A coalgebraic approach to Kleene algebra with tests

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Abstract

Kleene algebra with tests is an extension of Kleene algebra, the algebra of regular expressions, which can be used to reason about programs. We develop a coalgebraic theory of Kleene algebra with Tests, along the lines of the coalgebraic theory of regular expressions based on deterministic automata. Since the known automata-theoretic presentation of Kleene algebra with tests does not lend itself to a coalgebraic theory, we define a new interpretation of Kleene algebra with tests expressions and a corresponding automata-theoretic presentation. One outcome of the theory is a coinductive proof principle, that can be used to establish equivalence of our Kleene algebra with tests expressions.

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1. Introduction

Kleene algebra (KA) is the algebra of regular expressions [2,5]. As is well-known, the theory of regular expressions enjoys a strong connection with the theory of finite-state automata. This connection was used by Rutten [13] to give a coalgebraic treatment of regular expressions. One of the fruits of this coalgebraic treatment is coinduction, a proof technique for demonstrating the equivalence of regular expressions [15]. Other methods for
proving the equality of regular expressions have previously been established—for instance, reasoning by using a sound and complete axiomatization \[6,16\], or by minimization of automata representing the expressions \[3\]. However, the coinduction proof technique can give relatively short proofs, and is fairly simple to apply.

Recently, Kozen \[7\] introduced Kleene algebra with tests (\textsc{Kat}), an extension of \textsc{ka} designed for the particular purpose of reasoning about programs and their properties. The regular expressions of \textsc{kat} allow one to intersperse boolean tests along with program actions, permitting the convenient modelling of programming constructs such as conditionals and \textit{while} loops. The utility of \textsc{kat} is evidenced by the fact that it subsumes propositional Hoare logic, providing a complete deductive system for Hoare-style inference rules for partial correctness assertions \[9\].

The goal of this paper is to develop a coalgebraic theory of \textsc{kat}, paralleling the coalgebraic treatment of \textsc{ka}. Our coalgebraic theory yields a coinductive proof principle for demonstrating the equality of \textsc{kat} expressions, in analogy to the coinductive proof principle for regular expressions. The development of our coalgebraic theory proceeds as follows. We first introduce a form of deterministic automaton and define the language accepted by such an automaton. Next, we develop the theory of such automata, showing that coinduction can be applied to the class of languages representable by our automata. We then give a class of expressions, which play the same role as the regular expressions in classical automata theory, and fairly simple rules for computing derivatives of these expressions.

The difficulty of our endeavor is that the known automata-theoretic presentation of \textsc{kat} \[11\] does not lend itself to a coalgebraic theory. Moreover, the notion of derivative, essential to the coinduction proof principle in this context, is not readily definable for \textsc{kat} expressions as they are defined by Kozen \[7\]. Roughly, these difficulties arise from tests being commutative and idempotent, and suggest that tests need to be handled in a special way. In order for the coalgebraic theory to interact smoothly with tests, we introduce a \textit{type system} along with new notions of strings, languages, automata, and expressions, which we call \textit{mixed strings}, \textit{mixed languages}, \textit{mixed automata}, and \textit{mixed expressions}, respectively. \(\text{(We note that none of these new notions coincide with those already developed in the theory of \textsc{kat}.)}\) All well-formed instances of these notions can be assigned types by our type system. Our type system is inspired by the type system devised by Kozen \[8,10\] for \textsc{ka} and \textsc{kat}, but is designed to address different issues.

This paper is structured as follows. In the next section, we introduce mixed strings and mixed languages, which will be used to interpret our mixed expressions. In Section 3, we define a notion of mixed automaton that is used to accept mixed languages. We then impose a coalgebraic structure on such automata. In Section 4, we introduce a sufficient condition for proving equivalence that is more convenient than the condition that we derive in Section 3. In Section 5, we introduce our type system for \textsc{kat}, and connect typed \textsc{kat} expressions with the mixed language they accept. In Section 6, we give an example of how to use the coalgebraic theory, via the coinductive proof principle, to establish equivalence of typed \textsc{kat} expressions. In Section 7, we show that our technique is complete, that is, it can establish the equivalence of any two typed \textsc{kat} expressions that are in fact equivalent. We conclude in Section 8 with considerations of future work.
2. Mixed languages

In this section, we define the notions of mixed strings and mixed languages that we will use throughout the paper. Mixed strings are a variant of the guarded strings introduced by Kaplan [4] as an abstract interpretation for program schemes; sets of guarded strings were used by Kozen [11] as canonical models for Kleene algebra with tests. Roughly speaking, a guarded string can be understood as a computation where atomic actions are executed amidst the checking of conditions, in the form of boolean tests. Mixed strings will be used as an interpretation for the mixed expressions we introduce in Section 5.

Mixed strings are defined over two alphabets: a set of primitive programs (denoted $P$) and a set of primitive tests (denoted $B$). We allow $P$ to be infinite, but require that $B$ be finite. (We will see in Section 3 where this finiteness assumption comes in. Intuitively, this is because our automata will process each primitive test individually.) Primitive tests can be put together to form more complicated tests. A literal $l$ is a primitive test $b \in B$ or its negation $\overline{b}$; the underlying primitive test $b$ is said to be the base of the literal, and is denoted by $\text{base}(l)$. When $A$ is a subset of $B$, $\text{lit}(A)$ denotes the set of all literals over $A$. A test is a nonempty set of literals with distinct bases. Intuitively, a test can be understood as the conjunction of the literals it comprises. The base of a test $t$, denoted by $\text{base}(t)$, is defined to be the set $\{\text{base}(l) : l \in t\}$, in other words, the primitive tests the test $t$ is made up from.

We extend the notion of base to primitive programs, by defining the base of a primitive program $p \in P$ as $H_{11083}$.

Example 2.1. Let $P = \{p, q\}$, and $B = \{b, c, d\}$. The literals $\text{lit}(B)$ of $B$ are $\{b, \overline{b}, c, \overline{c}, d, \overline{d}\}$. Tests include $\{b, \overline{c}, d\}$ and $\{\overline{b}, \overline{d}\}$, but $\{b, \overline{b}, c\}$ is not a test, as $b$ and $\overline{b}$ have the same base $b$. The base of $\{b, \overline{c}, d\}$ is $\{b, \overline{c}, d\}$.

Primitive programs and tests are used to create mixed strings. A mixed string is either the empty string, denoted by $\epsilon$, or a sequence $a_1 \ldots a_n$ (where $n \geq 1$) with the following properties:

1. each $a_i$ is either a test or primitive program,
2. for $i = 1, \ldots, n - 1$, if $a_i$ is a test, then $a_{i+1}$ is a primitive program,
3. for $i = 1, \ldots, n - 1$, if $a_i$ is a primitive program, then $a_{i+1}$ is a test, and
4. for $i = 2, \ldots, n - 1$, if $a_i$ is a test, then $\text{base}(a_i) = B$.

Hence, a mixed string is an alternating sequence of primitive programs and tests, where each test in the sequence is a “complete” test, except possibly if it occurs as the first or the last element of the sequence. This allows us to manipulate mixed strings on a finer level of granularity; we can remove literals from the beginning of a mixed strings and still obtain a mixed string. The length of the empty mixed string $\epsilon$ is 0, while the length of a mixed string $a_1 \ldots a_n$ is $n$.

Example 2.2. Let $P = \{p, q\}$, and $B = \{b, c, d\}$. Mixed strings include $\epsilon$ (of length 0), $\{b\}$ and $p$ (both of length 1), and $\{b\}p\{b, \overline{c}, d\}q\{\overline{d}\}$ (of length 5). The sequence $\{b\}p\{b, d\}q\{\overline{d}\}$ is not a mixed string, since $\text{base}(\{b, d\}) \neq B$. 
Example 2.3. Let $\mathcal{P} = \{p, q\}$, and $\mathcal{B} = \{b, c, d\}$. The concatenation of the mixed strings $p$ and $\{b, c, d\}q$ is $p\{b, c, d\}q$. Similarly, the concatenation of the mixed strings $\{b\}p\{b, \bar{c}\}$ and $\{d\}q\{\bar{d}\}$ is the mixed string $\{b\}p\{b, \bar{c}, d\}q\{\bar{d}\}$. However, the concatenation of $\{b\}p\{b, \bar{c}\}$ and $\{b, d\}q$ is not defined, as $\{b, \bar{c}\} \cap \{b, d\} \neq \emptyset$. The concatenation of $\{b\}p\{b, \bar{c}\}$ and $q$ is also not defined, as $\text{base}(\{b, \bar{c}\}) \neq \mathcal{B}$, and thus $\{b\}p\{b, \bar{c}\}q$ is not a mixed string.

We assign one or more types to mixed strings in the following way. A type is of the form $A \rightarrow B$, where $A$ and $B$ are subsets of $\mathcal{B}$. Intuitively, a mixed string has type $A \rightarrow B$ if the first element of the string has base $A$, and it can be concatenated with an element with base $B$. It will be the case that a mixed string of type $A \rightarrow B$ can be concatenated with a mixed string of type $B \rightarrow C$ to obtain a mixed string of type $A \rightarrow C$.

