VA 2006

- Forelæsninger: tirsdage 13-15, fredage 10-12 i Lille UP1
- Øvelser: 3 hold, tirsdage og fredage (½ uge forskudt)
- Hjemmeside

Linear Programming

- Tirsdag, 14/11: **Introduction to LP**, afsnit 29.1 og 29.2
  - Opgaver til 1. øvelsesgang: 29.1-1,-2,-3,4,-5,-6,-7,-8,-9, 29.2-1,-2,-3, BBB (hjemmeopgave, udleveres ved 1. forelæsning, afleveres den 24/11, ej obligatorisk)
- Fredag, 17/11: **SIMPLEX algorithm**, afsnit 29.3 og 29.5
  - Opgaver til 2. øvelsesgang: 29.3-2,-3,-4,-5,-6, 29.5-3,-4,-5.
- Tirsdag, 21/11: **Duality**, afsnit 29.4
  - Opgaver til 3. øvelsesgang: 29.4-1, 29.4-5, 29-2, VA-P2 (hjemmeopgave, udleveres ved 3. forelæsning, afleveres den 1/12, ej obligatorisk).

Introduction

- Linear Programming by Example
- Geometric Interpratation
- Linear Programming – Brief History
- Standard and Slack Formulations
- SIMPLEX by Example

Diet Problem (after Chvatal)

- Every day Polly needs:
  - 2000 kcal,
  - 55g protein,
  - 800mg calcium.
- She will get other stuff (e.g., iron and vitamins) by taking pills. Not that this could not be included in the model – we just want to keep it simple.
- She wants a diet that will meet the requirements while being neither expensive nor boring.

Value and Price per Serving

Food	Energy (kcal)	Protein (g)	Calcium (mg)	Price per serving
Oatmeal	110	4	2	3
Chicken	205	32	12	24
Eggs	160	13	54	13
Whole milk	160	8	285	9
Cherry pie	420	4	22	20
Pork with beans	260	14	80	19

10 portions of pork with beans would cover her needs!
And would cost only 190. But ...

Limits to What Polly Can Stomach

- Oatmeal: at most 4 servings a day.
- Chicken: at most 3 servings a day.
- Eggs: at most 2 servings a day.
- Milk: at most 8 servings a day.
- Cherry pie: at most 2 servings a day.
- Pork with beans: at most 2 servings a day.

8 servings of milk and 2 servings of cherry pie would meet her needs. Boring but she could stomach it. Especially since it would cost 112.
Can she find a less expensive diet?

Variables

- X_1 : number of oatmeal servings.
- X_2 : number of chicken servings.
- X_3 : number of eggs servings.
- X_4 : number of milk servings.
- X_5 : number of cherry pie servings.
- X_6 : number of pork and pie servings.

Linear Constraints

$$
\begin{array}{ccccccccccccc}
x_1 & & & & & & & & & & & \leq & 4 \\
& & x_2 & & & & & & & & & \leq & 3 \\
& & & & x_3 & & & & & & & \leq & 2 \\
& & & & & & x_4 & & & & & \leq & 8 \\
& & & & & & & & x_5 & & & \leq & 2 \\
& & & & & & & & & & x_6 & \leq & 2 \\
110x_1 & + & 205x_2 & + & 160x_3 & + & 160x_4 & + & 420x_5 & + & 260x_6 & \geq & 2000 \\
4x_1 & + & 32x_2 & + & 13x_3 & + & 8x_4 & + & 4x_5 & + & 14x_6 & \geq & 55 \\
2x_1 & + & 12x_2 & + & 54x_3 & + & 285x_4 & + & 22x_5 & + & 80x_6 & \geq & 800 \\
x_1, & & x_2, & & x_3, & & x_4, & & x_5, & & x_6 & \geq & 0
\end{array}
$$

Linear Programming Problem

$$
\begin{array}{lccccccccccccc}
\min & 3x_1 & + & 24x_2 & + & 13x_3 & + & 9x_4 & + & 20x_5 & + & 19x_6 & & \\
s.t. & x_1 & & & & & & & & & & & \leq & 4 \\
 & & & x_2 & & & & & & & & & \leq & 3 \\
 & & & & & x_3 & & & & & & & \leq & 2 \\
 & & & & & & & x_4 & & & & & \leq & 8 \\
 & & & & & & & & & x_5 & & & \leq & 2 \\
 & & & & & & & & & & & x_6 & \leq & 2 \\
 & 110x_1 & + & 205x_2 & + & 160x_3 & + & 160x_4 & + & 420x_5 & + & 260x_6 & \geq & 2000 \\
 & 4x_1 & + & 32x_2 & + & 13x_3 & + & 8x_4 & + & 4x_5 & + & 14x_6 & \geq & 55 \\
 & 2x_1 & + & 12x_2 & + & 54x_3 & + & 285x_4 & + & 22x_5 & + & 80x_6 & \geq & 800 \\
 & x_1, & & x_2, & & x_3, & & x_4, & & x_5, & & x_6 & \geq & 0
\end{array}
$$

Linear Objective Function

$$\begin{aligned} \min \quad & 3x_1 + 24x_2 + 13x_3 + 9x_4 + 20x_5 + 19x_6 \\ s.t. \quad & \text{linear constraints} \end{aligned}$$

- The value of the objective function for a particular set of values for x_1, x_2, x_3, x_4, x_5, x_6 is called its **objective value**.
- If a particular set of values for x_1, x_2, x_3, x_4, x_5, x_6 satisfies all constraints, it is said to be a **feasible solution** (dansk: tilladt løsning). The set of all feasible solutions is called the **feasible region** (dansk: tilladt område). It can be shown to be convex.
- A feasible solution that has the maximum (or minimum) objective value is called an **optimal solution**.

