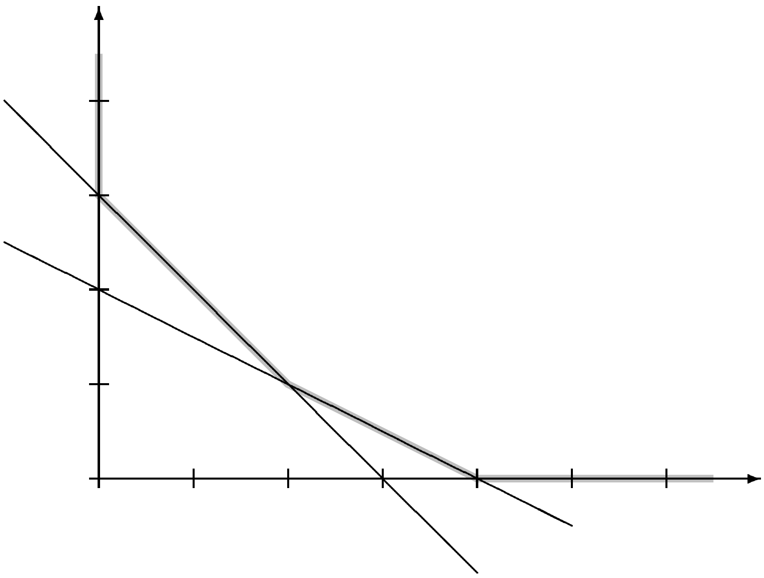
March 12, 2004

- Linear Programming, duality
- Simplex and its complexity
- Interior point methods — history
- Newton's method
- Obtaining $x \geq 0$ through barrier function
- Putting the pieces together
- Discussion

matrix notation, omit transposition x^T

Linear programming

$$\begin{array}{ll} \text{minimize} & 2x_1 + 3x_2 \\ \text{subject to} & 1x_1 + 2x_2 \geq 4 \\ & 1x_1 + 1x_2 \geq 3 \\ & x_1, x_2 \geq 0 \end{array}$$



In matrix form

$$\begin{array}{ll} \text{minimize} & cx \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array}$$

Linear programming

- Primal

$$\begin{array}{ll} \text{minimize} & cx \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array}$$

- Dual

$$\begin{array}{ll} \text{maximize} & by \\ \text{subject to} & yA + s = c \\ & s \geq 0, y \in \mathbb{R} \end{array}$$

- *Weak duality*: Assume that x primal feasible and y, s dual feasible

$$by \leq cx$$

  Duality gap: $cx - by \geq 0$

- *Strong duality*: If the problem has a feasible solution, then there exists a primal-dual feasible pair (x^*, y^*, s^*), so that

$$cx^* = by^*$$

- *Complementary slackness* (alternative formulation of strong duality). If $x \geq 0, s \geq 0, y \in \mathbb{R}$ satisfy

$$\begin{array}{lcl} Ax & = & b \\ yA + s & = & c \\ sx & = & 0 \end{array}$$

  then (x, y, s) optimal.