The mixed string $\varepsilon$ has many types, namely it has type $A \rightarrow A$, for all $A \in \wp(\mathcal{B})$. A mixed string of length 1 consisting of a single test $t$ has type $\text{base}(t) \cup A \rightarrow A$, for any $A \in \wp(\mathcal{B})$ such that $A \cap \text{base}(t) = \emptyset$. A mixed string of length 1 consisting of a single program $p$ has type $\emptyset \rightarrow \mathcal{B}$. A mixed string $a_1 \ldots a_n$ of length $n > 1$ has type $\text{base}(a_1) \rightarrow \mathcal{B} \setminus \text{base}(a_n)$.

Example 2.4. Let $\mathcal{P} = \{p, q\}$, and $\mathcal{B} = \{b, c, d\}$. The mixed string $p\{b, \bar{c}, d\}$ has type $\emptyset \rightarrow \emptyset$. The mixed string $\{\bar{d}\}p$ has type $\{d\} \rightarrow \mathcal{B}$. The mixed string $\{b\}p\{b, \bar{c}, d\}q\{b, \bar{c}\}$ has type $\{b\} \rightarrow \{d\}$. The concatenation of $\{b\}p\{b, \bar{c}, d\}q\{b, \bar{c}\}$ and $\{\bar{d}\}p$, namely $\{b\}p\{b, \bar{c}, d\}q\{b, \bar{c}, \bar{d}\}p$, has type $\{b\} \rightarrow \mathcal{B}$.

A mixed language is a set of mixed strings, and is typeable, with type $A \rightarrow B$, if all of the mixed strings it contains have type $A \rightarrow B$. In this paper, we will only be concerned with typeable mixed languages.

We will be interested in different operations on mixed languages in the following sections. When $L_1, L_2$, and $L$ are mixed languages, we use the notation $L_1 \cdot L_2$ to denote the set $\{\sigma_1 \cdot \sigma_2 : \sigma_1 \in L_1, \sigma_2 \in L_2\}$, $L^0$ to denote the set $\{\varepsilon\}$, and for $n \geq 1$, $L^n$ to denote the set $L \cdot L^{n-1}$. The following two operations will be useful in Section 5. The operator $T$, defined by

$$T(L) = \{\sigma : \sigma \in L, |\sigma| = 1, \sigma \text{ is a test}\}$$
extracts from a language all the mixed strings made up of a single test. The operator $\varepsilon$, defined by
\[ \varepsilon(L) = L \cap \{\varepsilon\} \]
especially checks if the empty mixed string $\varepsilon$ is in $L$, since $\varepsilon(L)$ is nonempty if and only if the empty mixed string is in $L$.

3. Mixed automata

Having introduced a notion of mixed strings, we now define a class of deterministic automata that can accept mixed strings. Mixed strings enforce a strict alternation between programs and tests, and this alternation is reflected in our automata. The transitions of the automata are labelled with primitive programs and literals. Given a mixed string, mixed automaton can process the tests in the string in many different orders; this reflects the fact that the tests that appear in mixed strings are sets of literals.

A mixed automaton over the set of primitive programs $P$ and set of primitive tests $B$ is a 3-tuple $M = (\langle S_A \rangle_{A \in \wp(B)}, o, \langle \delta_A \rangle_{A \in \wp(B)})$, consisting of a set $S_A$ of states for each possible base $A \neq \emptyset$ of a test as well as a set $S_{\emptyset}$ of program states, an output function $o : S_{\emptyset} \rightarrow \{0, 1\}$, and transition functions $\delta_A : S_A \times \text{lit}(A) \rightarrow \bigcup_{A \in \wp(B)} S_A$, subject to the following two conditions:

\begin{enumerate}
  \item $\delta_A(s, l) \in S_A\backslash\{\text{base}(l)\}$, and
  \item for every state $s$ in $S_A$, for every test $t$ with base $A$, and for any two orderings $\langle x_1, \ldots, x_m \rangle$, $\langle y_1, \ldots, y_m \rangle$ of the literals in $t$, if $s \xrightarrow{x_1} \ldots \xrightarrow{x_m} s_1$ and $s \xrightarrow{y_1} \ldots \xrightarrow{y_m} s_2$ then $s_1 = s_2$.
\end{enumerate}

(For convenience, we write $s \xrightarrow{l} s'$ if $\delta_A(s, l) = s'$ for $A$ the base of $s$.)

We give an example of a mixed automaton in Example 3.2. Intuitively, a state in $S_A$ can process a mixed string of type $A \rightarrow B$, for some $B$. Condition A1 enforces the invariant that, as a string is being processed, the current state is in $S_A$, for $A$ the base of the first element of the string. Condition A2 is a form of “path independence”: regardless of the order in which we process the literals of a test, we end up in the same program state. Condition A2, and basing transitions on literals rather than tests, allow the manipulation of mixed expressions at a finer level of granularity. This is related to a similar choice we made when allowing mixed strings to start with a test that is not “complete”. This flexibility will be useful when we analyze mixed expressions in Section 5.

The accepting states are defined via the output function $o(s)$, viewed as a characteristic function. Accepting states are in $S_{\emptyset}$.

As in the coalgebraic treatment of automata [13], and contrary to standard definitions, we allow both the state spaces $S_A$ and the set $P$ of primitive programs to be infinite. We also do not force mixed automata to have initial states, for reasons that will become clear.

We now define the mixed language accepted by a state of a mixed automaton. Call a sequence $\mu = e_1 \ldots e_m$ of primitive programs and literals a linearization of a mixed string $\sigma = a_1 \ldots a_n$ if $\mu$ can be obtained from $\sigma$ by replacing each test $a_i$ in $\sigma$ with a sequence of length $|a_i|$ containing exactly the literals in $a_i$. 
Example 3.1. Let $P = \{p, q\}$, and $B = \{b, c\}$. The mixed string $\{b\} p \{\bar{b}, c\} q \{b, \overline{c}\}$ (of type $\{b\} \rightarrow \emptyset$) has four linearizations: $bpbcqb\overline{c}$, $bpcbqb\overline{c}$, $bpbcq\overline{c}b$, and $bpcbq\overline{c}b$.

Intuitively, a mixed string $\sigma$ is accepted by an automaton if a linearization of $\sigma$ is accepted by the automaton according to the usual definition. Formally, a mixed string $\sigma$ is accepted by a state $s$ of an automaton $M$ if either

1. $\sigma$ is $\mathcal{E}$ and $s$ is a program state with $o(s) = 1$ (i.e., $s$ is an accepting program state), or
2. there exists a linearization $e_1 \ldots e_m$ of $\sigma$ such that $s \xrightarrow{e_1} \ldots \xrightarrow{e_m} s'$, $s'$ is a program state, and $o(s') = 1$.

If $\sigma$ is accepted (by a state $s$) in virtue of satisfying the second criterion, then every linearization is a witness to this fact—in other words, the existential quantification in the second criterion could be replaced with a universal quantification (over all linearizations of $\sigma$) without any change in the actual definition. This is because of condition A2 in the definition of a mixed automaton.

We define the mixed language accepted by state $s$ of automaton $M$, written $LM(s)$, as the set of mixed strings accepted by state $s$ of $M$. It is easy to verify that all the strings accepted by a state have the same type, namely, if $s$ is in $S_A$, then every string in $LM(s)$ has type $A \rightarrow \emptyset$, and hence $LM(s)$ has type $A \rightarrow \emptyset$.

Example 3.2. Let $P = \{p, q\}$, and $B = \{b, c\}$. Consider the mixed automaton over $P$ and $B$ pictured in Fig. 1, given by $M = \langle (S_A)_{A \in \mathcal{P}(B)}, o, (\delta_A)_{A \in \mathcal{P}(B)} \rangle$, where:

$$S_{\{b,c\}} = \{s_{2,\{b,c\}}, s_{sink,\{b,c\}}\},$$
$$S_{\{b\}} = \{s_{1,\{b\}}, s_{2,\{b\}}, s_{sink,\{b\}}\},$$
$$S_{\{c\}} = \{s_{2,\{c\}}, s_{sink,\{c\}}\},$$
$$S_{\emptyset} = \{s_{1,\emptyset}, s_{2,\emptyset}, s_{sink,\emptyset}\}$$

and

$$o(s_{1,\emptyset}) = 1,$$
$$o(s_{2,\emptyset}) = 1,$$
$$o(s_{sink,\emptyset}) = 0.$$

The transition function $\delta_A$ can be read off from Fig. 1; note that the sink states $s_{sink,A}$ as well as the transitions to the sink states are not pictured. Intuitively, any transition not pictured in the automaton can be understood as going to the appropriate sink state. For instance, we have $\delta_{\{b,c\}}(s_{2,\{b,c\}}, c) = s_{sink,\{b\}}$. We can check that the two conditions A1 and A2 hold in $M$. The language accepted by state $s_{1,\{b\}}$ is $LM(s_{1,\{b\}}) = \{(b), \{b\} p \{b, \overline{c}\}\}$. The language accepted by state $s_{1,\emptyset}$ is $LM(s_{1,\emptyset}) = \{e, p\{b, \overline{c}\}\}$.