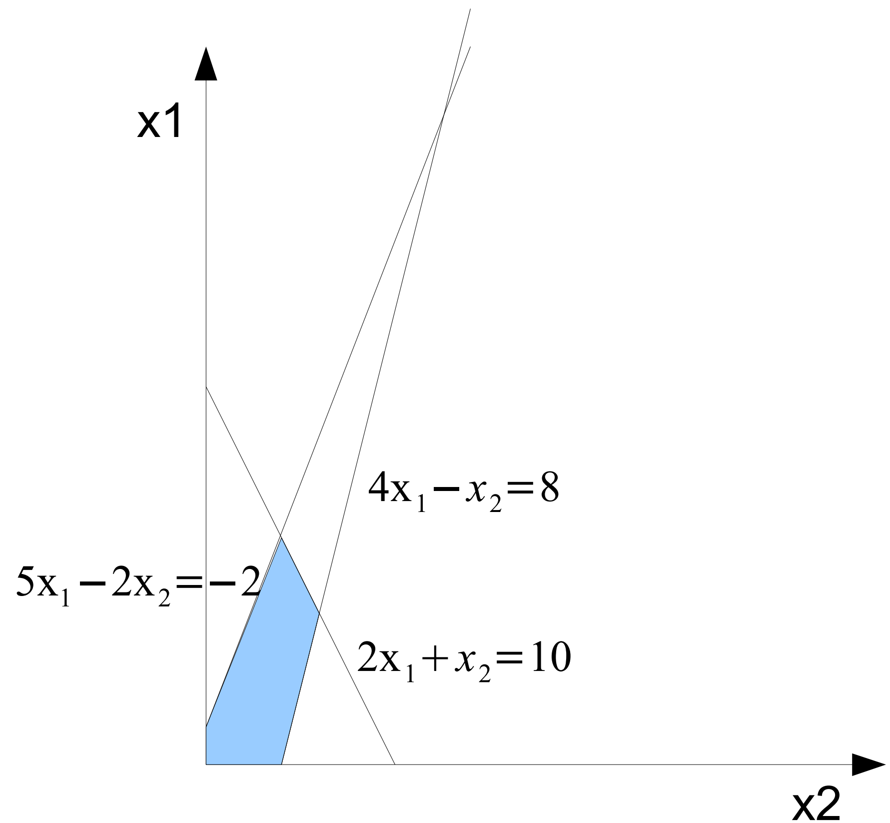
General LP Problem

$$\begin{aligned} &min \quad \sum_{j=1}^{n} c_j x_j \\ &s.t. \quad \text{m linear constraints} \end{aligned}$$

- Minimization or maximization of a linear objective function with *n* real-valued variables.
- An optimal solution must satisfy *m* linear constraints (inequalities or equalities).
- Strict inequalities are not allowed.
- ”programming” in ”linear programming” does not refer to any code. It was chosen before computer programming was born.

