Rotationally Optimal Spanning and Steiner Trees in Uniform Orientation Metrics*

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Abstract

We consider the problem of finding a minimum spanning and Steiner tree for a set of n points in the plane where the orientations of edge segments are restricted to λ uniformly distributed orientations, $\lambda = 2, 3, 4, \ldots$, and where the coordinate system can be rotated around the origin by an arbitrary angle. The most important cases with applications in VLSI design arise when $\lambda = 2$ or $\lambda = 4$. In the former, so-called rectilinear case, the edge segments have to be parallel to one of the coordinate axes, and in the latter, so-called octilinear case, the edge segments have to be parallel to one of the coordinate axes *or* to one of the lines making $45°$ with the coordinate axes (so-called diagonals). As the coordinate system is rotated — while the points remain stationary — the length and indeed the topology of the minimum spanning or Steiner tree changes. We suggest a straightforward polynomial-time algorithm to solve the rotational minimum spanning tree problem. We also give a simple algorithm to solve the rectilinear Steiner tree problem in the rotational setting, and a finite time algorithm for the general Steiner tree problem with λ uniform orientations. Finally, we provide some computational results indicating the average savings for different values of n and λ both for spanning and Steiner trees.

Keywords: Steiner trees in uniform orientation metrics; rotational problems; VLSI design

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1 Introduction

Suppose that we are given a set P of n points in the plane. We are interested in finding a minimum spanning tree (MST) or a Steiner minimum tree (SMT) for P under the assumption that the edge segments are permitted to have a limited number of orientations. In fact, we will assume that these orientations are evenly spaced. More specifically, let λ be a non-negative integer, $\lambda \geq 2$. Let $\omega = \pi/\lambda$. Legal orientations are then defined by the straight lines passing through the origin and making angles $i\omega$, $i = 0, 1, ..., \lambda - 1$ with the x-axis. Finding an MST in this so-called λ-geometry is not a challenging problem since it can be defined as a minimum spanning tree problem in a complete graph with appropriate edge lengths. On the other hand, computing an SMT in this setting is known to be an NP-hard problem [1].

Suppose that we are permitted to rotate the coordinate system, that is, to rotate all legal orientations simultaneously. Rotation by ω (or a multiple of ω) will have no impact on the lengths of the edges. But for any angle $\alpha \in [0, \omega[$, the edge lengths will change. What is the value of α minimizing the length of an MST or SMT for P? Once α is fixed, finding an MST is straightforward, while determining similar SMTs clearly remains NP-hard.

Our interest in rotated MSTs and SMTs with bounded orientations was motivated by recent developments in VLSI technology. It will soon be possible to manufacture chips with wires having more than two orientations. This makes the case $\lambda = 4$ important in practice. Furthermore, additional length savings seem to be available when the coordinate system can be rotated. This is in particular useful for small values of λ and for nets with limited number of terminals.

In this paper we give a straightforward polynomial-time algorithm for the rotational MST problem. Also, we give a finite time algorithm for the rotational SMT problem. In particular, for the most important rectilinear case ($\lambda = 2$), we show that no more than $O(n^2)$ rotation angles need to be considered in order to compute a rotationally optimal rectilinear SMT.

Our results on the structure of rotationally optimal MSTs and SMTs appear to give insight into the problem of determining the Steiner ratio in λ-geometry for all values of λ (consult the book by Cieslik [3] for a comprehensive survey).

The paper is organized as follows. As a warm up, in Section 2 we determine how the length of a single edge changes under rotation. In Section 3 we present our polynomial-time algorithm for the MST problem and in Section 4 a finite time algorithm for the SMT problem is given. Computational results for both randomly generated instances and instances from VLSI design are presented in Section 5. Finally, in Section 6 we give some concluding remarks and discuss related and open problems.

2 Length of a Segment

Consider a given λ and rotation angle $\alpha \in [0, \omega[$. In this section we examine the situation where there are only two points $u = (u_x, u_y)$ and $v = (v_x, v_y)$. We let $|uv|_\alpha$ denote the length of the shortest path between u and v with the restriction that the straight line segments of the path only use one of the λ legal orientations. We may restrict our attention to the cases where this path consists of either one or two line segments [10]. In the former case we say that the edge between u and v is *straight* (and $|uv|_\alpha$ is equal to the Euclidean distance $\|uv\|$ between u and v); otherwise the edge is *bent* and the two line segments form the *half-edges* of the edge between u and v. The two half-edges meet at the *corner point* of the bent edge, and the smaller meeting angle is $\pi - \omega$ [10].