We define a homomorphism between mixed automata $M$ and $M'$ to be a family $f = \langle f_A \rangle_{A \in \mathcal{P}(B)}$ of functions $f_A : S_A \rightarrow S'_A$ such that:

1. for all $s \in S'_A$, $o(s) = o'(f_A(s))$, and for all $p \in P$, $f_B(\delta_A(s, p)) = \delta'_{\{f_A(s), p\}}$,
2. for all $s \in S_A$ (where $A \neq \emptyset$) and all $l \in \text{lit}(A)$, $f_A \setminus \text{base}(l) \med(M(s, l)) = \delta'_{\{f_A(s), l\}}$. 
A homomorphism preserves accepting states and transitions. We write \( f : M \rightarrow M' \) when \( f \) is a homomorphism between automata \( M \) and \( M' \). For convenience, we often write \( f(s) \) for \( f_A(s) \) when the type \( A \) of \( s \) is understood. It is straightforward to verify that mixed automata form a category (denoted \( \mathcal{M}, \mathcal{A} \)), where the morphisms of the category are mixed automata homomorphisms.

We are interested in identifying states that have the same behaviour, that is, that accept the same mixed language. A bisimulation between two mixed automata \( M = (\langle S_A \rangle_{A \in \mathcal{P}(B)}, o, \langle \delta_A \rangle_{A \in \mathcal{P}(B)}), M' = (\langle S'_A \rangle_{A \in \mathcal{P}(B)}, o', \langle \delta'_A \rangle_{A \in \mathcal{P}(B)}) \) is a family of relations \( \langle R_A \rangle_{A \in \mathcal{P}(B)} \) such that the following two conditions hold:

1. for all \( s \in S_A \) and \( s' \in S'_A \), if \( s \mathcal{R} s' \) then \( o(s) = o'(s') \) and for all \( p \in \mathcal{P} \), \( \delta(s, p) R_B \delta'(s', p) \), and
2. for all \( s \in S_A \) and \( s' \in S'_A \) (where \( A \neq \emptyset \)), if \( s R_A s' \), then for all \( l \in \text{lit}(A) \), \( \delta_A(s, l) R_A \delta'_A(s', l) \).

A bisimulation between \( M \) and itself is called a bisimulation on \( M \). Two states \( s \) and \( s' \) of \( M \) having the same type \( B \) are said to be bisimilar, denoted by \( s \sim_M s' \), if there exists a bisimulation \( \langle R_A \rangle_{A \in \mathcal{P}(B)} \) such that \( s R_B s' \). (We simply write \( s \sim s' \) when \( M \) is clear from the context.) For each \( M \), the relation \( \sim_M \) is the union of all bisimulations on \( M \), and in fact is the greatest bisimulation on \( M \).

**Proposition 3.3.** If \( s \) is a state of \( M \) and \( s' \) is a state of \( M' \) with \( s \sim s' \), then \( L_M(s) = L_{M'}(s') \).

**Proof.** We show, by induction on the length of mixed strings that for all mixed strings \( \sigma \), and for all states \( s, s' \) such that \( s \sim s' \), then \( \sigma \in L_M(s) \) if and only if \( \sigma \in L_{M'}(s') \). For the empty mixed string \( \epsilon \), we have \( \epsilon \in L_M(s) \) if and only if \( o(s) = 1 \) if and only if \( o'(s') = 1 \) (by definition of bisimilarity) if and only if \( \epsilon \in L_{M'}(s') \). Assume inductively that the results holds for mixed strings of length \( n \). Let \( \sigma \) be a mixed string of length \( n + 1 \), of the form
\[ a \sigma'. \] Assume \( \sigma \in L_M(s) \). By definition, there is a linearization \( e_1 \ldots e_m \) of \( a \) and a state \( s_1 \) such that \( s \stackrel{e_1}{\rightarrow} \ldots \stackrel{e_m}{\rightarrow} s_1 \) and \( \sigma' \in L_M(s_1) \). By the definition of bisimilar states, we have \( s' \stackrel{e_1}{\rightarrow} \ldots \stackrel{e_m}{\rightarrow} s'_1 \) and \( s_1 \sim s'_1 \). By the induction hypothesis, \( \sigma' \in L_M(s'_1) \). By the choice of \( s'_1 \), we have that \( \sigma \in L_M(s') \), as desired. \( \Box \)

Conditions (1) and (2) of the definition of a bisimulation are analogous to the conditions in the definition of a homomorphism. Indeed, a homomorphism can be viewed as a bisimulation.

**Proposition 3.5.** If \( f : M \rightarrow M' \) is a mixed automaton homomorphism, then \( \langle R_A \rangle_{A \in \mathcal{P}(B)} \), defined by \( R_A = \{ (s, f_A(s)) : s \in S_A \} \) is a bisimulation.

**Proof.** First, for all \( s \in S_A \), \( s R_{A} s' \) implies \( s' = f_{A}(s) \), and \( o(s) = o'(f_{A}(s)) = o'(s') \). Moreover, for all \( p \in \mathcal{P} \), we have \( \delta'_{A}(s', p) = \delta'_{A}(f_{A}(s), p) = f_{B}(\delta_{A}(s, l)) \), so that \( \delta_{A}(s, l) R_{A} \delta_{A}(s', l) \), as required. Similarly, let \( s \in S_{A} \) (where \( A \neq \emptyset \)); \( s R_{A} s' \) implies \( s' = f_{A}(s) \), and thus for all \( l \in \text{lit}(A) \), \( \delta'_{A}(s', l) = \delta'_{A}(f_{A}(s), l) = f_{A}(\{ \text{base}(l) \}) \), so that \( \delta_{A}(s, l) R_{A} \delta_{A}(s', l) \), as required, proving that \( \langle R_A \rangle_{A \in \mathcal{P}(B)} \) is a bisimulation. \( \Box \)

An immediate consequence of this relationship is that homomorphisms preserve accepted languages.

**Proposition 3.5.** If \( f : M \rightarrow M' \) is a mixed automaton homomorphism, then \( L_M(s) = L_{M'}(f(s)) \) for all states \( s \) of \( M \).

**Proof.** Immediate from Propositions 3.4 and 3.3. \( \Box \)

It turns out that we can impose a mixed automaton structure on the set of all mixed languages with type \( A \rightarrow \emptyset \). We take as states mixed languages of type \( A \rightarrow \emptyset \). A state is accepting if the empty string \( \varepsilon \) is in the language. It remains to define the transitions between states; we adapt the idea of Brzozowski derivatives [1]. Our definition of derivative depends on whether we are taking the derivative with respect to a program element or a literal.

If the mixed language \( L \) has type \( \emptyset \rightarrow B \) and \( p \in \mathcal{P} \) is a primitive program, define

\[ D_p(L) = \{ \sigma : p \cdot \sigma \in L \}. \]

If the mixed language \( L \) has type \( A \rightarrow B \) (for \( A \neq \emptyset \)) and \( l \in \text{lit}(A) \) is a literal, then

\[ D_l(L) = \{ \sigma : \{ l \} \cdot \sigma \in L \}. \]

Define \( L_A \) to be the set of mixed languages of type \( A \rightarrow \emptyset \). Define \( L \) to be \( \langle L_A \rangle_{A \in \mathcal{P}(B)}, o_L, \langle \delta_A \rangle_{A \in \mathcal{P}(B)} \rangle \), where \( o_L(L) = 1 \) if \( \varepsilon \in L \), and 0 otherwise; \( \delta_{\emptyset}(L, p) = D_p(L) \); and
\[ \delta_A(L, l) = D_l(L), \text{ for } A \neq \emptyset \text{ and } l \in \text{lit}(A). \] It is easy to verify that \( \mathcal{L} \) is indeed a mixed automaton. The following properties of \( \mathcal{L} \) are significant.

**Proposition 3.6.** For a mixed automaton \( M \) with states \( \langle S_A \rangle_{A \in \mathcal{A}(B)} \), the maps \( f_A : S_A \to \mathcal{L} \) mapping a state \( s \) in \( S_A \) to the language \( L_M(s) \) form a mixed automaton homomorphism.

**Proof.** We check the two conditions for the family \( \{ f_A \}_{A \in \mathcal{A}(B)} \) to be a homomorphism. First, given \( s \in S_A \), \( o(s) = 1 \) if and only if \( \varepsilon \in L_M(s) \), which is equivalent to \( o_L(f_A(s)) = 1 \).

Moreover, given \( p \in B \), \( f_B(\delta_A(s, p)) = L_M(\delta_A(s, p)) = \{ \sigma : p \cdot \sigma \in L_M(s) \} = D_p(L_M(s)) = D_p(f_A(s)) \), as required. Similarly, given \( s \in S_A \) (where \( A \neq \emptyset \)), and \( l \in \text{lit}(A) \), \( f_A \{ [\text{base}(l)] \}(\delta_A(s, l)) = L_M(\delta_A(s, l)) = \{ \sigma : [l] \cdot \sigma \in L_M(s) \} = D_l(L_M(s)) = D_l(f_A(s)) \), as required. \( \square \)

**Proposition 3.7.** For any mixed language \( L \) in \( \mathcal{L} \), the mixed language accepted by state \( L \) in \( \mathcal{L} \) is \( L \) itself, that is, \( L_{\mathcal{L}}(L) = L \).