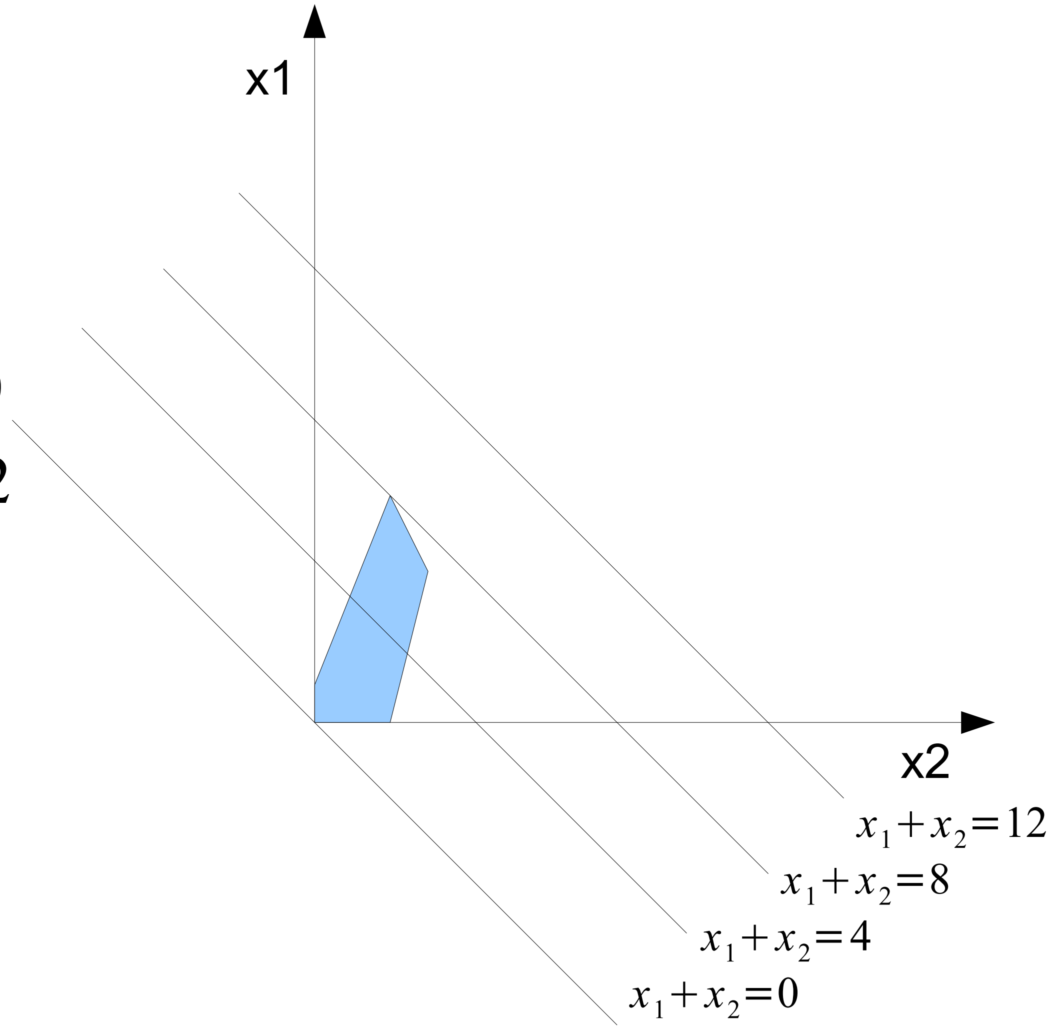
Geometric Interpretation

$$\begin{array}{lcccccc} max & x_1 & + & x_2 & & \\ s.t. & 4x_1 & - & x_2 & \leq & 8 \\ & 2x_1 & + & x_2 & \leq & 10 \\ & 5x_1 & - & 2x_2 & \geq & -2 \\ & x_1, & & x_2 & \geq & 0 \end{array}$$



Geometric Interpretation

$$
\begin{array}{llcccc}
max & x_1 & + & x_2 & & \\
s.t. & 4x_1 & - & x_2 & \leq & 8 \\
 & 2x_1 & + & x_2 & \leq & 10 \\
 & 5x_1 & - & 2x_2 & \geq & -2 \\
 & x_1, & & x_2 & \geq & 0
\end{array}
$$



Special Cases of LP

- LP may have no feasible solution (in case of conflicting constraints).
- LP may have feasible solutions but no optimal solution (in case of unboundedness).
- LP may have more than one optimal solution.

Geometric Interpretation in R^3

- 3 variables.
- Each constraint defines a half-space in R^3. The set of feasible solutions is the intersection of these half-spaces. It is convex. Can be unbounded or empty.
- The set of points in which the objective function has the same value *z* is a plane.
- The value of the objective function increases as the plane is translated in one normal direction and it decreases as it is translated in the other normal direction.
- If the set of feasible solutions is bounded and not empty, then there is an optimal solution in an extreme vertex of the convex set of feasible solutions.

Geometric Interpretation in R^d

- *d* variables.
- Each constraint defines a half-space in R^d. The set of feasible solutions is the intersection of these half-spaces, called **simplex**. It is convex. Can be unbounded or empty.
- The set of points in which the objective function has the same value *z* is a **hyperplane**.
- The value of the objective function increases or decreases as the hyperplane is translated.
- If the set of feasible solutions is bounded and not empty, then there is an optimal solution in an extreme vertex of the simplex.

General Idea Behind SIMPLEX Algorithm

- SIMPLEX starts with a feasible solution corresponding to some vertex of the simplex. We will show how to find such a vertex (or decide that the feasible region is empty).
- SIMPLEX keeps ”jumping” from a vertex of the simplex to a new vertex if the new vertex offers a feasible solution that is better (or at least not worse). We will show how SIMPLEX ”jumps”.
- When no more ”jumps” are possible, we will show that SIMPLEX is in an optimal vertex (or the LP is unbounded)..

History of LP

- L.V. Kantorovich pointed out in 1939 the importance of restricted classes of LPs.
- T.C. Koopmans realized in 1947 the importance of LP for the analysis of classical economic theories.
- G.B. Dantzig designed in 1947 the simplex method to solve LP for U.S. Air Force. Not a polynomial algorithm!
- Many applications followed over the years.
- In 1975 Kantorovich and Koopmans got the Nobel prize.
- L.G. Khachian discovered first polynomial algorithm in 1979. Terribly slow.
- N. Karmarkar discovered second polynomial algorithm in 1984. Practical.

Applications of LP

- Scheduling problems: airline wishes to schedule its flight crews on all flights while using as few crew members as possible.
- Location problems: Locating drills to maximize the amount of oil that will be extracted under given budget constraints.
- Many network and graph problems can be formulated as LP.
- Integer programming problems.

