

LP in Standard and Slack Forms

$$\begin{aligned} \max \quad & \sum_{j=1}^n c_j x_j \\ \text{s.t.} \quad & \sum_{j=1}^n a_{ij} x_j \leq b_i \quad \text{for } i=1,2,\dots,m \\ & x_j \geq 0 \quad \text{for } j=1,2,\dots,n \end{aligned}$$

$$z = 0 + \sum_{j=1}^n c_j x_j$$

$$x_{n+i} = b_i - \sum_{j=1}^n a_{ij} x_j \quad \text{for } i=1,2,\dots,m$$

Auxiliary Linear Program

- L: LP in standard form:

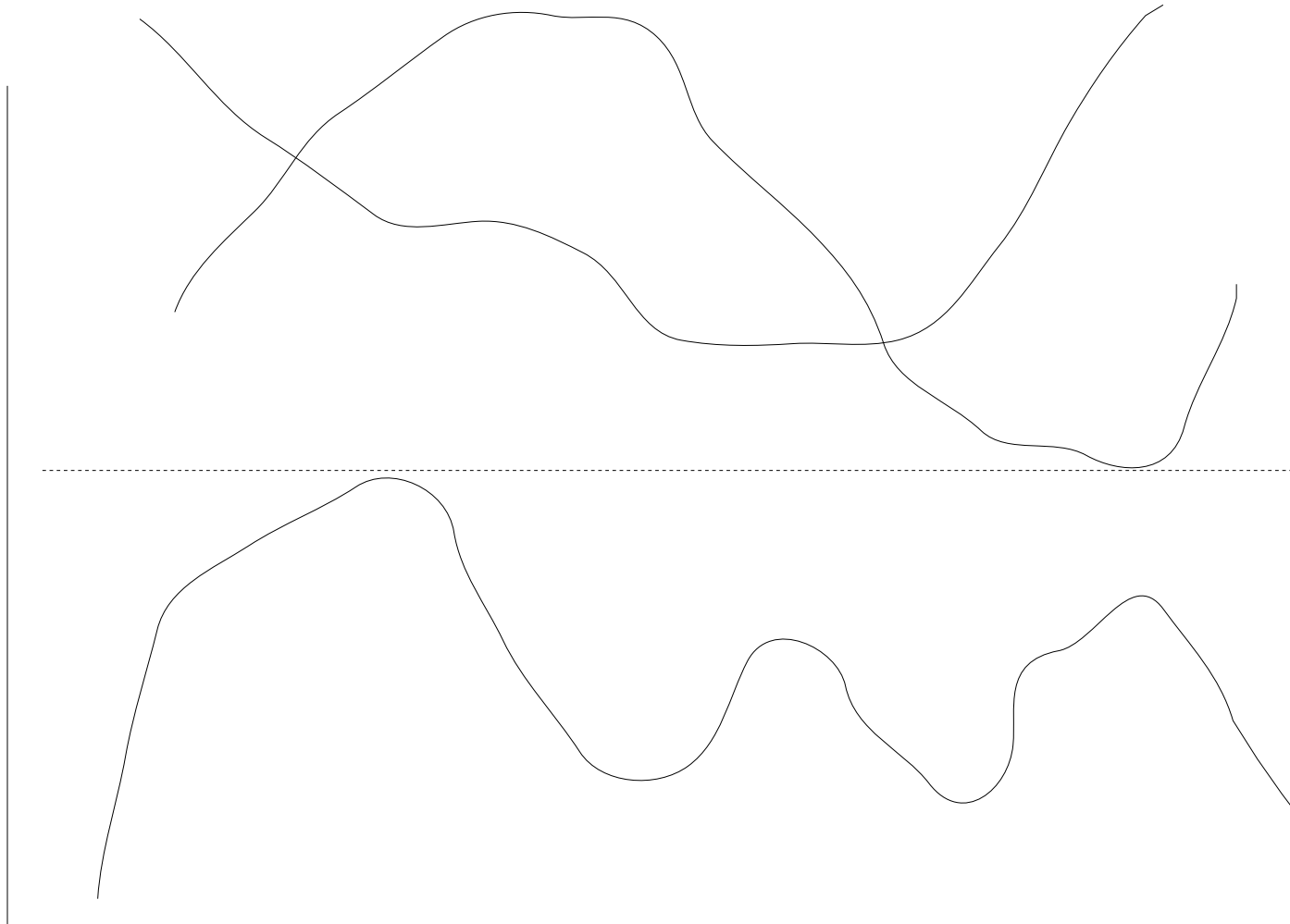
$$\begin{aligned} \max \quad & \sum_{j=1}^n c_j x_j \\ \text{s.t.} \quad & \sum_{j=1}^n a_{ij} x_j \leq b_i \quad \text{for } i=1,2,\dots,m \\ & x_j \geq 0 \quad \text{for } j=1,2,\dots,n \end{aligned}$$