**Proof.** We prove by induction on the length of linearizations of \( \sigma \) that for all mixed strings \( \sigma, \sigma' \in L \) if and only if \( \sigma \in L_{\mathcal{L}}(L) \). For the empty mixed string \( \varepsilon \), we have \( \varepsilon \in L \iff o_L(L) = 1 \iff \varepsilon \in L_{\mathcal{L}}(L) \). For \( \sigma \) of the form \( p\sigma' \), we have \( \sigma = p \cdot \sigma' \), and thus we have \( p \cdot \sigma' \in L \iff \sigma' \in D_p(L) \), which by the induction hypothesis holds if and only if \( \sigma' \in L_{\mathcal{L}}(D_p(L)) \iff \sigma' \in D_p(L_{\mathcal{L}}(L)) \) (because \( L_{\mathcal{L}} \) is a mixed automaton homomorphism from \( \mathcal{L} \) to \( \mathcal{L} \)), which is just equivalent to \( p \cdot \sigma' \in L_{\mathcal{L}}(L) \). For \( \sigma \) with a linearization \( le_1 \ldots e_m \), letting \( \sigma' \) denote a string with linearization \( e_1 \ldots e_m \), we have \( \sigma = [l] \cdot \sigma' \), and we can derive in an exactly similar manner that \( [l] \cdot \sigma' \in L \iff \sigma' \in D_l(L) \iff \sigma' \in L_{\mathcal{L}}(D_l(L)) \iff \sigma' \in D_l(L_{\mathcal{L}}(L)) \iff [l] \cdot \sigma' \in L_{\mathcal{L}}(L) \iff \sigma \in L_{\mathcal{L}}(L) \). \( \square \)

These facts combine into the following fundamental property of \( \mathcal{L} \), namely, that \( \mathcal{L} \) is a final automaton.

**Theorem 3.8.** \( \mathcal{L} \) is final in the category \( \mathcal{M} A \), that is, for every mixed automaton \( M \), there is a unique homomorphism from \( M \) to \( \mathcal{L} \).

**Proof.** Let \( M \) be a mixed automaton. By Proposition 3.6, there exists a homomorphism \( f \) from \( M \) to the final automaton \( \mathcal{L} \), mapping a state \( s \) to the language \( L_M(s) \) accepted by that state. Let \( f' \) be another homomorphism from \( M \) to \( \mathcal{L} \). To establish uniqueness, we need to show that for any state \( s \) of \( M \), we have \( f(s) = f'(s) \):

\[
\begin{align*}
f(s) &= L_M(s) \quad \text{(by definition of } f) \\
&= L_{\mathcal{L}}(f'(s)) \quad \text{(by Proposition 3.5)} \\
&= f'(s) \quad \text{(by Proposition 3.7)}.
\end{align*}
\]

Hence, \( f \) is the required unique homomorphism. \( \square \)
The finality of $L$ gives rise to the following coinduction proof principle for language equality, in a way which is by now standard [15].

Corollary 3.9. For two mixed languages $K$ and $L$ of type $A \rightarrow \emptyset$, if $K \sim L$ then $K = L$.

In other words, to establish the equality of two mixed languages, it is sufficient to exhibit a bisimulation between the two languages when viewed as states of the final automaton $L$. In the following sections, we will use this principle to analyze equality of languages described by a typed form of KAT expressions.

4. Pseudo-bisimulations

The “path independence” condition (A2) in the definition of a mixed automaton gives mixed automata a certain form of redundancy. It turns out that due to this redundancy, we can define a simpler notion than bisimulation that still lets us establish the bisimilarity of mixed automata a certain form of redundancy. It turns out that due to this redundancy, we can always complete a pseudo-bisimulation to a bisimulation.

A pseudo-bisimulation (relative to the ordering $b_1, \ldots, b_{|B|}$ of the primitive tests in $B$) between two mixed automata $M = ((S_A)_A \in \mathcal{P}(\mathcal{B}), o, \langle \delta_A \rangle_A \in \mathcal{P}(\mathcal{B}))$ and $M' = ((S'_A)_A \in \mathcal{P}(\mathcal{B}), o', \langle \delta'_A \rangle_A \in \mathcal{P}(\mathcal{B}))$ is a family of relations $(R_i)_{i = 0, \ldots, |B|}$ such that the following two conditions hold:

1. for all $s \in S_\emptyset$ and $s' \in S'_\emptyset$, if $sR_0s'$, then $o(s) = o'(s')$ and for all $p \in \mathcal{P}$, $\delta_\emptyset(s, p)R_i\emptyset\delta'_\emptyset(s', p)$, and
2. for all $i = 1, \ldots, |B|$, for all $s \in S_{A_i}$ and $s' \in S'_{A_i}$, if $sR_is'$, then for all $l \in \text{lit}(b_i)$, $\delta_{A_i}(s, l)R_{i-l}\delta'_{A_i}(s', l)$.

The sense in which pseudo-bisimulation is weaker than a bisimulation is that there need not be a relation for each element of $\mathcal{P}(\mathcal{B})$. As the following theorem shows, however, we can always complete a pseudo-bisimulation to a bisimulation.

Theorem 4.1. If $(R_i)_{i = 0, \ldots, |B|}$ is a pseudo-bisimulation (relative to the ordering $b_1, \ldots, b_{|B|}$ of the primitive tests in $B$), then there exists a bisimulation $(R'_A)$ such that $R'_{A_i} = R_i$ for all $i = 0, \ldots, |B|$ (with $A_i$ denoting $\{b_j : j \leq i, j \in \{1, \ldots, |B|\}\}$).

Proof. Let $(R_i)_{i = 0, \ldots, |B|}$ be a pseudo-bisimulation (relative to the ordering on primitive tests $b_1, \ldots, b_{|B|}$). We define a family of relations $R'_A \subseteq S_A \times S'_A$ for each $A \in \mathcal{P}(B)$, and show that it forms a bisimulation with the required property. The proof relies on the path independence condition A2 of mixed automata in a fundamental way. Given $A \in \mathcal{P}(B)$, let $i(A)$ be the largest $i \in \{1, \ldots, |B|\}$ such that $\{b_1, \ldots, b_i\} \subseteq A$, and let $c(A)$ be the relative complement of $\{b_1, \ldots, b_{i(A)}\}$ defined by $A \setminus \{b_1, \ldots, b_{i(A)}\}$. We say that a sequence of literals $l_1, \ldots, l_k$ is exhaustive over a set of bases $A$ if $A = \{\text{base}(l_1), \ldots, \text{base}(l_k)\}$ and $|A| = k$. Define $R'_A$ as follows: $sR'_As'$ holds if and only if for all literal sequences $l_1, \ldots, l_k$ exhaustive over $c(A)$, we have $s \overset{l_1}{\longrightarrow} \cdots \overset{l_k}{\longrightarrow} s_1, s' \overset{l_1}{\longrightarrow} \cdots \overset{l_k}{\longrightarrow} s'_1$, and $sR_i(A)s'_1$. Clearly, if $A = \{b_1, \ldots, b_{i(A)}\}$, then $R'_{s(A)} = R_{i(A)}$, as required. We now check that $(R'_A)_{A \in \mathcal{P}(B)}$ is a bisimulation. Clearly, since $R'_0 = R_0$, if $sR'_0s'$, then $sR_0s'$, and hence $o(s) = o'(s')$, and
for all \( p \in \mathcal{P} \), it holds that \( \delta_{\bigtriangleup}(s, p)R_{\mathcal{B}}\delta_{\bigtriangleup}(s', p) \), implying \( \delta_{\bigtriangleup}(s, p)R_{\mathcal{B}}\delta_{\bigtriangleup}(s', p) \). Now, let \( A \neq \emptyset, s \in S_A, s' \in S_A', l \in \text{litr}(A) \), and assume \( sR_AS_A's' \). Consider the following cases:

Case A = \( \{b_1, \ldots, b_{l_1}(A)\} \), base(l) = \( b_{j}(A) \): Since \( sR'_AS_A's' \), then \( sR_{l_1}(A)s' \), and by the properties of pseudo-bisimulations, we have \( \delta_A(s, l)R_{l_1}(A)\delta_A(s', l) \), which is exactly \( \delta_A(s, l)R_{l_1}(A)\delta_A(s', l) \).