Shortest Path as LP Problem

- **Given**: Weighted, directed graph $G = (V,E)$ with real-valued weights $w(u,v)$ on all edges $e=(u,v)$ in E, a source vertex s and a destination vertex t.
- **Find**: Shortest distance $d[t]$ from s to t.

- Bellman-Ford algorithm: $d[t]$ is the shortest distance from s to t if and anly if
  - no edge $e=(u,v)$ can be relaxed: $d[v] \leq d[u] + w(u,v)$
  - $d[s] = 0$.

Shortest Path as LP

- maximize d[t]

  subject to

  $d[v] \le d[u] + w(u,v),\ \forall\ (u,v) \in E.$

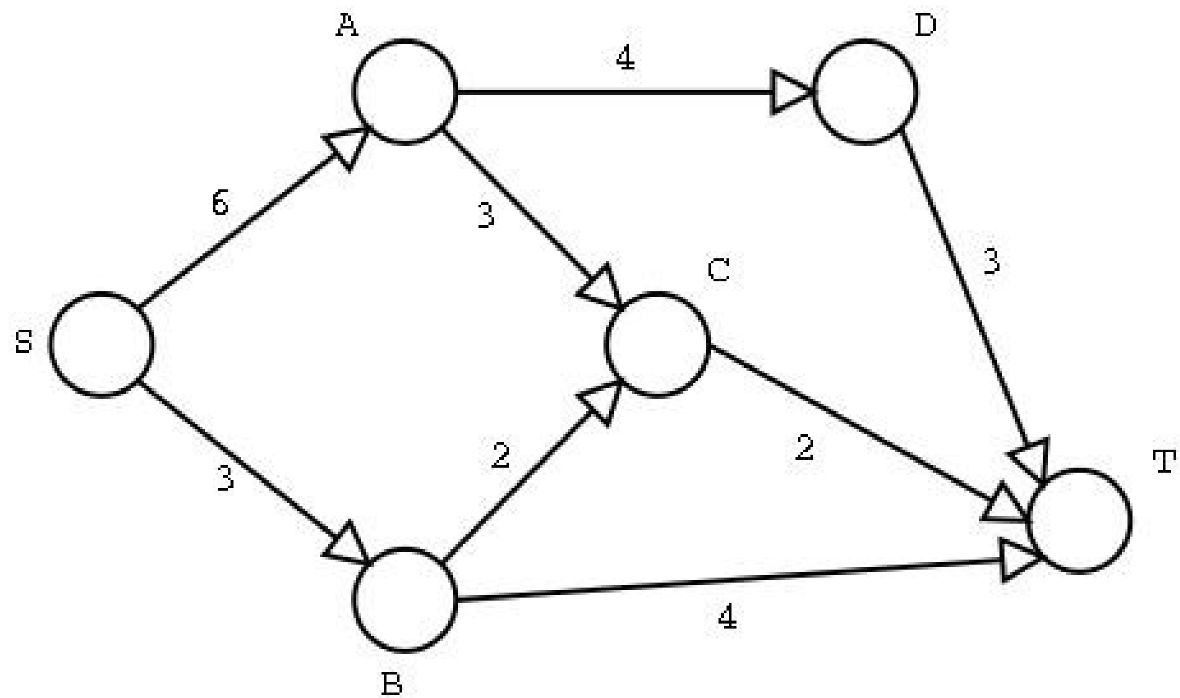
  $d[s] = 0.$

- In particular, at least one of the constraints $d[t] \le d[u] + w(u,t)$, $(u,t) \in E$ must be tight. So we have to maximize $d[t]$

$$
\begin{array}{llllll}
max & x_t & & & & \\
s.t. & x_v & \le & x_u & + & w_{uv}, \quad \forall (u,v) \in E \\
 & x_s & = & 0 & &
\end{array}
$$

Maximum Flow

- **Given**: A directed graph $G = (V,E)$ where each edge $(u,v) \in E$ has a real-valued, nonnegative capacity $c(u,v)$, a source vertex s and a destination vertex t.
- **Find**: A **maximum flow** $f: V \times V \rightarrow \boldsymbol{R}$ from s to t



Flow

- **Given**: A directed graph $G = (V,E)$ where each edge $(u,v) \in E$ has a real-valued, nonnegative **capacity** $c(u,v)$, a **source** vertex s and a **destination** vertex t.
- A **flow** from s to t in G is a real-valued function $f: V \times V \to \mathbf{R}$ satisfying:
  - Capacity constraints: $f(u,v) \leq c(u,v), \forall u, v \in V.$
  - Skew symmetry: $f(u,v) = - f(v,u), \forall u, v \in V.$
  - Flow conservation: $\sum_{v \in V} f(u,v) = 0, \forall u \in V \setminus \{s,t\}$
- Flow **value** $|f|$ is defined as

$$\sum_{v \in V} f(s,v)$$

Equivalent LPs

- Two maximization LPs *L* and *L*' are **equivalent** iff for each feasible solution ***x*** to *L* with the objective value z, there is a corresponding feasible solution ***x'*** to *L*' with the same objective value *z*, and vice versa.
- Similarly for two minimization LPs.
- A minimization LP *L* and a maximization LP *L*' are **equivalent** iff for each feasible solution ***x*** to *L* with the objective value *z* , there is a corresponding feasible solution *x*' to *L*' with the objective value -*z*.