The shortest MST or SMT for u and v is obtained when the coordinate system is rotated so that the orientation of the line segment uv overlaps with one of the legal orientations, i.e., the edge between u and v is straight. So our problem is trivial. However, it is still interesting to determine how the length of the edge uv changes as the coordinate system is rotated.

Assume that u and v are on the horizontal x-axis and $u_x < v_x$. When the coordinate system is rotated by increasing angle α, $0 \leq \alpha \leq \omega$, the length of uv increases until $\alpha = \omega/2$, and then decreases until $\alpha = \omega$ (Figure 1). More specifically,

$$|uv|_\alpha = \|uv\| \frac{\sin \alpha + \sin(\omega - \alpha)}{\sin(\omega)}$$

In particular, if $\lambda = 2$ and $\omega = \pi/2$, then

$$|uv|_\alpha = \|uv\|(\sin \alpha + \cos \alpha)$$

It is obvious that the function $f_{uv}(\alpha) = |uv|_\alpha$ is periodically strictly concave (for $\alpha \in [0, 2\pi[$ and with period $\omega = \pi/\lambda$). Furthermore, the minimum is attained when the orientation of the line segment uv overlaps with one of the legal orientations.

3 Minimum Spanning Tree

Consider a collection S of m edges $u_1v_1, \ldots, u_mv_m$. Their total length as a function of the rotation angle α is $|S|_\alpha = \sum_{i=1}^{m} |u_iv_i|_\alpha$.

Lemma 3.1 *The total length $|S|_\alpha$ of the edges in S is minimized when one of the edges u_iv_i is* straight, *i.e., when the orientation of the line segment from u_i to v_i overlaps with one of the legal orientations.*

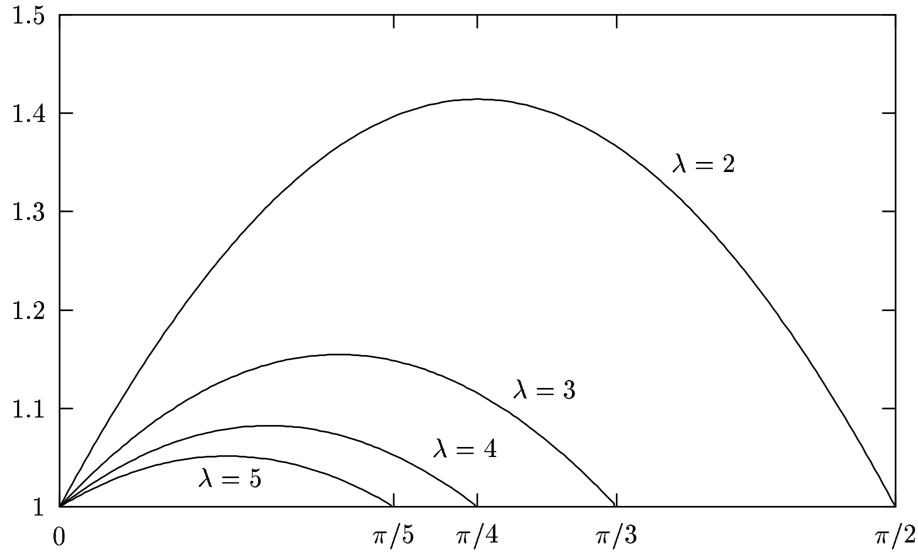

Figure 1: $f_{uv}(\alpha) = |uv|_\alpha$ for $\|uv\| = 1$ and $\alpha \in [0, \frac{\pi}{\lambda}]$, $\lambda = 2, 3, 4, 5$.

Proof. Since $|S|_\alpha$ is a sum of piecewise strictly concave functions, $|S|_\alpha$ is also piecewise strictly concave. Therefore, the minimum is attained at one of the non-concave points (or breakpoints), that is, when one of the edges is straight. ∎

Consider a set P of n points in the plane. Assume that the coordinate system has been rotated in such a way that an MST T for P is shortest possible.