Case A = \( \{b_1, \ldots, b_{l_1}(A)\} \), base(l) = \( b_j, j < i(A) \): Since \( sR'_AS_A's' \), then \( sR_{l_1}(A)s' \). Let \( l_1, \ldots, l_k \) be an arbitrary exhaustive sequence of literals over \( \{b_{l_1}(A), \ldots, b_{l_j+1}\} \). Let \( l_{i_1}(A), \ldots, l_{i_1+1} \) be the arrangement of \( l_1, \ldots, l_k \) such that \( \text{base}(l_{i_1}) = b_{m} \). Consider the states \( s_1, s_2, s', s'' \) such that \( s \xrightarrow{l_1} \ldots \xrightarrow{l_{i_1}} s_1 \xrightarrow{l} s_2 \), and \( s' \xrightarrow{l_{i_1}} \ldots \xrightarrow{l_{i_1+1}} s'_1 \xrightarrow{l} s'_2 \). By the definition of pseudo-bisimulation, we have that \( s_2R_{l_{1-1}}(A)s'_2 \). Now, by condition A2, we have states \( s_3, s'_3 \) such that \( s \xrightarrow{l_1} \ldots \xrightarrow{l_{i_1}} s_3 \xrightarrow{l} s_2 \) and \( s' \xrightarrow{l_{i_1}} \ldots \xrightarrow{l_{i_1+1}} s'_3 \). Since \( l_1, \ldots, l_k \) was arbitrary, \( s_2R_{l_{1-1}}(A)s'_2 \) and \( i(A \setminus \{\text{base}(l)\}) = j - 1 \), we have \( s_3R_{l_1}(A)\delta_A(s', l) \). That is, \( \delta_A(s, l)R_{l_1}(A)\delta_A(s', l) \).

Case A \( \supset \{b_1, \ldots, b_{l_1}(A)\} \), base(l) \( \in c(A) \): Pick an arbitrary sequence \( l_1, \ldots, l_k \) of literals that is exhaustive over \( c(A \setminus \{\text{base}(l)\}) \), and states \( s_1, s_2, s'_1, s'_2 \) such that \( s \xrightarrow{l_1} \ldots \xrightarrow{l_k} s_1 \) and \( s' \xrightarrow{l_1} \ldots \xrightarrow{l_k} s'_1 \). By definition of \( R_{l_1}(A) \), we have \( s_1R_{l_1}(A)s'_1 \). Since the sequence of literals \( l_1, \ldots, l_k \) was arbitrary, and since \( i(A) = i(A \setminus \{\text{base}(l)\}) \), we have that \( s_2R_{l_1}(A)\delta_A(s', l) \) that is, \( \delta_A(s, l)R_{l_1}(A)\delta_A(s', l) \).

Case A \( \supset \{b_1, \ldots, b_{l_1}(A)\} \), base(l) = \( b_j, j < i(A) \): Pick an arbitrary sequence \( l_1, \ldots, l_k \) of literals that is exhaustive over \( c(A \setminus \{\text{base}(l)\}) \), and states \( s_1, s'_1 \) such that \( s \xrightarrow{l_1} \ldots \xrightarrow{l_k} s_1 \) and \( s' \xrightarrow{l_1} \ldots \xrightarrow{l_k} s'_1 \). By definition of \( R_{l_1}(A) \), we have \( s_1R_{l_1}(A)s'_1 \). By definition of pseudo-bisimulation, if \( s \xrightarrow{l_1} s_2 \) and \( s'_1 \xrightarrow{l_1} s'_2 \), then we have \( s_2R_{l_1}(A)\delta_A(s', l) \). By condition A2, we have that for states \( s_3, s'_3, s \xrightarrow{l_1} \ldots \xrightarrow{l_k} s_3 \xrightarrow{l} s_2 \) and \( s' \xrightarrow{l_1} \ldots \xrightarrow{l_k} s'_3 \). Since \( l_1, \ldots, l_k \) was arbitrary, and \( i(A \setminus \{\text{base}(l)\}) = i(A) - 1 \), we have \( s_3R_{l_1}(A)\delta_A(s', l) \), that is, \( \delta_A(s, l)R_{l_1}(A)\delta_A(s', l) \).
Let us say that two states $s, s'$ are pseudo-bisimilar if they are related by some $R_i$ in a pseudo-bisimulation $(R_i)$; it follows directly from Theorem 4.1 that pseudo-bisimilar states are bisimilar.

5. Mixed expressions and derivatives

A mixed expression (over the set of primitive programs $P$ and the set of primitive tests $B$) is any expression built via the following grammar:

$$e ::= 0 \mid 1 \mid p \mid l \mid e_1 + e_2 \mid e_1 \cdot e_2 \mid e^*$$

(with $p \in P$ and $l \in \text{lit}(B)$). For simplicity, we often write $e_1 e_2$ for $e_1 \cdot e_2$. We also freely use parentheses when appropriate. Intuitively, the constants 0 and 1 stand for failure and success, respectively. The expression $p$ represents a primitive program, while $l$ represents a primitive test. The operation $+$ is used for choice, $\cdot$ for sequencing, and $^*$ for iteration. These are a subclass of the KAT expressions as defined by Kozen [7]. (In addition to allowing negated primitive tests, Kozen also allows negated tests.) We call them mixed expressions to emphasize the different interpretation we have in mind.

In a way similar to regular expressions denoting regular languages, we define a mapping $M$ from mixed expressions to mixed languages inductively as follows:

$$M(0) = \emptyset,$$

$$M(1) = \{\varepsilon\},$$

$$M(p) = \{p\},$$

$$M(l) = \{\{l\}\},$$

$$M(e_1 + e_2) = M(e_1) \cup M(e_2),$$

$$M(e_1 \cdot e_2) = M(e_1) \cdot M(e_2),$$

$$M(e^*) = \bigcup_{n \geq 0} M(e)^n.$$

The mapping $M$ is a rather canonical homomorphism from mixed expressions to mixed languages. (It is worth noting that we have not defined any axioms for deriving the “equivalence” of mixed expressions, and it is quite possible for distinct mixed expressions to give rise to the same mixed language.)

Inspired by a type system devised by Kozen [8,10] for KA and KAT expressions, we impose a type system on mixed expressions. The types have the form $A \to B$, where $A, B \in \wp(B)$, the same types we assigned to mixed strings in Section 2. We shall soon see that this is no accident. We assign a type to a mixed expression via a type judgment written $\vdash e : A \to B$. The following inference rules are used to derive the type of a mixed expression:

$$\vdash 0 : A \to B \quad \vdash 1 : A \to A \quad \vdash p : \emptyset \to B$$

$$\vdash l : A \cup \{\text{base}(l)\} \to A \setminus \{\text{base}(l)\}$$

$$\vdash e_1 : A \to B \quad \vdash e_2 : A \to B$$

$$\vdash e_1 + e_2 : A \to B$$

$$\vdash e_1 : A \to B \quad \vdash e_2 : B \to C$$

$$\vdash e_1 \cdot e_2 : A \to C$$
\[ \vdash e : A \rightarrow A \]
\[ \vdash e^* : A \rightarrow A \]

It is clear from these rules that any subexpression of a mixed expression having a type judgment also has a type judgment.

The typeable mixed expressions (which intuitively are the “well-formed” expressions) induce typeable mixed languages via the mapping \( M \), as formalized by the following proposition.

**Proposition 5.1.** If \( \vdash e : A \rightarrow B \), then \( M(e) \) is a mixed language of type \( A \rightarrow B \).

**Proof.** A straightforward induction on the structure of mixed expressions. \( \square \)

Our goal is to manipulate mixed languages by manipulating the mixed expressions that represent them via the mapping \( M \). (Of course, not every mixed language is in the image of \( M \).) In particular, we are interested in the operations \( T(L) \) and \( \delta(L) \), as defined in Section 2, as well as the language derivatives \( D_P \) and \( D_l \) introduced in the last section.

We now define operators on mixed expressions that capture those operators on the languages denoted by those mixed expressions. We define \( \hat{T} \) inductively on the structure of mixed expressions, as follows:

\[
\begin{align*}
\hat{T}(0) &= 0, \\
\hat{T}(1) &= 1, \\
\hat{T}(p) &= 0, \\
\hat{T}(l) &= l, \\
\hat{T}(e_1 + e_2) &= \hat{T}(e_1) + \hat{T}(e_2), \\
\hat{T}(e_1 \cdot e_2) &= \hat{T}(e_1) \cdot \hat{T}(e_2), \\
\hat{T}(e^*) &= \hat{T}(e)^*
\end{align*}
\]

(where \( p \in \mathcal{P} \) and \( l \in \text{lit}(B) \)). The operator \( \hat{T} \) “models” the operator \( T(L) \), as is made precise in the following way.

**Proposition 5.2.** If \( \vdash e : A \rightarrow B \), then \( \hat{T}(e) \) is a typeable mixed expression such that \( T(M(e)) = M(\hat{T}(e)) \).

**Proof.** A straightforward induction on the structure of mixed expressions. \( \square \)

We define \( \hat{\delta} \) inductively on the structure of mixed expressions, as follows:

\[
\begin{align*}
\hat{\delta}(0) &= 0, \\
\hat{\delta}(1) &= 1, \\
\hat{\delta}(p) &= 0, \\
\hat{\delta}(l) &= 0,
\end{align*}
\]
\[ \hat{\delta}(e_1 + e_2) = \begin{cases} 0 & \text{if } \hat{\delta}(e_1) = \hat{\delta}(e_2) = 0, \\ 1 & \text{otherwise}, \end{cases} \]
\[ \hat{\delta}(e_1 \cdot e_2) = \begin{cases} 1 & \text{if } \hat{\delta}(e_1) = \hat{\delta}(e_2) = 1, \\ 0 & \text{otherwise}, \end{cases} \]
\[ \hat{\delta}(e^*) = 1 \]

(where \( p \in \mathcal{P} \) and \( l \in \text{lit}(\mathcal{B}) \)). Note that \( \hat{\delta}(e) \) is always the mixed expression 0 or 1. In analogy to Proposition 5.2, we have the following fact connecting the \( \varepsilon \) and \( \hat{\delta} \) operators.