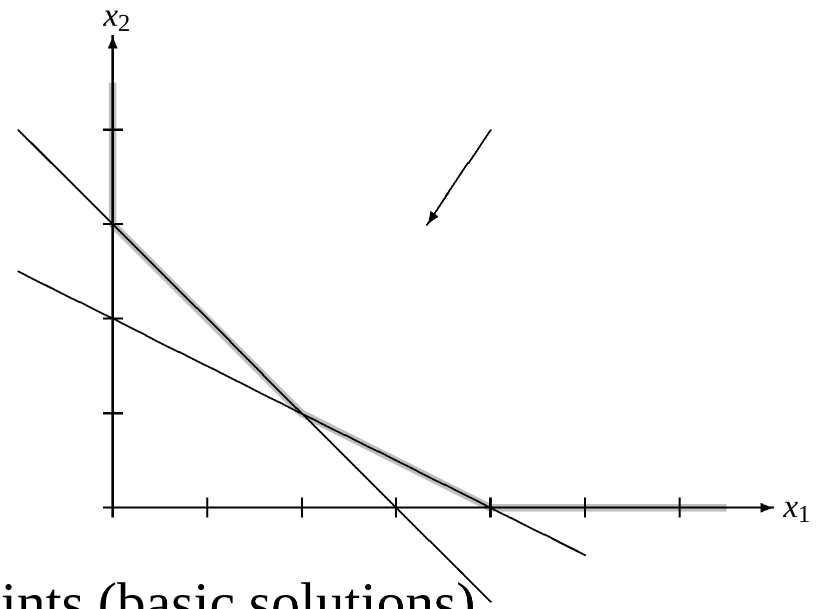
$$cx - by = (yA + s)x - y(Ax) = sx$$

Solving Linear Programs

Optimization problem with slack variables added

$$\begin{array}{lrcrcrcrcl} \text{maximize} & 2x_1 & + & 3x_2 & & & & & & \\ \text{subject to} & 1x_1 & + & 2x_2 & - & x_3 & & & = & 4 \\ & 1x_1 & + & 1x_2 & & & - & x_4 & = & 3 \\ & & & & & & x_1, x_2, & x_3, x_4 & \geq & 0 \end{array}$$

The set of constraints form a polyhedral.
Optimal solution is found at extreme points



Extreme points (basic solutions)

$$(2,1,0,0) \quad (0,3,2,0) \quad (4,0,0,1)$$

Basic set can be chosen in $\binom{n}{m}$ ways (i.e. exponential).

Complexity of Simplex

Klee and Minty (1975) proved that the Simplex algorithm may use exponential time

$$\begin{array}{l}
\text{maximize} \\
2^{n-1}x_1 + 2^{n-2}x_2 + \ldots + 2x_{n-1} + 1x_n \\
\text{subject to}
\end{array}$$

$$\begin{array}{rcrcrcrcrcl}
1x_1 & + & & + & & + & & + & & \leq & 5 \\
4x_1 & + & 1x_2 & + & & + & & + & & \leq & 5^2 \\
8x_1 & + & 4x_2 & + & 1x_3 & + & & + & & \leq & 5^3 \\
\vdots & + & & + & & + & & + & & \leq & \vdots \\
2^n x_1 & + & 2^{n-1}x_2 & + & \ldots & + & 4x_{n-1} & + & 1x_n & \leq & 5^n
\end{array}$$

$$x_i \geq 0, i = 1, \ldots, n$$

The problem has

- n variables
- n constraints
- 2^n extreme points
- Simplex, starting at $x = (0, \ldots, 0)$, visits all extreme points
- optimal solution $(0, 0, \ldots, 0, 5^n)$

Complexity of Simplex

For $n = 3$ simplex visits $2^3 = 8$ extreme points
Assume (s_1, s_2, s_3) slack variables:

	nonbasis			
basis	x_1	x_2	x_3	RHS
s_1	1^*			5
s_2	4	1		25
s_3	8	4	1	125
$-z$	4	2	1	0

	nonbasis			
basis	s_1	x_2	x_3	RHS
x_1	1			5
s_2	-4	1^*		5
s_3	-8	4	1	85
$-z$	-4	2	1	-20

	nonbasis			
basis	s_1	s_2	x_3	RHS
x_1	1^*			5
x_2	-4	1		5
s_3	8	-4	1	65
$-z$	4	-2	1	-30

	nonbasis			
basis	x_1	s_2	x_3	RHS
s_1	1			5
x_2	4	1		25
s_3	-8	-4	1^*	25
$-z$	-4	-2	1	-50

	nonbasis			
basis	x_1	s_2	s_3	RHS
s_1	1^*			5
x_2	4	1		25
x_3	-8	-4	1	25
$-z$	4	2	-1	-75