LP in Standard Form

- Maximization of a linear function.
- *n* non-negative real-valued variables.
- *m* linear inequalities (”less than or equal to”).

$$
\begin{aligned}
&max && \sum_{j=1}^{n} c_j x_j && \\
&s.t. && \sum_{j=1}^{n} a_{ij} x_j \leq b_i && for \quad i = 1, 2, \ldots, m \\
& && x_j \geq 0 && for \quad j = 1, 2, \ldots, n
\end{aligned}
$$

Converting LP into Standard Form

$$
\begin{array}{lrcrcl}
min & -2x_1 & + & 3x_2 & & \\
s.t. & x_1 & + & x_2 & = & 7 \\
 & x_1 & - & 2x_2 & \leq & 4 \\
 & & & x_1 & \geq & 0
\end{array}
$$

- Minimization LP is converted to an equivalent maximization problem by negating the coefficients of the objective function.

$$
\begin{array}{lrcrcl}
max & 2x_1 & - & 3x_2 & & \\
s.t. & x_1 & + & x_2 & = & 7 \\
 & x_1 & - & 2x_2 & \leq & 4 \\
 & & & x_1 & \geq & 0
\end{array}
$$

Converting LP into Standard Form

$$\begin{array}{lccccc} max & 2x_1 & - & 3x_2 & & \\ s.t. & x_1 & + & x_2 & = & 7 \\ & x_1 & - & 2x_2 & \leq & 4 \\ & x_1 & & & \geq & 0 \end{array}$$

- Every variable x_j without non-negativity constraint is replaced by two non-negative variables x'_j and x''_j and each occurrence of x_j is replaced by $x'_j - x''_j$.

$$\begin{array}{lccccccc} max & 2x_1 & - & 3x'_2 & + & 3x''_2 & & \\ s.t. & x_1 & + & x'_2 & - & x''_2 & = & 7 \\ & x_1 & - & 2x'_2 & + & 2x''_2 & \leq & 4 \\ & x_1, & & x'_2, & & x''_2 & \geq & 0 \end{array}$$

Converting LP into Standard Form

$$
\begin{array}{llllllll}
max & 2x_1 & - & 3x_2' & + & 3x_2'' & & \\
s.t. & x_1 & + & x_2' & - & x_2'' & = & 7 \\
 & x_1 & - & 2x_2' & + & 2x_2'' & \leq & 4 \\
 & x_1, & & x_2', & & x_2'' & \geq & 0
\end{array}
$$

- Each equality constraint is replaced by a pair of ”opposite” inequality constraints.

$$
\begin{array}{llllllll}
max & 2x_1 & - & 3x_2' & + & 3x_2'' & & \\
s.t. & x_1 & + & x_2' & - & x_2'' & \leq & 7 \\
 & x_1 & + & x_2' & - & x_2'' & \geq & 7 \\
 & x_1 & - & 2x_2' & + & 2x_2'' & \leq & 4 \\
 & x_1, & & x_2', & & x_2'' & \geq & 0
\end{array}
$$

Converting LP into Standard Form

$$
\begin{array}{lccccccc}
max & 2x_1 & - & 3x_2' & + & 3x_2'' & & \\
s.t. & x_1 & + & x_2' & - & x_2'' & \leq & 7 \\
 & x_1 & + & x_2' & - & x_2'' & \geq & 7 \\
 & x_1 & - & 2x_2' & + & 2x_2'' & \leq & 4 \\
 & x_1, & & x_2', & & x_2'' & \geq & 0
\end{array}
$$

- Inequalities are ”turned around” by multiplying both sides by -1.

$$
\begin{array}{lccccccc}
max & 2x_1 & - & 3x_2' & + & 3x_2'' & & \\
s.t. & x_1 & + & x_2' & - & x_2'' & \leq & 7 \\
 & -x_1 & - & x_2' & + & x_2'' & \leq & -7 \\
 & x_1 & - & 2x_2' & + & 2x_2'' & \leq & 4 \\
 & x_1, & & x_2', & & x_2'' & \geq & 0
\end{array}
$$

Converting LP into a Standard Form

$$
\begin{array}{lccccccc}
max & 2x_1 & - & 3x_2' & + & 3x_2'' & & \\
s.t. & x_1 & + & x_2' & - & x_2'' & \leq & 7 \\
 & -x_1 & - & x_2' & + & x_2'' & \leq & -7 \\
 & x_1 & - & 2x_2' & + & 2x_2'' & \leq & 4 \\
 & x_1, & & x_2', & & x_2'' & \geq & 0
\end{array}
$$