- L_{aux} : Auxiliary LP:

$$\begin{aligned} \max \quad & -x_0 \\ \text{s.t.} \quad & \sum_{j=1}^n a_{ij} x_j - x_0 \leq b_i \quad \text{for } i=1,2,\dots,m \\ & x_j \geq 0 \quad \text{for } j=0,1,2,\dots,n \end{aligned}$$

- L_{aux} is bounded and feasible.

Duality



Upper Bounds on Maximization LP

$$\begin{array}{rcllclclcl}
 \text{max} & 4x_1 & + & x_2 & + & 5x_3 & + & 3x_4 & & \\
 \text{s.t.} & x_1 & - & x_2 & - & x_3 & + & 3x_4 & \leq & 1 \\
 & 5x_1 & + & x_2 & + & 3x_3 & + & 8x_4 & \leq & 55 \\
 & -x_1 & + & 2x_2 & + & 3x_3 & - & 5x_4 & \leq & 3 \\
 & x_1, & & x_2, & & x_3, & & x_4 & \geq & 0
 \end{array}$$

- Construct a linear combination of the constraints using nonnegative multipliers y_1 , y_2 , and y_3 :

$$(y_1 + 5y_2 - y_3)x_1 + (-y_1 + y_2 + 2y_3)x_2 + (-y_1 + 3y_2 + 3y_3)x_3 + (3y_1 + 8y_2 - 5y_3)x_4 \leq y_1 + 55y_2 + 3y_3$$

- Left-hand side will be an upper bound for the LP if the coefficients at each x_j are at least as big as the corresponding coefficients in the objective function

$$y_1 + 5y_2 - y_3 \geq 4 \quad -y_1 + y_2 + 2y_3 \geq 1 \quad 3y_1 + 8y_2 - 5y_3 \geq 3 \quad -y_1 + 3y_2 + 3y_3 \geq 5$$

- Any set of nonnegative multipliers y_i satisfying these inequalities also satisfies

$$4x_1 + x_2 + 5x_3 + 3x_4 \leq y_1 + 55y_2 + 3y_3$$

- Good upper bound: minimize right-hand side s.t. constraints.

Good Upper Bound

$$\begin{array}{llllll} \min & y_1 & + & 55y_2 & + & 3y_3 \\ \text{s.t.} & y_1 & + & 5y_2 & - & y_3 & \geq & 4 \\ & -y_1 & + & y_2 & + & 2y_3 & \geq & 1 \\ & -y_1 & + & 3y_2 & + & 3y_3 & \geq & 5 \\ & 3y_1 & + & 8y_2 & - & 5y_3 & \geq & 3 \\ & y_1, & & y_2, & & y_3 & \geq & 0 \end{array}$$

Duality

- The identification of a dual problem is almost always coupled with the discovery of a polynomial-time algorithm.
- Duality provides a proof that a solution is optimal.

LP in Standard Form and Its Dual

$$\begin{array}{llllll} \text{max} & 3x_1 & + & x_2 & + & 2x_3 \\ \text{s.t.} & x_1 & + & x_2 & + & 3x_3 & \leq & 30 \\ & 2x_1 & + & 2x_2 & + & 5x_3 & \leq & 24 \\ & 4x_1 & + & x_2 & + & 2x_3 & \leq & 36 \\ & x_1 & , & x_2 & , & x_3 & \geq & 0 \end{array}$$

$$\begin{array}{llllll} \text{min} & 30y_1 & + & 24y_2 & + & 36y_3 \\ \text{s.t.} & y_1 & + & 2y_2 & + & 4y_3 & \geq & 3 \\ & y_1 & + & 2y_2 & + & y_3 & \geq & 1 \\ & 3y_1 & + & 5y_2 & + & 2y_3 & \geq & 2 \\ & y_1 & , & y_2 & , & y_3 & \geq & 0 \end{array}$$