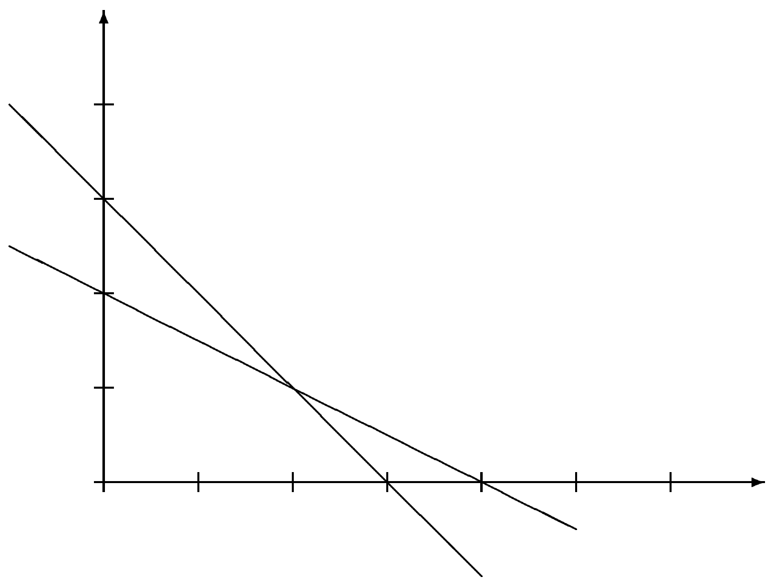
	nonbasis			
basis	s_1	s_2	s_3	RHS
x_1	1			5
x_2	-4	1^*		5
x_3	8	-4	1	65
$-z$	-4	2	-1	-95

	nonbasis			
basis	s_1	x_2	s_3	RHS
x_1	1^*			5
s_2	-4	1		5
x_3	-8	4	1	85
$-z$	4	-2	-1	-105

	nonbasis			
basis	x_1	x_2	s_3	RHS
s_1	1^*			5
s_2	4	1		25
x_3	8	4	1	125
$-z$	-4	-2	-1	-125

Interior-point methods — history

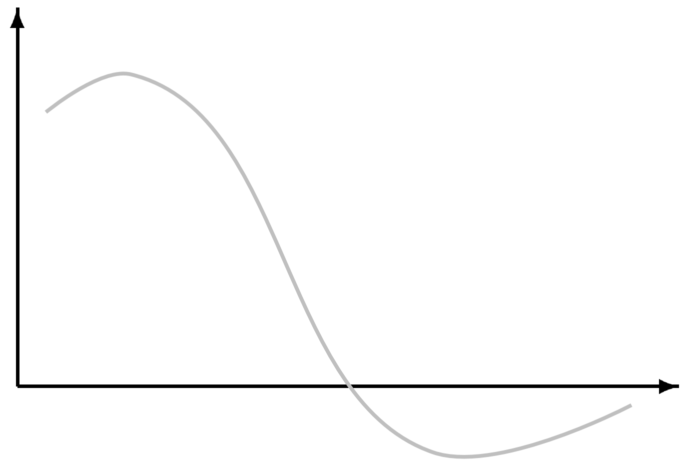
- 40ies: Simplex algorithm, Dantzig.
- Klee and Minty (1975) proved that a variant of Simplex may use exponential time. Stimulated research in alternatives.
- Khachiyan (1979) polynomial algorithm, *ellipsoid method* Bad performance in practice.
- Karmarkar (1984) polynomial algorithm *path-following*
- Path following method now described by Newton's method, barrier function



Newton's method

$f(x) : \mathbb{R}^n \to \mathbb{R}^n$ smooth nonlinear function, solve:

$$f(x) = 0$$



Taylor's theorem (linearization)

$$f(x^0 + d_x) \approx f(x^0) + \nabla f(x^0) d_x$$

If x^0 initial guess, compute d_x such that $f(x^0 + d_x) = 0$.

$$f(x^0) + \nabla f(x^0) d_x = 0 \qquad\qquad d_x = -(\nabla f(x^0))^{-1} f(x^0)$$

d_x defines search direction, new point x^+

$$x^+ = x^0 + \alpha d_x$$

where $0 < \alpha < 1$ is step size.