Theorem 3.2 *One of the edges in a rotationally optimal MST T overlaps with one of the legal orientations.*

Proof. Follows immediately from Lemma 3.1. ∎

The algorithm to determine T is therefore straightforward. For each pair of points u and v of P, consider the edge uv. Rotate the coordinate system so that the orientation of one of the legal orientations overlaps with the line segment from u to v. Compute an MST for P and store it provided that it is shorter than any MST found so far. Figure 2 shows how the lengths of MSTs for a set of 10 points and $\lambda = 4$ changes when the coordinate system is rotated. The histogram was generated by computing 1000 MSTs for values of α evenly distributed in the interval $[0, \frac{\pi}{4}]$. The vertical dashed lines indicate a subset (see Section 5) of the MSTs computed by the algorithm, corresponding to those for which the MST length might not be a strictly concave function of the rotation angle. The optimum is on the x-axis.

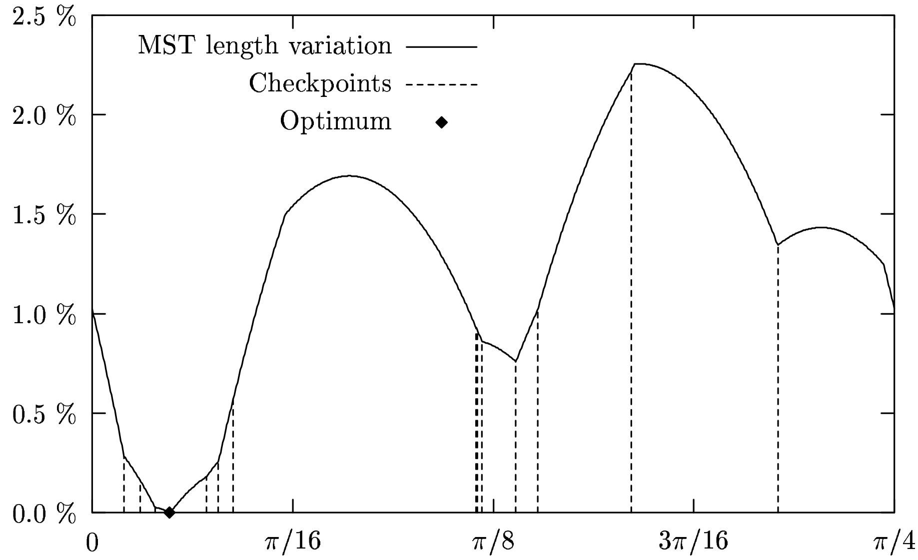

Figure 2: MST length variation as a function of rotation angle.

For a fixed rotation of the coordinate system, an MST can be computed in $O(n \log n)$ time [10]. Since it is only necessary to consider $O(n^2)$ different rotation angles, the total running time for computing a rotationally optimal MST is $O(n^3 \log n)$. In Section 5 we will address the issue of making this algorithm much more efficient in practice.

4 Steiner Minimum Tree

Consider the problem of interconnecting the set P of points by a tree of minimum total length, but *allowing* additional junctions, so-called Steiner points. When a set of λ uniformly spaced legal orientations is given, this is denoted the Steiner tree problem in uniform orientation metrics [1, 6].

A Steiner minimum tree (SMT) is a union of full Steiner trees (FSTs) in which all terminals are leaves (and all interior nodes are Steiner points). In λ-geometry FSTs are denoted λ-FSTs. Our aim in this section is to prove that for a rotationally optimal SMT there exists a λ-FST with no bent edges. We first prove this for the case where λ is a multiple of 3. The proof relies on an elegant theorem showing the relationship between the lengths of such λ-FSTs and equivalent FSTs in Euclidean geometry. This theorem is likely to be of independent interest.

We then prove the result for general λ. Note that a separate proof for the rectilinear case ($\lambda = 2$) appears in [8].

4.1 The case where λ is a multiple of 3

A Simpson line for a *Euclidean* FST F is a line segment extending one of the edges of F such that the Euclidean distance between the endpoints equals $\|F\|$, the total (Euclidean) length of edges in F. It is well known that a Simpson line for F can be constructed in linear time [4, 5]. For λ being a multiple of 3 it turns out that the λ-geometry distance between the endpoints of the Simpson line also equals the length of the corresponding λ-FST:

Theorem 4.1 *Suppose that λ is a multiple of 3. Let F be a Euclidean FST with terminal set N and topology $\mathcal{T}$, and let F_λ be a λ-FST, also with terminal set N and topology $\mathcal{T}$. Let s_1s_2 be a Simpson line of F of length $\|s_1s_2\| = \|F\|$, and let $|s_1s_2|$ be the distance in λ-geometry between s_1 and s_2, the endpoints of the Simpson line. Then the length of F_λ is $|s_1s_2|$.*

Proof. The result is clearly true if $|N| = 2$; in this case the endpoints of the Simpson line coincide with the two terminals in N. So we may assume that $|N| \geq 3$ and that $\mathcal{T}$ contains at least one Steiner point. By [7], we can choose F_λ so that the Steiner points of F_λ exactly coincide with the Steiner points of F (Figure 3).