- Renaming the variables

$$
\begin{array}{lccccccc}
max & 2x_1 & - & 3x_2 & + & 3x_3 & & \\
s.t. & x_1 & + & x_2 & - & x_3 & \leq & 7 \\
 & -x_1 & - & x_2 & + & x_3 & \leq & -7 \\
 & x_1 & - & 2x_2 & + & 2x_3 & \leq & 4 \\
 & x_1, & & x_2, & & x_3 & \geq & 0
\end{array}
$$

LP in Standard Form

- *n* variables, *m* constraints

$$\begin{aligned} max \quad & \sum_{j=1}^{n} c_j x_j \\ s.t. \quad & \sum_{j=1}^{n} a_{ij} x_j \quad \leq \quad b_i \quad for \quad i = 1, 2, \ldots, m \\ & \qquad\quad x_j \quad \geq \quad 0 \quad for \quad j = 1, 2, \ldots, n \end{aligned}$$

Slack Variables (Overskudsvariable)

- Consider one of the constraints, for example

$$2x_1 + 3x_2 + x_3 \leq 5$$

- For every feasible solution x_1, x_2, x_3 , the value of the left-hand side is at most the value of the right-hand side.
- Often there can be a **slack** between these two values.
- Denote the slack by x_4.
- By requiring that $x_4 \geq 0$, we can replace the inequality by the equality

$$2x_1 + 3x_2 + x_3 + x_4 = 5$$

Slack Variables

$$
\begin{aligned}
max \quad & \sum_{j=1}^{n} c_j x_j \\
s.t. \quad & \sum_{j=1}^{n} a_{ij} x_j \leq b_i \quad & for \quad i=1,2,\ldots,m \\
& x_j \geq 0 \quad & for \quad j=1,2,\ldots,n
\end{aligned}
$$

$$
\begin{aligned}
max \quad & \sum_{j=1}^{n} c_j x_j \\
s.t. \quad & \sum_{j=1}^{n} a_{ij} x_j + x_{n+i} = b_i \quad & for \quad i=1,2,\ldots,m \\
& x_j \geq 0 \quad & for \quad j=1,2,\ldots,n+m
\end{aligned}
$$

LP in Slack Form

$$
\begin{array}{llll}
max & \sum_{j=1}^{n} c_j x_j & & \\
s.t. & \sum_{j=1}^{n} a_{ij} x_j + x_{n+i} = b_i & for & i=1,2,\ldots,m \\
 & x_j \geq 0 & for & j=1,2,\ldots,n+m
\end{array}
$$

$$
\begin{array}{lllll}
max & z & = & \sum_{j=1}^{n} c_j x_j & \\
s.t. & x_{n+i} & = & b_i - \sum_{j=1}^{n} a_{ij} x_j & for \quad i=1,2,\ldots,m \\
 & x_j & \geq & 0 & for \quad j=1,2,\ldots,n+m
\end{array}
$$

$$
\begin{array}{llll}
z & = & 0 + \sum_{j=1}^{n} c_j x_j & \\
x_{n+i} & = & b_i - \sum_{j=1}^{n} a_{ij} x_j & for \quad i=1,2,\ldots,m
\end{array}
$$

Standard to Slack Form - Example

$$
\begin{array}{lccccccc}
max & 2x_1 & - & 3x_2 & + & 3x_3 & & \\
s.t. & x_1 & + & x_2 & - & x_3 & \leq & 7 \\
 & -x_1 & - & x_2 & + & x_3 & \leq & -7 \\
 & x_1 & - & 2x_2 & + & 2x_3 & \leq & 4
\end{array}
$$

$$
\begin{array}{lcccccccccccc}
max & 2x_1 & - & 3x_2 & + & 3x_3 & & & & & & & \\
s.t. & x_1 & + & x_2 & - & x_3 & + & x_4 & & & & = & 7 \\
 & -x_1 & - & x_2 & + & x_3 & & & + x_5 & & & = & -7 \\
 & x_1 & - & 2x_2 & + & 2x_3 & & & & + & x_6 & = & 4
\end{array}
$$

$$
\begin{array}{ccccccccc}
z & = & 0 & + & 2x_1 & - & 3x_2 & + & 3x_3 \\
x_4 & = & 7 & - & x_1 & - & x_2 & + & x_3 \\
x_5 & = & -7 & + & x_1 & + & x_2 & - & x_3 \\
x_6 & = & 4 & - & x_1 & + & 2x_2 & - & 2x_3
\end{array}
$$

Basic Solutions

- Any solution of LP in the standard form yields a solution of LP in the corresponding slack form (with the same objective value) and vice versa.
- Setting right-hand side variables of the slack form to 0 yields a **basic solution.**
- Left-hand side variables are called **basic**. Right-hand side variables are called **nonbasic**.
- The basic variables are said to constitute a **basis.**
- Note that a basic solution does not need to be feasible.