Nonlinear programming

Assume $c : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}^m$ smooth functions.

$$\begin{array}{ll} \text{minimize} & c(x) \\ \text{subject to} & g(x) = 0 \end{array}$$

To use Newton's method formulate as

$$f(x) = 0$$

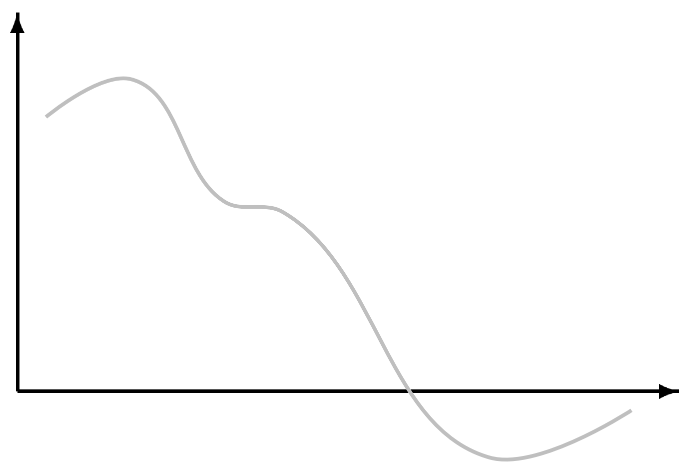
Lagrangian function, y lagrangian multipliers:

$$L(x,y) = c(x) - y g(x)$$

First order optimality criterion

$$\begin{array}{rclcl} \nabla_x L(x,y) & = & \nabla c(x) - \nabla g(x) y & = & 0 \\ \nabla_y L(x,y) & = & -g(x) & = & 0 \end{array}$$

Necessary but not sufficient condition



If $c(x)$ is convex, and $g(x)$ is affine, then sufficient

Newton's method for optimality criterion

Looking for solution to "$f(x,y) = 0$" i.e.

$$\begin{array}{rcccl} \nabla_x L(x,y) & = & \nabla c(x) - \nabla g(x) y & = & 0 \\ \nabla_y L(x,y) & = & -g(x) & = & 0 \end{array}$$

Taylor expansion in $(x,y) = (x^0, y^0)$:

$$f(x^0, y^0) + \nabla f(x^0, y^0)(d_x, d_y) = 0$$

thus

$$\begin{pmatrix} \nabla c(x^0) - \nabla g(x^0)^T y^0 \\ -g(x^0) \end{pmatrix} + \begin{pmatrix} \nabla^2 c(x^0) - \sum_{i=1}^m y_i \nabla^2 g_i(x^0) & -\nabla g(x^0)^T \\ -\nabla g(x^0) & 0 \end{pmatrix} \begin{pmatrix} d_x \\ d_y \end{pmatrix} = 0$$

next point

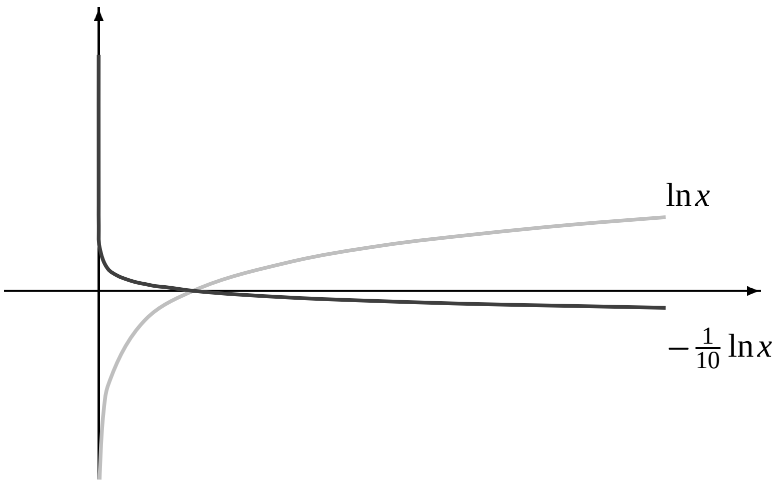
$$\begin{pmatrix} x^+ \\ y^+ \end{pmatrix} = \begin{pmatrix} x^0 \\ y^0 \end{pmatrix} + \alpha \begin{pmatrix} d_x \\ d_y \end{pmatrix}$$

Primal problem, barrier function

Linear programming model

$$\begin{aligned} \text{minimize} \quad & cx \\ \text{subject to} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

Newton's method cannot handle inequalities.