Consider an edge uv of F and the corresponding λ-edge uv_λ in F_λ. Let $\theta_1, \theta_2 \in [0, \omega[$ be the smaller angles that the two half-edges of uv_λ make with the straight edge between u and v. Since λ is a multiple of 3 and the edges of the Euclidean FST F use exactly 3 equally spaced orientations (separated by angles which are multiples of $\pi/3$), the angles θ_1 and θ_2 will be the same for all edges in F_λ. Furthermore, if an edge uv is rotated through any multiple of $\pi/3$, the rotated half-edges still overlap with legal orientations.

Consider a transformation of the edges of F, composed of rotations and translations, such that they are placed end-to-end along the Simpson line. Since $\|F\| = \|s_1s_2\|$, the edges will exactly cover the Simpson line. Now, consider the corresponding transformation of λ-edges. In view of the above, all transformed edges will use the same two (neighbouring) legal orientations. Therefore, their total length will be exactly $|s_1s_2|$, the λ-geometry distance between the endpoints of the Simpson line. ∎

Theorem 4.2 *Suppose λ is a multiple of 3. A rotationally optimal SMT in λ-geometry is a union of λ-FSTs, at least one of which contains no bent edges.*

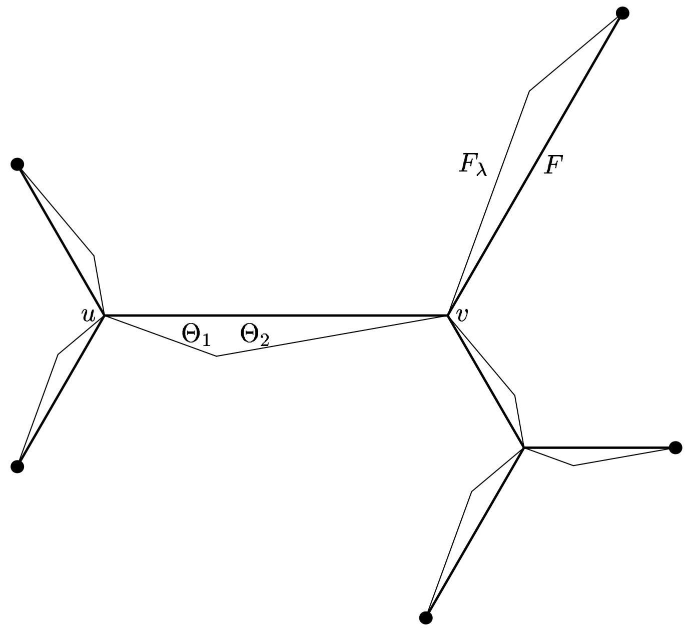

Figure 3: Illustration of Theorem 4.1. Straight edges between terminals and/or Steiner points belong the F while half-edges belong to F_λ.

Proof. Replace each λ-FST in the SMT by a geodesic in λ-geometry between the endpoints of a Simpson line for the corresponding Euclidean FST. (Note that this replaces each λ-FST by a simple λ-edge between two fixed points.) By Theorem 4.1, for any fixed rotation angle, the sum of the lengths of these geodesics in λ-geometry equals the length of the SMT. Hence, we can apply exactly the same argument as in the MST case. ∎

4.2 The case for general λ

In this subsection we show that Theorem 4.2 holds for arbitrary λ. First, assume that the coordinate system is rotated by a fixed angle $\alpha \in [0, \omega[$. In this case the problem to be solved is the Steiner tree problem in uniform orientation metrics [1, 6]. Brazil et al. [1] proved that for any given instance of this problem, there exists an SMT for which every λ-FST has *at most* one bent edge.

Now let us assume that the rotational λ-geometry Steiner tree problem has an optimal solution for a given rotation angle $\alpha^* \in [0, \omega[$. We wish to show that at least one of the λ-FSTs in the SMT contains no bent edge. First we note that if an

SMT contains a Steiner point s with degree more than 3, then the λ-FST for which s is an internal node contains *no* bent edges [1]. Thus we may restrict our attention to SMTs for which all Steiner points have degree 3.