SIMPLEX - Example

- LP problem in standard form:

$$
\begin{array}{lccccccc}
max & 3x_1 & + & x_2 & + & 2x_3 & & \\
s.t. & x_1 & + & x_2 & + & 3x_3 & \leq & 30 \\
 & 2x_1 & + & 2x_2 & + & 5x_3 & \leq & 24 \\
 & 4x_1 & + & x_2 & + & 2x_3 & \leq & 36 \\
 & x_1 & , & x_2 & , & x_3 & \geq & 0
\end{array}
$$

SIMPLEX – Example Continued

- LP in slack form:

$$
\begin{array}{lclclclcl}
z & = & 0 & + & 3x_1 & + & x_2 & + & 2x_3 \\
x_4 & = & 30 & - & x_1 & - & x_2 & - & 3x_3 \\
x_5 & = & 24 & - & 2x_1 & - & 2x_2 & - & 5x_3 \\
x_6 & = & 36 & - & 4x_1 & - & x_2 & - & 2x_3
\end{array}
$$

- Set all **nonbasic** variables (right-hand side) to 0.
- Compute values of **basic** variables: x_4=30, x_5=24, x_6=36.
- Compute the objective value z (= 0).
- This gives the feasible basic solution (0,0,0,30,24,36).
- It is feasible; not always the case – we were lucky.

SIMPLEX: 1. Pivoting

- Can x_1 be increased without violating feasibility?

$$
\begin{array}{ccccccccc}
z & = & 0 & + & 3x_1 & + & x_2 & + & 2x_3 \\
x_4 & = & 30 & - & x_1 & - & x_2 & - & 3x_3 \\
x_5 & = & 24 & - & 2x_1 & - & 2x_2 & - & 5x_3 \\
x_6 & = & 36 & - & 4x_1 & - & x_2 & - & 2x_3
\end{array}
$$

- If x_1 is increased to 1, then x_4=29, x_5=22, x_6=32 while z=3. (1,0,0,29,22,32) is a feasible solution.
- If x_1 is increased to 2, then x_4=28, x_5=20, x_6=28 while z=6. (2,0,0,28,20,28) is a feasible solution.
- If x_1 is increased to 3, then x_4=27, x_5=18, x_6=24 while z=9. (3,0,0,27,18,24) is a feasible solution.

SIMPLEX: 1. Pivoting

- Can x_1 be increased without violating feasibility? By how much?

$$
\begin{array}{ccccccccc}
z & = & 0 & + & 3x_1 & + & x_2 & + & 2x_3 \\
x_4 & = & 30 & - & x_1 & - & x_2 & - & 3x_3 \\
x_5 & = & 24 & - & 2x_1 & - & 2x_2 & - & 5x_3 \\
x_6 & = & 36 & - & 4x_1 & - & x_2 & - & 2x_3
\end{array}
$$

- If x_1 is increased beyond 30 then x_4 becomes negative.
- If x_1 is increased beyond 12 then x_5 becomes negative.
- If x_1 is increased beyond 9 then x_6 becomes negative.
- Constraint defining x_6 is **binding**.

SIMPLEX: 1. Pivoting

- So x_1 can be increased to 9 without losing feasibility. The feasible solution is (9,0,0,21,6,0).
- We will now rewrite the slack form to an equivalent slack form with x_1, x_4, x_5 as basic variables and with (9,0,0,21,6,0) being its feasible basic solution.
- This rewriting is called **pivoting.**
- Binding constraint defining x_6 is rewritten so that it has x_1 on its left-hand side.
- All occurences of x_1 in other constraints and in the objective function are replaced by the right-hand side of the binding constraint.

SIMPLEX: 1. Pivoting

$$
\begin{array}{llllllll}
z & = & 0 & + & 3(9 - x_2/4 - x_3/2 - x_6/4) & + & x_2 & + & 2x_3 \\
x_4 & = & 30 & - & (9 - x_2/4 - x_3/2 - x_6/4) & - & x_2 & - & 3x_3 \\
x_5 & = & 24 & - & 2(9 - x_2/4 - x_3/2 - x_6/4) & - & 2x_2 & - & 5x_3 \\
x_1 & = & 9 & - & x_2/4 & - & x_3/2 & - & x_6/4
\end{array}
$$

$$
\begin{array}{lllllllll}
z & = & 27 & + & x_2/4 & + & x_3/2 & - & 3x_6/4 \\
x_4 & = & 21 & - & 3x_2/4 & - & 5x_3/2 & + & x_6/4 \\
x_5 & = & 6 & - & 3x_2/2 & - & 4x_3 & + & x_6/2 \\
x_1 & = & 9 & - & x_2/4 & - & x_3/2 & - & x_6/4
\end{array}
$$

New basic variables: $x_1=9$, $x_4=21$, $x_5=6$

New objective value $z = 27$

New feasible basic solution: (9,0,0,21,6,0)

SIMPLEX: 2. Pivoting

- Can x_3 be increased without violating feasibility? By how much?

$$
\begin{array}{ccccccccc}
z & = & 27 & + & x_2/4 & + & x_3/2 & - & 3x_6/4 \\
x_4 & = & 21 & - & 3x_2/4 & - & 5x_3/2 & + & x_6/4 \\
x_5 & = & 6 & - & 3x_2/2 & - & 4x_3 & + & x_6/2 \\
x_1 & = & 9 & - & x_2/4 & - & x_3/2 & - & x_6/4
\end{array}
$$

- If x_3 is increased beyond 42/5 then $x_4 < 0$.
- If x_3 is increased beyond 3/2 then $x_5 < 0$.
- If x_3 is increased beyond 9/2 then $x_1 < 0$.
- Constraint defining x_5 is binding.