New objective:

$$\begin{aligned} \text{minimize} \quad & cx - \mu \textstyle\sum_{j=1}^{n} \ln(x_j) \\ \text{subject to} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

where $\mu > 0$ is a small constant.

$$\lim_{x_j \to 0} -\mu \ln(x_j) = \infty$$

If $x > 0$ initially, then *barrier function* maintains $x > 0$.

Solution to barrier problem is approx. of original problem

Primal problem, optimality condition

Define optimality conditions for barrier problem

$$L(x,y) = cx - \mu \sum_{j=1}^{n} \ln(x_j) - y(Ax - b)$$

Differentiation gives

$$\frac{\partial L}{\partial x_j} = c_j - \mu x_j^{-1} - A_{:j}^T y \quad \text{and} \quad \frac{\partial L}{\partial y_i} = b_i - A_{i:} x$$

In vector notation

$$\begin{array}{rcl} \nabla_x L(x,y) & = & c - \mu X^{-1} e - A^T y \;=\; 0 \\ \nabla_y L(x,y) & = & b - Ax \;=\; 0 \end{array}$$

where

$$X = \begin{pmatrix} x_1 & 0 & \cdots & 0 \\ 0 & x_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x_n \end{pmatrix}$$

since $x > 0$ the inverse X^{-1} exists
If we introduce $s = \mu X^{-1} e$ then

$$\begin{array}{rcl} c - s - A^T y & = & 0 \\ b - Ax & = & 0 \\ s & = & \mu X^{-1} e \end{array}$$

or equivalent (Kuhn-Tucker optimality condition)

$$\left\{ \begin{array}{rcl} A^T y + s & = & c \\ Ax & = & b \\ Xs & = & \mu e \end{array} \right. \quad \boxed{x > 0}$$

Convexity of barrier function

If objective function is convex, then optimality condition is sufficient

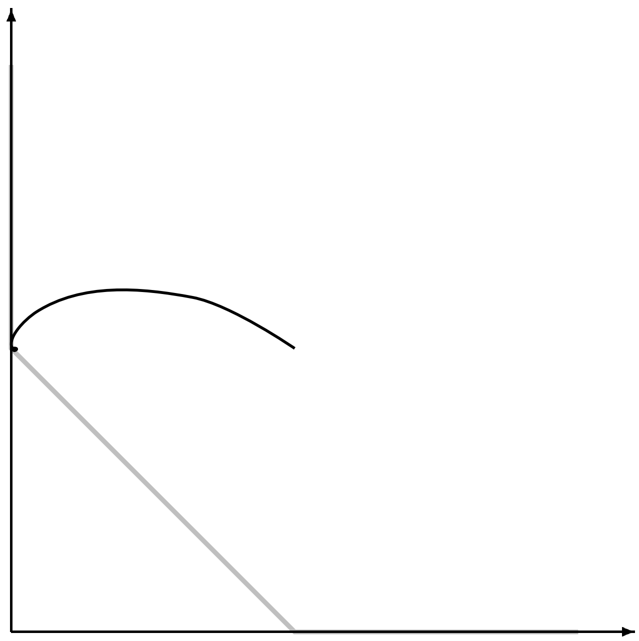
$$cx - \mu \sum_{j=1}^{n} \ln(x_j)$$

- The function $\ln(x)$ is concave, i.e. $-\ln(x)$ is convex
- Barrier function is sum of convex functions
- Barrier function is convex

Path following methods

primal *central path* is $\{x(\mu) : \mu > 0\}$
dual *central path* is $\{y(\mu), s(\mu) : \mu > 0\}$

Example



How large is the error?

$$\begin{array}{rcl} A^T y + s & = & c \\ Ax & = & b \\ Xs & = & \mu e \end{array}$$