Let F be a λ-FST with a single bent edge. Consider a rotation of the coordinate system through an angle α. We define F_α to be a λ-FST under this rotated coordinate system on the same terminals as F such that 1) the topology of F_α is the same as the topology of F, and 2) the angles between edges at every Steiner point of F_α are the same as the corresponding angles of F. It is clear that F_α always exists for a sufficiently small angle α (i.e., until the length of some edge or half edge goes to 0). In fact, below we give a construction that proves the existence of F_α. Let $|uv|_\alpha$ be the length of some straight edge or a half-edge uv in F_α as a function of α (for $\alpha \in I_\epsilon =]\alpha^* - \epsilon, \alpha^* + \epsilon[$ where $\epsilon > 0$ is sufficiently small).

Lemma 4.3 *Let uv be a straight edge or a half-edge in an FST F with a single bent edge. Then $|uv|_\alpha$ is a strictly concave function of α in a sufficiently small neighbourhood I_ϵ of α^*.*

Proof. In the following we assume that the FST F contains at least one Steiner point, otherwise the lemma is trivially true by our discussion in Section 2.

By the definition of F, we know that F contains exactly one bent edge. We consider F to be a binary tree rooted at the *corner point* of the bent edge. Note that all edges in this tree are straight edges.

There exists a pair of terminals (leaves) t_1 and t_2 adjacent to a common Steiner point (parent) s in the binary tree. Let C_1 be the circle through t_1, s and t_2 (Figure 4). Since the angle $\angle t_1 s t_2$ is fixed, s moves along the circumference of the circle C_1 as α varies.

Let v be the third vertex adjacent to s. Since F contains a bent edge, v must be either a Steiner point or the corner point. We next show that we can replace t_1 and t_2 by a single point e_1, which we will refer to as a *pseudo-terminal*, such that e_1 lies on the extension of vs for all $\alpha \in I_\epsilon$. As indicated in Figure 4, we choose e_1 to be the intersection of the extension of vs with C_1. Since $\angle t_1 s v$ is fixed, it follows that $\angle e_1 s t_1$ is fixed (since it equals $\pi - \angle t_1 s v$). Hence the length of the arc $t_1 e_1$ of C_1 is fixed (since, by elementary trigonometry, equal subtended angles in a circle must come from equal length arcs). But, since t_1 is fixed, it now easily follows that e_1 must also be fixed for all $\alpha \in I_\epsilon$.

This construction can be recursively repeated bottom-up in the binary tree until we end up with a single bent edge between two points, each of which is a terminal or pseudo-terminal.

Consider any straight edge or half-edge uv in the tree, where v is the parent of u in the binary tree. If u is a terminal, then the construction given above shows that

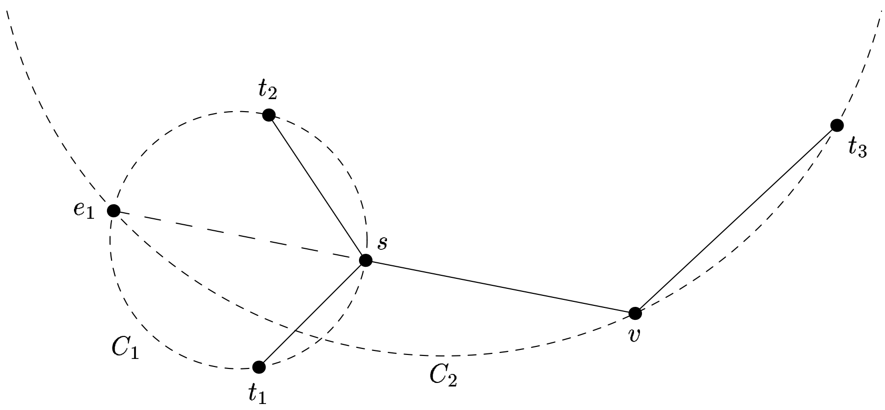

Figure 4: Replacing a pair of terminals with a pseudo-terminal.

v moves along some fixed circle that also passes through u. Therefore, in this case $|uv|_\alpha$ is strictly concave.

Now assume that u is a Steiner point, and v is either a Steiner point or the corner point. Let e_1 be the pseudo-terminal that lies on the extension of the edge uv, and assume that u moves along the circle C_1 that passes through e_1 (Figure 5). The construction above shows that v moves along another circle C_2 that also passes through e_1.