LP in Standard Form and Its Dual

$$\begin{aligned} \max \quad & \sum_{j=1}^n c_j x_j \\ \text{s.t.} \quad & \sum_{j=1}^n a_{ij} x_j \leq b_i \quad \text{for } i=1,2,\dots,m \\ & x_j \geq 0 \quad \text{for } j=1,2,\dots,n \end{aligned}$$

$$\begin{aligned} \min \quad & \sum_{i=1}^m b_i y_i \\ \text{s.t.} \quad & \sum_{i=1}^m a_{ij} y_i \geq c_j \quad \text{for } j=1,2,\dots,n \\ & y_i \geq 0 \quad \text{for } i=1,2,\dots,m \end{aligned}$$

Weak Duality

- \mathbf{x}^* : feasible solution to the primal LP.
- \mathbf{y}^* : feasible solution to the dual LP.

- Claim
$$\sum_{j=1}^n c_j x_j^* \leq \sum_{i=1}^m b_i y_i^*$$

- Proof:

$$\begin{aligned} \sum_{j=1}^n c_j x_j^* &\leq \sum_{j=1}^n \left(\sum_{i=1}^m a_{ij} y_i^* \right) x_j^* && \text{from the dual} \\ &= \sum_{i=1}^m \left(\sum_{j=1}^n a_{ij} x_j^* \right) y_i^* \\ &\leq \sum_{i=1}^m b_i y_i^* && \text{from the primal} \end{aligned}$$

Importance of Weak Duality

- \mathbf{x}^* : feasible solution to the primal LP.
- \mathbf{y}^* : feasible solution to the dual LP.
- If

$$\sum_{j=1}^n c_j x_j^* = \sum_{i=1}^m b_i y_i^*$$

then \mathbf{x}^* and \mathbf{y}^* are optimal solutions to the primal and to the dual LPs, respectively.

Final Feasible Basic Solution and Corresponding Dual Solution

$$\begin{aligned}
 \max \quad z &= 28 - x_3/6 - x_5/6 - 2x_6/3 \\
 \text{s.t.} \quad x_4 &= 18 - x_3/2 + x_5/2 + 0x_6 \\
 x_2 &= 4 - 8x_3/3 - 2x_5/3 + x_6/3 \\
 x_1 &= 8 + x_3/6 + x_5/6 - x_6/3
 \end{aligned}$$

Basic variables: $x_1=8$, $x_2=4$, $x_4=18$

Objective value $z = 28$

$$y_i = \begin{cases} -c'_{n+i} & \text{if } (n+i) \in N \\ 0 & \text{otherwise} \end{cases}$$

$y_1 = 0$ (since x_4 is basic), $y_2 = 1/6$, $y_3 = 2/3$

$$\begin{aligned}
 \min \quad & 30y_1 + 24y_2 + 36y_3 \\
 \text{s.t.} \quad & y_1 + 2y_2 + 4y_3 \geq 3 \\
 & y_1 + 2y_2 + y_3 \geq 1 \\
 & 3y_1 + 5y_2 + 2y_3 \geq 2 \\
 & y_1, y_2, y_3 \geq 0
 \end{aligned}$$

Feasible Solution to the Dual

$$\begin{array}{llllll} \min & 30y_1 & + & 24y_2 & + & 36y_3 \\ \text{s.t.} & y_1 & + & 2y_2 & + & 4y_3 & \geq & 3 \\ & y_1 & + & 2y_2 & + & y_3 & \geq & 1 \\ & 3y_1 & + & 5y_2 & + & 2y_3 & \geq & 2 \\ & y_1 & , & y_2 & , & y_3 & \geq & 0 \end{array}$$

$$y_1 = 0 \text{ (since } x_4 \text{ is basic)}$$

$$y_2 = 1/6$$

$$y_3 = 2/3$$

$$\text{Objective value: } 30 \times 0 + 24 \times 1/6 + 36 \times 2/3 = 28$$

$$1 \times 0 + 2 \times 1/6 + 4 \times 2/3 = 3$$

$$1 \times 0 + 2 \times 1/6 + 1 \times 2/3 = 1$$

$$3 \times 0 + 5 \times 1/6 + 2 \times 2/3 = 13/6$$

Duality Theorem

- Suppose that SIMPLEX terminates with a feasible basic solution $\mathbf{x}^* = (x_1^*, x_2^*, \dots, x_n^*)$ with:
 - N and B denoting the nonbasic and basic variables for the final slack form
 - c' denoting the coefficients of the objective function in the final slack form.
- Let $\mathbf{y}^* = (y_1^*, y_2^*, \dots, y_m^*)$ be defined by
$$y_i^* = \begin{cases} -c'_{n+i} & \text{if } (n+i) \in N \\ 0 & \text{otherwise} \end{cases}$$
- Then \mathbf{x}^* is an optimal solution to the primal LP, \mathbf{y}^* is an optimal solution to the dual LP and