**Proposition 5.3.** If \( \vdash e : A \rightarrow B \), then \( \hat{\delta}(e) \) is a typeable mixed expression such that \( \varepsilon(M(e)) = M(\hat{\delta}(e)) \).

**Proof.** A straightforward induction on the structure of mixed expressions. \( \square \)

Finally, we define, by induction on the structure of mixed expressions, the derivative operator \( \hat{D} \) for typeable mixed expressions. There are two forms of the derivative, corresponding to the two forms of derivative for mixed languages: the derivative \( \hat{D}_l \) with respect to a literal \( l \in \text{lit}(\mathcal{B}) \), and the derivative \( \hat{D}_p \) with respect to a primitive program \( p \in \mathcal{P} \). The two forms of derivative are defined similarly, except on the product of two expressions. (Strictly speaking, since the definition of the derivative depends on the type of the expressions being differentiated, \( \hat{D} \) should take type derivations as arguments rather than simply expressions. To lighten the notation, we write \( \hat{D} \) as though it took mixed expressions as arguments, with the understanding that the appropriate types are available.)

The derivative \( \hat{D}_p \) with respect to a primitive program \( p \in \mathcal{P} \) is defined as follows:

\[ \hat{D}_p(0) = 0, \]
\[ \hat{D}_p(1) = 0, \]
\[ \hat{D}_p(q) = \begin{cases} 1 & \text{if } p = q, \\ 0 & \text{otherwise}, \end{cases} \]
\[ \hat{D}_p(l) = 0, \]
\[ \hat{D}_p(e_1 + e_2) = \hat{D}_p(e_1) + \hat{D}_p(e_2), \]
\[ \hat{D}_p(e_1 \cdot e_2) = \begin{cases} \hat{D}_p(e_1) \cdot e_2 & \text{if } B \neq \emptyset, \\ \hat{D}_p(e_1) \cdot e_2 + \hat{\delta}(e_1) \cdot \hat{D}_p(e_2) & \text{otherwise}, \end{cases} \]
where \( \vdash e_1 : A \rightarrow B \) and \( \vdash e_2 : B \rightarrow C \),
\[ \hat{D}_p(e^*) = \hat{D}_p(e) \cdot e^*. \]

The derivative \( \hat{D}_l \) with respect to a literal \( l \in \text{lit}(\mathcal{B}) \) is defined as follows:

\[ \hat{D}_l(0) = 0, \]
\[ \hat{D}_l(1) = 0, \]
\[ \hat{D}_l(p) = 0, \]
\[ \hat{D}_l(l') = \begin{cases} 1 & \text{if } l = l', \\ 0 & \text{otherwise}, \end{cases} \]
We have the following proposition, similar to the previous two, connecting the derivative \( \hat{D} \) to the previously defined derivative \( D \) on mixed languages.

**Proposition 5.4.** Suppose that \( \vdash e : A \rightarrow B \).

If \( A = \emptyset \), then for all \( p \in P \), \( D_p(M(e)) = M(\hat{D}_p(e)) \).

If \( A \neq \emptyset \), then for all \( l \in \text{lit}(A) \), \( D_l(M(e)) = M(\hat{D}_l(e)) \).

**Proof.** The proof is by induction on the structure of the mixed expression \( e \). To illustrate the proof technique, we give one case of the proof.

Suppose that \( \vdash e_1 : A \rightarrow B \) and \( \vdash e_2 : B \rightarrow C \), and \( e = e_1 \cdot e_2 \). Suppose further that \( l \in \text{lit}(B) \) is a literal such that \( \text{base}(l) \in A \) and \( \text{base}(l) \in B \). We will show that the proposition holds for the expression \( e \), assuming (by the induction hypothesis) that the proposition holds for all subexpressions of \( e \).

We first establish three claims that will be needed.

**Claim 1.** If \( t \) is a test which (as a mixed string) can be judged to have type \( A \rightarrow B \), then \( \{ t \} \cdot \{ \sigma : [l] \cdot \sigma \in M(e_2) \} = \{ \sigma' : [l] \cdot \sigma' \in \{ t \} \cdot M(e_2) \} \).

First suppose that \( \sigma \) is a mixed string such that \( [l] \cdot \sigma \in M(e_2) \). Then \( \sigma \) can be judged to have type \( B \setminus \{ \text{base}(l) \} \rightarrow C \), and so \( [l] = \{ t \} \cdot \sigma = \{ t \} \cdot \sigma \in [l] \cdot M(e_2) \). It follows that \( [l] \cdot \sigma \in \{ \sigma' : [l] \cdot \sigma' \in [t] \cdot M(e_2) \} \). For the other direction, suppose that \( \sigma' \) is a mixed string such that \( [l] \cdot \sigma' \in [t] \cdot M(e_2) \). Then there exists a mixed string \( \tau \in M(e_2) \) such that \( [l] \cdot \sigma' = [t] \cdot \tau \). Since \( t \) can be judged to have type \( A \rightarrow B \) and \( \text{base}(l) \in A \cap B \), \( \text{base}(l) \notin t \) and there exists a mixed string \( \sigma \) such that \( [l] \cdot \sigma = [t] \cdot \sigma \). Thus \( \sigma = [t] \cdot \sigma \) where \( [l] \cdot \sigma \in M(e_2) \).

**Claim 2.** If \( \sigma \) is a mixed string such that \( l \cdot \sigma \in M(e_1) \), then \( l \cdot \sigma \in M(e_1) \setminus T(M(e_1)) \).

This claim holds because \( [l] \cdot \sigma \in M(e_1) \) implies that \( \sigma \) has type \( A \setminus \{ \text{base}(l) \} \rightarrow B \); since \( B \not\subseteq A' \), by the definition of the type of a mixed string, \( |\sigma| > 1 \) and so \( |[l] \cdot \sigma| > 1 \).

**Claim 3.** \( \{ \sigma : [l] \cdot \sigma \in M(e_1) \setminus T(M(e_1)) \} \cdot M(e_2) = \{ \sigma : [l] \cdot \sigma \in (M(e_1) \setminus T(M(e_1))) \cdot M(e_2) \} \).

The \( \subseteq \) direction is straightforward. For the \( \supseteq \) direction, let \( \sigma \) be a mixed string in the second set; then, there exist strings \( \tau_1 \in M(e_1) \setminus T(M(e_1)) \) and \( \tau_2 \in M(e_2) \) such that \( [l] \cdot \sigma = \tau_1 \cdot \tau_2 \). All strings in \( M(e_1) \) have type \( A \rightarrow B \); since \( \text{base}(l) \in B \), there are no
strings in $M(e_1)$ of length one consisting of a primitive program, and so $|\tau_1| > 3$. Hence $\sigma = \sigma' \cdot \tau_2$ for some mixed string $\sigma'$ such that $\{l\} \cdot \sigma' \in M(e_1) \setminus T(M(e_1))$.

Using these three claims, we show that $D_l(M(e)) = M(\hat{D}_l(e))$:

$$
\begin{align*}
M(\hat{D}_l(e_1 \cdot e_2)) &= M(\hat{D}_l(e_1) \cdot e_2 + \hat{T}(e_1) \cdot \hat{D}_l(e_2)) \quad \text{(by definition of $\hat{D}_l$)} \\
&= M(\hat{D}_l(e_1)) \cdot M(e_2) \cup M(\hat{T}(e_1)) \cdot M(\hat{D}_l(e_2)) \quad \text{(by definition of $M$)} \\
&= D_l(M(e_1)) \cdot M(e_2) \cup T(M(e_1)) \cdot D(M(e_2)) \quad \text{(by induction hypothesis)} \\
&= \{\sigma : [l] \cdot \sigma \in M(e_1)\} \cdot M(e_2) \cup T(M(e_1)) \cdot \{\sigma : [l] \cdot \sigma \in M(e_2)\} \quad \text{(by definition of $D_l$)} \\
&= \{\sigma : [l] \cdot \sigma \in M(e_1)\} \cdot M(e_2) \cup \{\sigma : [l] \cdot \sigma \in T(M(e_1)) \cdot M(e_2)\} \quad \text{(by Claim 1)} \\
&= \{\sigma : [l] \cdot \sigma \in (M(e_1) \setminus T(M(e_1))) \cdot M(e_2)\} \cup \{\sigma : [l] \cdot \sigma \in T(M(e_1)) \cdot M(e_2)\} \quad \text{(by Claim 2)} \\
&= \{\sigma : [l] \cdot \sigma \in M(e_1) \cdot M(e_2)\} \quad \text{(by definition of $D_l$)} \\
&= D_l(M(e_1 \cdot e_2)) \quad \text{(by definition of $M$)}.
\end{align*}
$$

The other cases are similar. □

### 6. Example

In this section, we use the notions of pseudo-bisimulation and the coinduction proof principle (Corollary 3.9), along with the derivative operator $\hat{D}$, to prove the equivalence of two mixed languages specified as mixed expressions.