By rescaling, translating and rotating, we can assume, without loss of generality, that C_2 is a unit circle, centered at $(1, 0)$, such that the point e_1 is at the origin (Figure 5). In terms of polar coordinates, the equation of C_2 is $r = 2\cos\theta$. If the centre of C_1 in polar coordinates is (R, ϕ) then C_1 has the polar equation $r = 2R\cos(\theta - \phi)$, since e_1 also lies on C_1. Hence $|uv| = 2\cos\theta - 2R\cos(\theta - \phi)$, and

$$\frac{d^2|uv|}{d\theta^2} = -2\cos\theta + 2R\cos(\theta - \phi) = -|uv|$$

which is strictly negative whenever uv exists. It immediately follows that $|uv|_\alpha$ is strictly concave over a sufficiently small interval I_ϵ. ∎

Theorem 4.4 *A rotationally optimal SMT in λ-geometry is a union of λ-FSTs, at least one of which contains no bent edges.*

Proof. Assume that every λ-FST of an SMT contains a bent edge. Lemma 4.3 shows that the length of every edge in the SMT is strictly concave under sufficiently

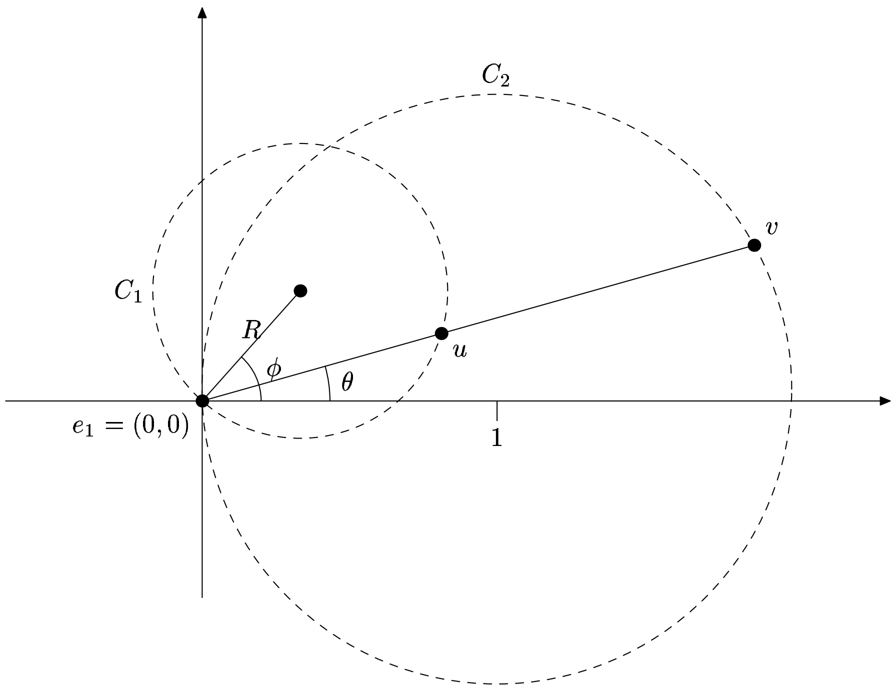

Figure 5: Computing length of edge uv using polar coordinates.

small rotations. Hence, the same holds for the sum of all edge lengths. Therefore, the SMT can in fact be shortened, which is a contradiction. ∎

A consequence of Theorem 4.4 is that there exists a *finite time algorithm* for the rotational SMT problem in λ-geometry. For every subset $N \subseteq P$ of terminals, every topology $\mathcal{T}$ on N, and every legal angle distribution around Steiner points in $\mathcal{T}$, a λ-FST without bent edges is constructed (if it exists). For a given terminal set N, topology $\mathcal{T}$ and angle distribution, a bottom-up construction similar to the one given in the proof of Lemma 4.3 can be used. Whenever such a tree exists, the orientations used by the edges in the tree give a feasible rotation angle α for which the SMT problem in uniform orientation problems is solved in finite time [1].

For general λ a super-exponential number of rotation angles must be tried. However, for $\lambda = 2$ only $O(n^2)$ rotation angles need to be considered — namely those given by the lines through each pair of terminals. This follows from the fact that for $\lambda = 2$ an SMT can be assumed to consist of Hwang-topology FSTs. When no bent edge is present in such an FST, it will contain a straight line segment between a pair of terminals [8].