$$\sum_{j=1}^n c_j x_j^* = \sum_{i=1}^m b_i y_i^*$$

Proof of Duality Theorem

- We have to show that
 - y^* is feasible solution for the dual, and
 - $$\sum_{j=1}^n c_j x_j^* = \sum_{i=1}^m b_i y_i^*$$

Proof of Duality Theorem

- Objective function of the final slack form of the primal is:

$$v^* + \sum_{j \in N} c_j^* x_j = v^* + \sum_{j \in N} c_j^* x_j + \sum_{j \in B} 0 x_j = v^* + \sum_{j=1}^{n+m} c_j^* x_j$$

- Optimal value of the primal objective function: $v^* = \sum_{j=1}^n c_j x_j^*$

$$\begin{aligned} \sum_{j=1}^n c_j x_j &= v^* + \sum_{j=1}^n c_j^* x_j + \sum_{j=n+1}^{n+m} c_j^* x_j = v^* + \sum_{j=1}^n c_j^* x_j - \sum_{i=1}^m y_i^* (b_i - \sum_{j=1}^n a_{ij} x_j) \\ &= (v^* - \sum_{i=1}^m b_i y_i^*) + \sum_{j=1}^n (c_j^* + \sum_{i=1}^m a_{ij} y_i^*) x_j \end{aligned}$$

- This must hold for every choice of x_1, x_2, \dots, x_n . Hence

$$v^* = \sum_{i=1}^m b_i y_i^* \quad \text{and} \quad c_j = c_j^* + \sum_{i=1}^m a_{ij} y_i^*, \quad \forall j = 1, 2, \dots, n$$

- Since $c_j^* \leq 0, \forall j = 1, 2, \dots, n+m$, we get

$$\sum_{i=1}^m a_{ij} y_i^* \geq c_j, \quad \forall j = 1, 2, \dots, n \quad \text{and} \quad y_i^* \geq 0, \quad \forall i = 1, 2, \dots, m$$

Primal Dual Combinations

		Dual		
		Optimal	Infeasible	Unbounded
Primal	Optimal	<i>Possible</i>	<i>Impossible</i>	<i>Impossible</i>
	Infeasible	<i>Impossible</i>	<i>Possible</i>	<i>Possible</i>
	Unbounded	<i>Impossible</i>	<i>Possible</i>	<i>Impossible</i>

Both Primal and Dual Infeasible

$$\begin{array}{llll} \max & 2x_1 & - & x_2 \\ \text{s.t.} & x_1 & - & x_2 \leq 1 \\ & -x_1 & + & x_2 \leq -2 \\ & x_1, & & x_2 \geq 0 \end{array}$$

$$\begin{array}{llll} \min & y_1 & - & 2y_2 \\ \text{s.t.} & y_1 & - & y_2 \geq 2 \\ & -y_1 & + & y_2 \geq -1 \\ & y_1, & & y_2 \geq 0 \end{array}$$

Practical Implications

- If $m \gg n$ then the number of constraint in the dual will be much smaller than in the primal.
- Number of pivots in SIMPLEX is usually less than $1.5m$ and only rarely is higher than $3m$.
- Number of pivots increases very slowly with n .
- Solving dual will in such cases be more efficient.