Fix $P$ to be the set of primitive programs $\{p, q\}$, and $B$ to be the set of primitive tests $\{b, c\}$. Let $[b]$ be a shorthand for $(b + b)$. Define $\alpha$ to be the mixed expression

$$
(bp([b]cq)^*\epsilon)^*\overline{b}
$$

and $\beta$ to be the mixed expression

$$
bp([b]cq + b\overline{c}p)^*\epsilon\overline{b} + \overline{b}.
$$

Our goal is to prove that $\alpha$ and $\beta$ are equivalent, in the sense that they induce the same language via the mapping $M$. In other words, we want to establish that $M(\alpha) = M(\beta)$. This example demonstrates the equivalence of the program

```plaintext
while b do {
    p;
    while c do q
}
```
and the program

\[
\text{if } b \text{ then } \{ \\
\quad p; \\
\quad \text{while } b + c \text{ do} \\
\quad \quad \text{if } c \text{ then } q \text{ else } p \\
\} \]

This equivalence is a component of the proof of the classical result that every \textit{while} program can be simulated by a \textit{while} program with at most one \textit{while} loop, as presented by Kozen [7]. We refer the reader there for more details.

There are a few ways to establish this equivalence. One is to rely on a sound and complete axiomatization of the equational theory of \textit{KAT}, and derive the equivalence of \( \alpha \) and \( \beta \) algebraically [12]. Another approach is to first construct for each expression an automaton that accepts the language it denotes, and then minimize both automata [11]. Two expressions are then equal if the two resulting automata are isomorphic.

In this paper, we describe a third approach, using the coinductive proof principle for mixed languages embodied by Corollary 3.9. Since the theory we developed in Section 3 applies only to mixed languages of type \( A \rightarrow \emptyset \), we verify that indeed we have \( \alpha = M(\beta) \) and \( \beta = M(\alpha) \):\( \{b\} \rightarrow \emptyset \), so that, by Proposition 5.1, \( M(\alpha) \) and \( M(\beta) \) are languages of type \( \{b\} \rightarrow \emptyset \).

We prove the equivalence of \( \alpha \) and \( \beta \) by showing that the mixed languages \( M(\alpha) \) and \( M(\beta) \) are pseudo-bisimilar, that is, they are related by some pseudo-bisimulation. More specifically, we exhibit a pseudo-bisimulation, relative to the ordering \( b_1 = b, b_2 = c \), on the final automaton \( L \), such that \( M(\alpha) \) and \( M(\beta) \) are pseudo-bisimilar. This is sufficient for proving equivalence, since by Theorem 4.1, the languages \( M(\alpha) \) and \( M(\beta) \) are then bisimilar, and by Corollary 3.9, \( M(\alpha) = M(\beta) \).

Define \( \alpha' \) to be the mixed expression

\[ ((b)cq)^*c\alpha \]

and define \( \beta' \) to be the mixed expression

\[ ((b)cq + b\overline{c}p)^*c\beta. \]

Notice that \( \beta = bp\beta' + \overline{b} \).

We note that (using the notation of the definition of pseudo-bisimulation), \( A_0 = \emptyset \), \( A_1 = \{b\} \), and \( A_2 = \{b, c\} \). We claim that the following three relations form a pseudo-bisimulation:

\[
R_2 = \{(M(\alpha'), M(\beta)'), (M(0), M(0))\}, \quad R_1 = \{(M([b]q\alpha'), M([b]q\beta')), (M(\alpha), M(\beta))\},
\]

\[
R_0 = \{(M(p\alpha'), M(p\beta')), (M(q\alpha'), M(q\beta')), (M(1), M(1)), (M(0), M(0))\}.
\]
It is straightforward to verify that \( \langle R_0, R_1, R_2 \rangle \) is a pseudo-bisimulation on \( L \), using the operators defined in the previous section. For instance, consider \( D_b(M(\chi)) \), which is equal to \( M(\hat{D}_b(\chi)) \) by Proposition 5.4. We compute \( \hat{D}_b(\chi) \) here:

\[
\hat{D}_b(\chi) = \hat{D}_b((bp((b|cq)^\ast\tau)^\ast)\beta + \hat{T}((bp((b|cq)^\ast\tau)^\ast)\beta)0) \\
= p((b|cq)^\ast\tau(bp((b|cq)^\ast\tau)^\ast\beta) = p\chi'.
\]

Hence, \( D_b(M(\chi)) = M(\hat{D}_b(\chi)) = M(p\chi') \). The other cases are similar.

As we shall see shortly, there is a way to mechanically construct such a bisimulation to establish the equivalence of two mixed expressions.

We remark that an alternative approach to establish equivalence of while programs based on coalgebras is described by Rutten [14]. This approach uses the operational semantics of the programs instead of an algebraic framework.

7. Completeness

Thus far, we have established a coinductive proof technique for establishing the equality of mixed languages (Section 3), and illustrated its use by showing the equality of two particular mixed languages specified by mixed expressions (Section 6), making use of the derivative calculus developed in Section 5. A natural question about this proof technique is whether or not it can establish the equivalence of any two mixed expressions that are equivalent (in that they specify the same mixed language). In this section, we answer this question in the affirmative by formalizing and proving a completeness theorem for our proof technique. In particular, we show that given two equivalent mixed expressions, a finite bisimulation relating them can be effectively constructed, by performing only simple syntactic manipulations. In fact, we exhibit a deterministic procedure for deciding whether or not two mixed expressions are equivalent.

In order to state our completeness theorem, we need a few definitions. We say that two mixed expressions \( e_1 \) and \( e_2 \) are equal up to ACI properties, written \( e_1 \equiv_{\text{ACI}} e_2 \), if \( e_1 \) and \( e_2 \) are syntactically equal, up to the associativity, commutativity, and idempotence of \( + \). That is, \( e_1 \) and \( e_2 \) are equal up to ACI properties if the following three rewriting rules can be applied to subexpressions of \( e_1 \) to obtain \( e_2 \):

\[
e + (f + g) = (e + f) + g \\
e + f = f + e \\
e + e = e.
\]

Given a relation \( \hat{R} \) between mixed expressions, we define an induced relation \( \hat{R}^{\text{ACI}} \) as follows: \( e_1 \hat{R}^{\text{ACI}} e_2 \) if and only if there exists \( e_1', e_2' \) such that \( e_1 \equiv_{\text{ACI}} e_1' \), \( e_2 \equiv_{\text{ACI}} e_2' \), and \( e_1' \hat{R} e_2' \).

We define a syntactic bisimulation between two mixed expressions \( e_1 \) and \( e_2 \) having the same type \( B \rightarrow \emptyset \) (for some \( B \subseteq \mathcal{B} \)) to be a family \( \hat{R} = (\hat{R}_A)_{A \in \mathcal{B}} \) of relations such that

1. for all mixed expressions \( e, e' \), if \( e \hat{R}_A e' \), then \( \vdash e : A \rightarrow \emptyset \) and \( \vdash e' : A \rightarrow \emptyset \);
2. \( e \hat{R}_B e' \).
(3) for all mixed expressions \( e, e' \), if \( e \not\sim_R e' \), then \( \hat{e}(e) = \hat{e}(e') \), and for all \( p \in \mathcal{P} \), \( \hat{D}_p(e) \hat{R}^\text{ACI}_B \hat{D}_p(e') \), and

(4) for all mixed expressions \( e, e' \), if \( e \not\sim_R e' \) (for \( A \neq \emptyset \)), then for all \( l \in \text{lit}(A) \), \( \hat{D}_l(e) \hat{R}^\text{ACI}_{A \setminus \{\text{base}(l)\}} \hat{D}_l(e') \).

A syntactic bisimulation resembles a bisimulation, but is defined over mixed expressions, rather than over mixed languages. The next theorem shows that any two equivalent mixed expressions are related by a finite syntactic bisimulation, that is, a syntactic bisimulation \( \hat{R} \) where the number of pairs in each relation \( \hat{R}_A \) is finite.

**Theorem 7.1.** For all mixed expressions \( e_1, e_2 \), of type \( A \to \emptyset \), \( M(e_1) = M(e_2) \) if and only if there exists a finite syntactic bisimulation between \( e_1 \) and \( e_2 \).

**Proof.** (\( \Rightarrow \)) It is easy to check that a syntactic bisimulation \( \hat{R} \) induces a bisimulation \( R \) such that \( e_1 \not\sim_R e_2 \) if and only if \( M(e_1) \backslash M(e_2) \). The result then follows by Corollary 3.9.