A feasible solution (x, y, s) to the system is

- Primal feasible: $Ax = b$ and $x > 0$.
- Dual feasible: $A^T y + s = c$ and $s = \mu X^{-1} e > 0$
- Duality gap:

$$\begin{array}{rcl} cx - yb & = & (yA + s)x - y(Ax) \\ & = & xs \\ & = & x(\mu X^{-1} e) \\ & = & \mu ee \; = \; \mu n \end{array}$$

Since an optimal solution x^* must satisfy $by \leq cx^* \leq cx$

$$cx - cx^* \leq n\mu$$

Overall principle

Problem: find primal-dual pair (x,y,s) with $x,s \geq 0$ such that

$$F(x,y,s) = \begin{pmatrix} Ax-b \\ yA+s-c \\ sx \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Algorithm:

- Select a tolerance:

$$\begin{array}{lll} ||Ax-b|| & \leq & \varepsilon_P \\ ||yA+s-c|| & \leq & \varepsilon_D \\ sx & \leq & \varepsilon_G \end{array}$$

- Choose an initial solution $x > 0, y > 0, s > 0$
- Based on the tolerance choose a sufficiently small μ
- Use Newton's method on the barrier function
- Terminate when the above tolerance is satisfied

Note

$$\nabla F(x,y,s) = \begin{pmatrix} A & 0 & 0 \\ 0 & A & I \\ S & 0 & X \end{pmatrix}$$

Step size, and convergence

$$\alpha^{\max} = \arg\max_{0\leq\alpha} \left\{ \begin{pmatrix} x^k \\ s^k \end{pmatrix} + \alpha \begin{pmatrix} d_x \\ d_s \end{pmatrix} \geq 0 \right\}$$

and for some $\theta \in]0,1[$

$$\alpha := \min(\theta\alpha^{\max}, 1)$$

ensures convergence. In practice $\theta = 0.9$ is good (but no polynomial guarantee!)

Number of steps

"interior point methods solve an LP in less than 100 iterations even if the problem contains millions of variables"

Complexity of each step

Derive d_x, d_y, d_s from

$$\begin{pmatrix} A & 0 & 0 \\ 0 & A & I \\ S & 0 & X \end{pmatrix} \begin{pmatrix} d_x \\ d_y \\ d_s \end{pmatrix} = \begin{pmatrix} Ax - b \\ yA + s - c \\ sx \end{pmatrix}$$

next point

$$\begin{pmatrix} x^+ \\ y^+ \\ s^+ \end{pmatrix} = \begin{pmatrix} x^0 \\ y^0 \\ s^0 \end{pmatrix} + \alpha \begin{pmatrix} d_x \\ d_y \\ d_s \end{pmatrix}$$

Dual problem, barrier function

Linear programming model

$$\begin{array}{ll} \text{maximize} & by \\ \text{subject to} & yA + s = c \\ & s \geq 0, y \in \mathbb{R} \end{array}$$

Introduce barrier function

$$\begin{array}{ll} \text{maximize} & by + \mu \sum_{j=1}^{n} \ln(s_j) \\ \text{subject to} & yA + s = c \\ & s > 0 \end{array}$$

Lagrangian function

$$L(x, y, s) = by + \mu \sum_{j=1}^{n} \ln(s_j) - x(yA + s - c)$$

Optimality conditions

$$\begin{array}{rclcl} \nabla_x L(x, y, s) & = & c - s - yA & = & 0 \\ \nabla_y L(x, y, s) & = & b - Ax & = & 0 \\ \nabla_s L(x, y, s) & = & \mu S^{-1} e - x & = & 0 \end{array}$$

which can be reduced to

$$\left\{ \begin{array}{rcll} yA + s & = & c, & \boxed{s > 0} \\ Ax & = & b & \\ Xs & = & \mu e & \end{array} \right.$$