It is possible to give a stronger version of Theorem 4.4 using some powerful characterizations of λ-FSTs from [2].

Theorem 4.5 *A rotationally optimal SMT in λ-geometry is a union of λ-FSTs, at least one of which uses at most 3 edge orientations.*

Proof. First we may assume by Theorem 4.4 that a rotationally optimal SMT has some λ-FST with no bent edges. Assume that all FSTs with no bent edges use 4 or more edge orientations. Let F be one of these λ-FSTs. Since the edges of F use 4 or more orientations, there exists a length-preserving perturbation of the Steiner points (a so-called zero-shift) in F such that exactly *two* bent edges are created [2].

Consider the path between the two bent edges in F. Let h be one of the half-edges that is closest to the other bent edge. Pick an interior point on h and fix it (i.e., make it into a pseudo-terminal). Now we have divided F into two (pseudo) FSTs, each with one bent edge, and can apply the result of Lemma 4.3 to show that the total length of both (pseudo) FSTs is strictly concave under sufficiently small rotations. This holds for all FSTs in the SMT with no bent edges, and therefore the SMT can in fact be shortened, which is a contradiction. ∎

It should be noted that another consequence of the results in [2] is that there in fact must be a λ-FST that uses 3 edge orientations that have a certain well-defined distribution. For λ being a multiple of 3 this means that there must exist a λ-FST in a rotationally optimal SMT that uses 3 edge orientations separated by an angle of $\pi/3$; such a λ-FST will be identical to the *Euclidean* FST spanning the same set of terminals and having the same topology.

5 Computational Results

In this section we give some computational results indicating the effect of allowing rotations of the coordinate system when computing MSTs and rectilinear SMTs. We used two sets of problem instances: VLSI design instances and randomly generated instances (uniformly distributed in a square). The VLSI instances were made available by courtesy of IBM and Research Institute for Discrete Mathematics, University of Bonn. In this study we have focused on one particular chip from 1996.

5.1 Minimum Spanning Tree

Generally most of the edges for a given point set will not be part of a MST for any rotation angle. It is a waste of time to consider the rotation angle given by such pairs of points (or edges). Fortunately, most of these edges can be pruned by using so-called bottleneck Steiner distances.

Consider the complete graph K_n where the vertices represent the points in the plane. Define the weight of the edge between vertices u and v to be the maximum distance between u and v in λ-geometry under any rotation angle. Compute bottleneck (Steiner) distances in K_n. It is defined as the minimum over the largest-weight edges in all paths between u and v in K_n, and can be computed in time $O(n^2)$ for all pairs of points; see [5, 11] for details. Let B_{uv} denote the bottleneck (Steiner) distance between u and v. Whenever $\|uv\| > B_{uv}$, then the edge uv cannot be in any minimum spanning tree generated during the rotation of the coordinate system. Figure 6 indicates how many edges survive the pruning for $n = 50$ and $\lambda = 2, 3, 4, 5$.

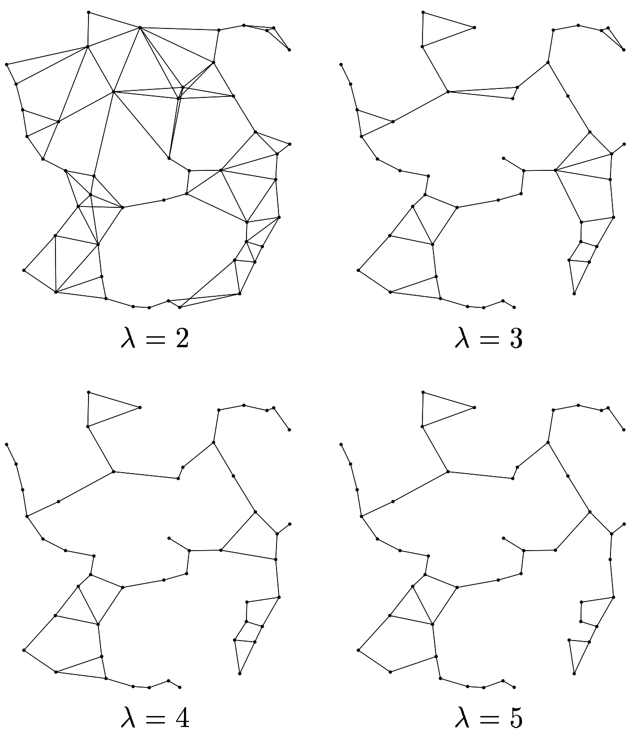

Figure 6: Edges surviving pruning of K_{50} for $\lambda = 2, 3, 4, 5$.