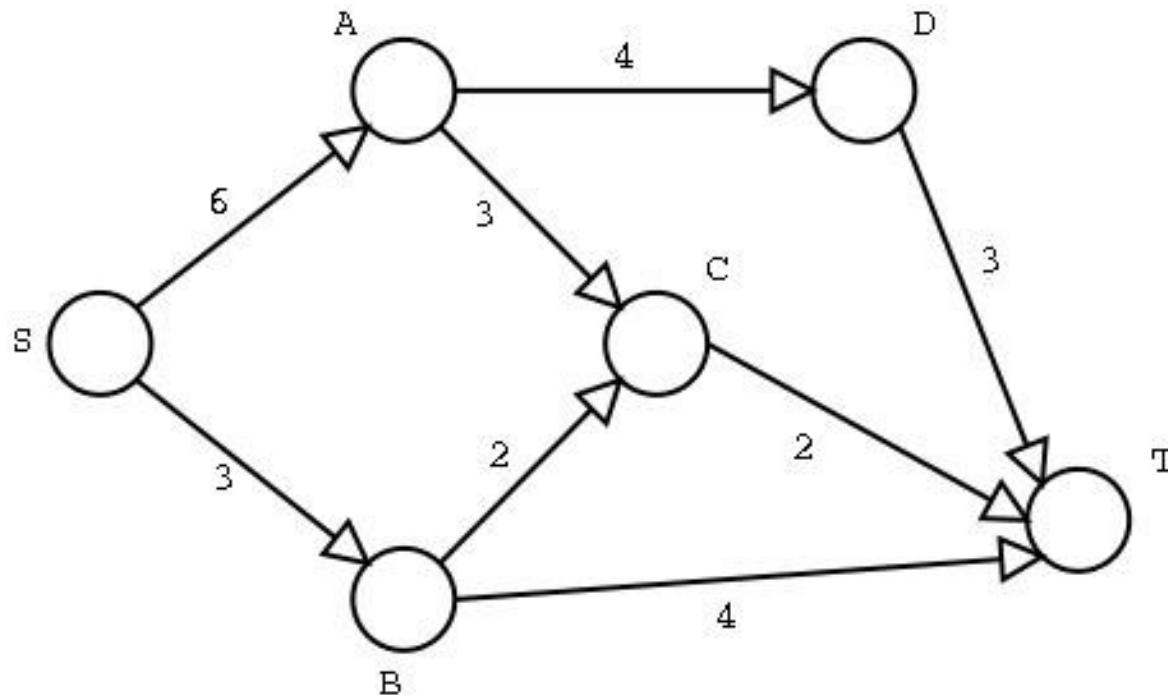
Certificate of Optimality

$$\begin{aligned} \max \quad & \sum_{j=1}^n c_j x_j \\ \text{s.t.} \quad & \sum_{j=1}^n a_{ij} x_j \leq b_i \quad \text{for } i=1, 2, \dots, m \\ & x_j \geq 0 \quad \text{for } j=1, 2, \dots, n \end{aligned}$$

$$\begin{aligned} \min \quad & \sum_{i=1}^m b_i y_i \\ \text{s.t.} \quad & \sum_{i=1}^m a_{ij} y_i \geq c_j \quad \text{for } j=1, 2, \dots, n \\ & y_i \geq 0 \quad \text{for } i=1, 2, \dots, m \end{aligned}$$

Maximum Flow

- **Given:** A directed graph $G = (V, E)$ where each edge $(u, v) \in E$ has a real-valued, nonnegative capacity $c(u, v)$, a source vertex s and a destination vertex t .
- **Find:** A **maximum flow** $f: V \times V \rightarrow \mathbf{R}$ from s to t



Flow

- **Given:** A directed graph $G = (V, E)$ where each edge $(u, v) \in E$ has a real-valued, nonnegative **capacity** $c(u, v)$, a **source** vertex s and a **destination** vertex t .
- A **flow** from s to t in G is a real-valued function $f: V \times V \rightarrow \mathbf{R}$ satisfying:
 - Capacity constraints: $f(u, v) \leq c(u, v), \forall u, v \in V$.
 - Skew symmetry: $f(u, v) = -f(v, u), \forall u, v \in V$.
 - Flow conservation:
- Flow **value** $|f|$ is defined as

$$\sum_{v \in V} f(s, v)$$

Maximum Flow as LP Problem

- maximize $\sum_{v \in V} f(s, v)$

subject to

$$f(u, v) \leq c(u, v) \text{ for all } u, v \in V$$

$$f(u, v) = -f(v, u) \text{ for all } u, v \in V$$

$$\sum_{v \in V} f(u, v) = 0 \quad \text{for all } u \in V - \{s, t\}$$

Maximum Flow as LP Problem

- maximize $\sum_{v \in V} x_{sv}$

subject to

$$x_{uv} \leq c_{uv} \quad \text{for all } u, v \in V$$

$$x_{uv} = -x_{vu} \quad \text{for all } u, v \in V$$

$$\sum_{v \in V} x_{uv} = 0 \quad \text{for all } u \in V \setminus \{s, t\}$$