Essentially the same as for primal problem

Primal-dual approach

Two sets of optimality conditions

$$\begin{cases} Ax &= b, \quad \boxed{x > 0} \\ yA + s &= c, \quad \boxed{s > 0} \\ Xs &= \mu e \end{cases}$$

perturbed KKT conditions. Let

$$F_\gamma(x,y,s) = \begin{pmatrix} Ax - b \\ yA + s - c \\ Xs - \gamma\mu e \end{pmatrix}$$

where $\mu = xs/n$ and $\gamma \geq 0$. Assume $(\bar{x},\bar{y},\bar{s})$ given. One iteration of Newton method to $F_\gamma(\bar{x},\bar{y},\bar{s}) = 0$ i.e.

$$\nabla F_\gamma(\bar{x},\bar{y},\bar{s}) \begin{pmatrix} d_x \\ d_y \\ d_s \end{pmatrix} = -F_\gamma(\bar{x},\bar{y},\bar{s})$$

Since

$$\nabla F_\gamma(\bar{x},\bar{y},\bar{s}) = \begin{pmatrix} A & 0 & 0 \\ 0 & A & I \\ \bar{S} & 0 & \bar{X} \end{pmatrix}$$

we obtain

$$\begin{pmatrix} A & 0 & 0 \\ 0 & A & I \\ \bar{S} & 0 & \bar{X} \end{pmatrix} \begin{pmatrix} d_x \\ d_y \\ d_s \end{pmatrix} = \begin{pmatrix} \bar{r}_P \\ \bar{r}_D \\ -\bar{X}\bar{s} + \gamma\bar{\mu}e \end{pmatrix}$$

primal residual $\bar{r}_P = b - A\bar{x}$
dual residual $\bar{r}_D = c - \bar{y}A - \bar{s}$

The algorithm

1 choose (x^0, y^0, s^0) with $x^0, s^0 > 0$, and $\varepsilon_P, \varepsilon_D, \varepsilon_G > 0$

2 for $k := 0$ to ∞

3 calculate residuals

$$\begin{aligned} \bar{r}_P^k &= b - A\bar{x}^k \\ \bar{r}_D^k &= c - \bar{y}^k A - \bar{s}^k \\ \mu^k &= x^k s^k / n \end{aligned}$$

4 if $\|\bar{r}_P^k\| \leq \varepsilon_P, \|\bar{r}_D^k\| \leq \varepsilon_D, \|\mu^k\| \leq \varepsilon_G$ then stop

5 choose $\gamma < 1$ and solve

$$\begin{pmatrix} A & 0 & 0 \\ 0 & A & I \\ \bar{S} & 0 & \bar{X} \end{pmatrix} \begin{pmatrix} d_x \\ d_y \\ d_s \end{pmatrix} = \begin{pmatrix} \bar{r}_P \\ \bar{r}_D \\ -\bar{X}\bar{s} + \gamma\bar{\mu}e \end{pmatrix}$$

5 compute

$$\alpha^{\max} = \arg\max_{0 \leq \alpha} \left\{ \begin{pmatrix} x^k \\ s^k \end{pmatrix} + \alpha \begin{pmatrix} d_x \\ d_s \end{pmatrix} \geq 0 \right\}$$