(\( \Leftarrow \)) We first show how to construct, for every mixed expression \( e \) with \( \vdash e : A_e \to B_e \), a finite-state automaton \( M = ((S_A)_{A \in \mathcal{P}(B)}, (\hat{\delta}_A)_{A \in \mathcal{P}(B)}) \) with transition functions \( \delta_{\emptyset} : S_{\emptyset} \times \mathcal{P} \to S_B \) and (for \( A \neq \emptyset \)) \( \delta_A : S_A \times \text{lit}(A) \to \bigcup_{A \in \mathcal{P}(B)} S_A \), satisfying the conditions

1. \( \delta_A(s, l) \in S_A \setminus \{\text{base}(l)\} \),
2. the states of \( S_A \) are mixed expressions having type \( A \to B_e \),
3. \( e \) is a state of \( S_A \),
4. if \( \delta_{\emptyset}(s_1, p) = s_2 \), then \( \hat{D}_p(s_1)^{\text{ACI}} = s_2 \), and
5. if \( \delta_A(s_1, l) = s_2 \), then \( \hat{D}_l(s_1)^{\text{ACI}} = s_2 \).

We define the automata by induction on the structure of \( e \). The cases for \( 0, 1, p, l \) are straightforward. We focus on the remaining cases:

**Case** \( e = e_1 + e_2 \): Assume by induction that we have automata \( M_1, M_2 \) for \( e_1 \) and \( e_2 \). Define:

\[
S_A = \{ f_1 + f_2 : f_1 \in S_{1,A}, f_2 \in S_{2,A} \}
\]

\[
\delta_{\emptyset}(f_1 + f_2, p) = \delta_{\emptyset}(f_1, p) + \delta_{\emptyset}(f_2, p)
\]

\[
\delta_A(f_1 + f_2, l) = \delta_{1,A}(f_1, l) + \delta_{2,A}(f_2, l), \text{ for } A \neq \emptyset, l \in \text{lit}(A).
\]

**Case** \( e = e_1 \cdot e_2 \): Let \( \vdash e_1 : A_1 \to B_1 \). Assume by induction that we have automata \( M_1, M_2 \) for \( e_1 \) and \( e_2 \). Define:

\[
S_A = \{ f \cdot e_2 + \sum_{(t,g) \in E} t \cdot g + \sum_{g \in G} g : f \in S_{1,A}, E \subseteq \{l_1 \ldots l_k : A \to B_1 \} \times S_{2,B_1}, G \subseteq S_{2,A} \}
\]

\[
\delta_{\emptyset} \left( f \cdot e_2 + \sum_{g \in G} g, p \right) =
\]

\[
\begin{cases} 
\delta_{1,\emptyset}(f, p) \cdot e_2 + \delta_{2,\emptyset}(e_2, p) + \sum_{g \in G} \delta_{2,\emptyset}(g, p) & \text{if } B = \emptyset, \hat{e}(f) = 1 \\
\delta_{1,\emptyset}(f, p) \cdot e_2 + \sum_{g \in G} \delta_{2,\emptyset}(g, p) & \text{otherwise}
\end{cases}
\]
\[
\delta_A \left( f \cdot e_2 + \sum_{(t,g) \in E} t \cdot g + \sum_{g \in G} g, l \right) =
\]
\[
\begin{cases}
\delta_{1, A}(f, l) \cdot e_2 + \sum_{(t,g) \in E} D_1(t) \cdot g + \sum_{g \in G} \delta_{2, A}(g, l) & \text{if } base(l) \in A \setminus B_1 \\
\delta_{1, A}(f, l) \cdot e_2 + \sum_{(t,g) \in E} t \cdot \delta_{2, B_1}(g, l) + \sum_{g \in G} \delta_{2, A}(g, l) & \text{if } base(l) \notin A \cup B_1 \\
\delta_{1, A}(f, l) \cdot e_2 + \hat{T}(f) \cdot \delta_{2, B_1}(e_2, l) & \text{if } base(l) \in B_1 \\
+ \sum_{(t,g) \in E} t \cdot \delta_{2, B_1}(g, l) + \sum_{g \in G} \delta_{2, A}(g, l)
\end{cases}
\]

for \( A \neq \emptyset, l \in \text{lit}(A) \).

**Case** \( e = e_A^* \): Let \( e_1 : A_1 \rightarrow A_1 \). Assume by induction that we have an automaton \( M_1 \) for \( e_1 \). Define:

\[
S_A = \begin{cases}
\{ \gamma \cdot e_1^* + \sum_{f \in F} f \cdot e_1^* : \gamma \in \{0, 1\}, F \subseteq S_{1, A_1} \} & \text{if } A = A_1 \\
\sum_{f \in F} f \cdot e_1^* : F \subseteq S_{1, A} & \text{otherwise}
\end{cases}
\]

\[
\delta_\emptyset \left( \gamma \cdot e_1^* + \sum_{f \in F} f \cdot e_1^*, p \right) =
\gamma \cdot \delta_{1, \emptyset}(e_1, p) \cdot e_1^* + \sum_{f \in F} \delta_{1, \emptyset}(f, p) \cdot e_1^* + \sum_{f \in F} \hat{\delta}(f) \cdot \delta_{1, \emptyset}(e, p) \cdot e_1^*,
\]

for \( A = A_1 \)

\[
\delta_\emptyset \left( \sum_{f \in F} f \cdot e_1^*, p \right) = \sum_{f \in F} \delta_{1, \emptyset}(f, p) \cdot e_1^*, \text{ for } A \neq A_1,
\]

\[
\delta_A \left( \gamma \cdot e_1^* + \sum_{f \in F} f \cdot e_1^*, l \right) =
\gamma \cdot \delta_{1, A}(e_1, l) \cdot e_1^* + \sum_{f \in F} \delta_{1, A}(f, l) \cdot e_1^* + \sum_{f \in F} \hat{\delta}(f) \cdot \delta_{1, A}(e, l) \cdot e_1^*,
\]

for \( A \neq \emptyset, A = A_1, l \in \text{lit}(A) \)

\[
\delta_A \left( \sum_{f \in F} f \cdot e_1^*, l \right) = \sum_{f \in F} \delta_{1, A}(f, l) \cdot e_1^*, \text{ for } A \neq \emptyset, A \neq A_1, l \in \text{lit}(A).
\]

It is straightforward (if tedious) to verify that the resulting automaton satisfies properties (1)–(5) given above.

This completes the construction of the finite state mixed automaton corresponding to \( e \).

Given equivalent mixed expressions \( e_1 \) and \( e_2 \) of type \( A \rightarrow \emptyset \), a finite syntactic bisimulation \( \hat{R} \) can be constructed as follows. First, construct the automata \( M_1 \) and \( M_2 \) corresponding
to $e_1$ and $e_2$. Then, initialize $\hat{R}$ to contain the pair $(e_1, e_2)$, and iterate the following process: for every $(e, e')$ in $\hat{R}$, add the pairs $(\delta_1_B(e, x), \delta_2_B(e', x))$ (where $e, e'$ have type $B \rightarrow \emptyset$), for all $x$. Perform this iteration until no new pairs are added to $\hat{R}$. This must terminate, because there are finitely many pairs of states $(e, e')$ with $e$ in $M_1$ and $e'$ in $M_2$. It is straightforward to check that $\hat{R}$ is a syntactic bisimulation, under the assumption that $M(e_1) = M(e_2)$. □

The procedure described in the proof of Theorem 7.1 can in fact be easily turned into a procedure for deciding if two mixed expressions are equivalent. To perform this decision, construct $\hat{R}$, and verify that at all pairs of states $(e, e')$ in $\hat{R}$, $\hat{R}(e) = \hat{R}(e')$. If this verification fails, then the two mixed expressions are not equivalent; otherwise, they are equivalent.

The bisimulation in Section 6 is indeed a bisimulation induced by a syntactic bisimulation on the mixed expressions $\alpha$ and $\beta$.

8. Conclusions and future work

We believe that proofs of equivalence between mixed expressions such as $\alpha$ and $\beta$ via bisimulation are in general more easily derived than ones obtained through a sound and complete axiomatization of KAT. Given two equivalent mixed expressions, we can exhibit a bisimulation using the purely mechanical procedure underlying Theorem 7.1: use the derivative operators to construct a finite bisimulation in which the two expressions are paired. In contrast, equational reasoning typically requires creativity.

The “path independence” of a mixed automaton (condition A2) gives any mixed automaton a certain form of redundancy. This redundancy persists in the definition of bisimulation, and is the reason why a pseudo-bisimulation, a seemingly weaker notion of bisimulation, gives rise to a bisimulation. An open question is to cleanly eliminate this redundancy; a particular motivation for doing this would be to make proofs of expression equivalence as simple as possible. Along these lines, it would be of interest to develop other weaker notions of bisimulation that give rise to bisimulations; pseudo-bisimulations require a sort of “fixed variable ordering” that does not seem absolutely necessary.

Another issue for future work would be to give a class of expressions wider than our mixed expressions for which there are readily understandable and applicable rules for computing derivatives. In particular, a methodology for computing derivatives of the KAT expressions defined by Kozen [7] would be nice to see. Intuitively, there seems to be a tradeoff between the expressiveness of the regular expression language and the simplicity of computing derivatives (in the context of KAT). Formal work towards understanding this tradeoff could potentially be quite useful.

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