SIMPLEX: 2. Pivoting

$$
\begin{array}{lllllllll}
z & = & 27 & + & x_2/4 & + & \frac{1}{2}(3/2-3x_2/8+x_6/8-x_5/4) & - & 3x_6/4 \\
x_4 & = & 21 & - & 3x_2/4 & - & \frac{5}{2}(3/2-3x_2/8+x_6/8-x_5/4) & + & x_6/4 \\
x_3 & = & 3/2 & - & 3x_2/8 & - & x_5/4 & + & x_6/8 \\
x_1 & = & 9 & - & x_2/4 & - & \frac{1}{2}(3/2-3x_2/8+x_6/8-x_5/4) & - & x_6/4
\end{array}
$$

$$
\begin{array}{lllllllll}
z & = & 111/4 & + & x_2/16 & - & x_5/8 & - & 11x_6/16 \\
x_4 & = & 69/4 & + & 3x_2/16 & + & 5x_5/8 & - & x_6/16 \\
x_3 & = & 3/2 & - & 3x_2/8 & - & x_5/4 & + & x_6/8 \\
x_1 & = & 33/4 & - & x_2/16 & + & x_5/8 & - & 5x_6/16
\end{array}
$$

New basic variables: $x_1=33/4$, $x_3=3/2$, $x_4=69/4$

New objective value z = 27.75

New feasible basic solution: (33/4, 0, 3/2, 69/4, 0, 0)

SIMPLEX: 3. Pivoting

- Can x_2 be increased without violating feasibility? By how much?

$$
\begin{array}{ccccccccc}
z & = & 111/4 & + & x_2/16 & - & x_5/8 & - & 11x_6/16 \\
x_4 & = & 69/4 & + & 3x_2/16 & + & 5x_5/8 & - & x_6/16 \\
x_3 & = & 3/2 & - & 3x_2/8 & - & x_5/4 & + & x_6/8 \\
x_1 & = & 33/4 & - & x_2/16 & + & x_5/8 & - & 5x_6/16
\end{array}
$$

- If x_2 is increased then x_4 also increases.
- If x_2 is increased beyond 4 then $x_3 < 0$.
- If x_2 is increased beyond 132 then $x_1 < 0$.
- Constraint defining x_3 is binding.

SIMPLEX: 3.Pivoting

$$
\begin{array}{lllllllll}
z & = & 111/4 & + & \frac{1}{16}(4-8x_3/3-2x_5/3+x_6/3) & - & x_5/8 & - & 11x_6/16 \\
x_4 & = & 69/4 & + & \frac{1}{16}(4-8x_3/3-2x_5/3+x_6/3) & + & 5x_5/8 & - & x_6/16 \\
x_2 & = & 4 & - & 8x_3/3 & - & 2x_5/3 & + & x_6/3 \\
x_1 & = & 33/4 & - & \frac{1}{16}(4-8x_3/3-2x_5/3+x_6/3) & + & x_5/8 & - & 5x_6/16
\end{array}
$$

$$
\begin{array}{lllllllll}
z & = & 28 & - & x_3/6 & - & x_5/6 & - & 2x_6/3 \\
x_4 & = & 18 & - & x_3/2 & + & x_5/2 & + & 0x_6 \\
x_2 & = & 4 & - & 8x_3/3 & - & 2x_5/3 & + & x_6/3 \\
x_1 & = & 8 & + & x_3/6 & + & x_5/6 & - & x_6/3
\end{array}
$$

New basic variables: $x_1=8$, $x_2=4$, $x_4=18$

New objective value z = 28

New feasible basic solution: (8, 4, 0, 18, 0, 0) is optimal

Pivoting in General

- PIVOT(*N*,*B*,*A*,*b*,*c*,*v*,*l*,*e*)
  - Compute the coefficients of the bounding constraint so that the **entering** basic variable x_e is expressed as a linear combination of the other variables.

$$b_e = b_l / a_{le} \qquad a_{ej} = a_{lj} / a_{le}, \forall j \in N \setminus e \qquad a_{el} = 1 / a_{le}$$

  - Compute the coefficients of the remaining constraints and the objective function (by substituting x_e by the right-hand side of the rewritten binding equation).

$$b_i = b_i - a_{ie} b_e, \forall i \in B \setminus l \quad a_{ij} = a_{ij} - a_{ie} a_{ej}, \forall j \in N \setminus e \quad a_{il} = -a_{ie} a_{el}$$

$$v = v + c_e b_e \qquad c_j = c_j - c_e a_{ej}, \forall j \in N \setminus e \qquad c_l = -c_e a_{el}$$

  - Compute new sets of basic and nonbasic variables (remove x_e from *N* and add it to *B*, remove x_l from *B* and add it to *N*).

SIMPLEX – Open Issues

- How to decide that LP is feasible?
- What to do if the initial basic solution is infeasible?
- How to decide that LP is unbounded?
- How to select entering and leaving variables?
- Does SIMPLEX terminate?
- Does it terminate with an optimal solution?