6 for some $\theta \in]0, 1[$

$$\alpha := \min(\theta\alpha^{\max}, 1)$$

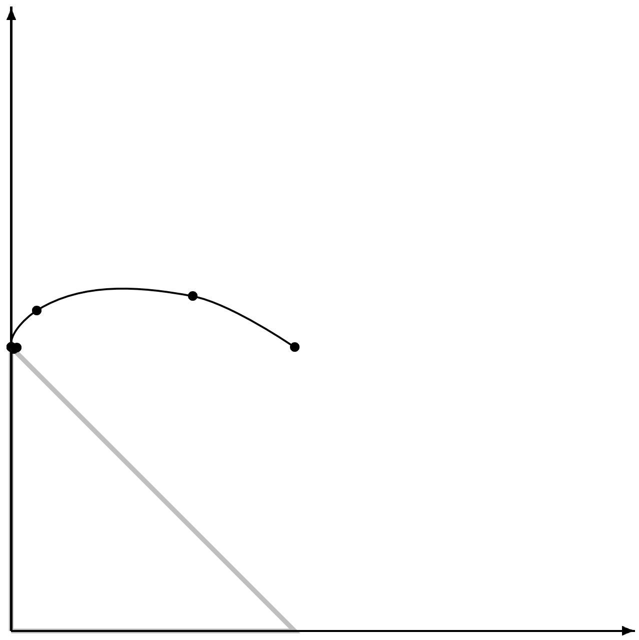
7 update

$$\begin{aligned} x^{k+1} &:= x^k + \alpha d_x, \\ y^{k+1} &:= y^k + \alpha d_y, \\ s^{k+1} &:= s^k + \alpha d_s \end{aligned}$$

Numerical Example

Problem:

$$\begin{array}{ll} \text{minimize} & -1x_1 - 2x_2 \\ \text{subject to} & x_1 + x_2 \leq 1 \\ & x_1, x_2 \geq 0 \end{array}$$



Iterations

k	x_1^k	x_2^k	$x^k s^k$
0	1.00e0	1.00e0	3.0e0
1	6.40e-1	1.18e0	1.1e0
2	0.91e-2	1.13e0	2.5e-1
3	0.91e-3	9.93e-1	4.1e-2
4	1.91e-3	9.98e-1	6.1e-3
5	2.42e-4	1.00e0	7.2e-4
6	2.45e-5	1.00e0	7.3e-5
7	2.45e-6	1.00e0	7.4e-6
8	2.56e-7	1.00e0	7.3e-7

Worst-case time complexity

- *Simplex*: exponential, but heuristics often get round this problem
- *Interior*: approximate solution, polynomial time in n and ε

Characterisation of the iterative sequence

- *Simplex*: generate a sequence of feasible basic solutions
- *Interior*: generate a sequence of feasible primal and dual solutions

- *Simplex*: primal solution decreases monotonically
- *Interior*: Duality gap decreaseas monotonically

- *Simplex*: many iterations
- *Interior*: few iterations (20-30)

- *Simplex*: one simplex iteration $O(n^2)$
- *Interior*: one iteration $O(n^3)$

Generated solutions

- *Simplex*: Returns an optimal basic solution
- *Interior*: Returns an ε-optimal solution $cx - by \leq \varepsilon$
- *Interior*: Solution is at the analytic center of the optimal face
- *Interior*: Basis solution can be generated constructed in strongly polynomial time

Initialization

- *Simplex*: Needs a feasible solution to start
- *Simplex*: First phase of algorithm finds feasible solution
- *Interior*: Self-dual version can be initialized by any positive vector

Degeneracy

Basis solution x_B is degenerate if $\exists i \in I_B : x_i = 0$

- *Simplex*: objective function remains same in simplex iteration, cycling
- *Interior*: theoretical and practical complexity is not affected by degeneracy

Practical performance

- *Simplex*: often good performance despite worst-case exponential time
- *Interior*: best suited for huge problems, higly degenerate problems

Warm-start

- *Simplex*: easy to warm-start from previous basis solution
- *Interior*: not as efficient as simplex for warm-start

Integer-linear problems

- *Simplex*: clear winner, generate cuts from basic solutions, warm-start in branch-and-bound
- *Interior*: basic solution *can* be constructed, no warm-start

Generalizations

- *Simplex*: generalized to nonlinear, and semi-infinite optimization problems
- *Interior*: same generalizations as simplex, moreover *conic* optimization

Perspectivation

- Interior point methods is one of the best examples of theoritcal goals leading to practical algorithms
- Interior point methods has started competition leading to innovation
- Interior point methods can be extended to a number of cones (*self-dual homogeneous cones*)
  - $\mathbb{R}^n$ (linear programming)
  - vectorized symmetric matrices over real numbers (semidefinite programming)
  - vectorized Hermitian matrices over complex numbers
  - vectorized Hermitian matrices over quaternions
  - vectorized Hermitian 3×3 matrices over octonions
- With conic optimization we can solve “more” problems than we asked for
- Challenge to model builders to use new relaxations