This pruning turns out to be extremely efficient in general. Table 1 shows the number of edges that survive for different values of λ and n. Each entry is an average over 100 instances.

λ	n=3	n=4	n=5	n=10	n=20	n=50	n=100
2	2.76	4.86	7.38	20.12	44.77	122.09	253.84
3	2.34	3.84	5.50	12.62	26.77	71.43	146.81
4	2.22	3.44	4.77	10.77	23.05	60.51	123.05
5	2.13	3.34	4.54	10.17	21.28	56.11	113.59
6	2.07	3.27	4.38	9.83	20.54	53.89	108.57
7	2.06	3.21	4.28	9.54	20.05	52.50	105.88
8	2.03	3.12	4.18	9.36	19.76	51.74	104.49
9	2.02	3.09	4.14	9.32	19.64	51.09	103.29
10	2.02	3.07	4.10	9.29	19.56	50.69	102.49
16	2.01	3.03	4.03	9.08	19.17	49.73	100.33
32	2.00	3.00	4.00	9.01	19.03	49.14	99.41

Table 1: Number of surviving edges after pruning for randomly generated instances (averages over 100 instances).

The extraordinary efficiency of pruning makes it possible to solve the problem of finding the best rotated MST much faster (especially for higher values of λ). It will on average require $O(n^2 \log n)$ time. However, it should be noted that it is possible to construct point sets of any size where the number of edges after pruning is $\Omega(n^2)$.

Table 2 shows the MST improvement for various values of n and λ where each entry is an average over 100 random instances. The table contains two measures of improvement. The first one is calculated as $1 - \frac{|T_{\min}|}{|T_{\max}|}$ in percent. $T_{\min}$ is the shortest MST while $T_{\max}$ is the longest MST taken over 600 uniformly distributed values of α in the interval $[0, \omega[$. In the second measure of improvement the value $T_{\max}$ has been replaced by the length of the MST without rotating the points ($\alpha = 0$); a natural alternative.

5.2 Rectilinear Steiner Tree

We used GeoSteiner [9] to compute RSMTs for each of the $O(n^2)$ orientations given by the pairs of terminals (where n is the number of terminals). The RSMT improvement obtained by rotating the coordinate system can be seen in Table 3. The values are percentages which express the improvement compared to not rotating at all. Results for both random and VLSI instances are given. For small problem instances the improvements are highest for randomly generated instances, while for large instances, the VLSI problems result in a higher improvement.

6 Concluding Remarks

We addressed the problem of determining MSTs and SMTs when edge segment orientations are limited to as set of uniformly distributed orientations and where the coordinate system is permitted to rotate by any angle. We suggested a simple polynomial-time algorithm to solve the MST problem. We also provided some computational results indicating how big the savings can be. As it could be expected, the savings become negligible when λ and n grows. On the other hand, for all practical applications, λ is very small. Nets occurring in VLSI design are also rather small (in terms of the number of terminals involved). However, when many nets are to be routed, the overall savings will not be as impressive as for small isolated nets.

There are several other geometric combinatorial optimization problems which require a selection of a subset of edges and can in the rotational setting be approached in the same way as the MST problem. The *travelling salesman problem* and the *matching problem* are probably the most well-known. Another straightforward generalization occurs when the orientations are fixed but not necessarily evenly spaced. Determination of rotated MSTs in higher dimensions seems also to require a straightforward generalization of the 2-dimensional case.

Computing rotationally optimal SMTs for any fixed, but not necessarily uniform set of orientations is still partially an open research problem. However, if we may limit our attention to FSTs with at most one bent edge and having Steiner points with degree 3 only, Lemma 4.3 can immediately be applied — resulting in a finite time algorithm.

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	$n=3$		$n=4$		$n=5$		$n=10$	
λ	$T_{\max}$	$\alpha=0$	$T_{\max}$	$\alpha=0$	$T_{\max}$	$\alpha=0$	$T_{\max}$	$\alpha=0$
2	21.86	14.48	17.42	9.67	15.86	8.38	10.29	4.89
3	8.86	5.34	7.38	4.34	6.37	3.69	4.31	2.52
4	5.55	3.48	4.32	2.50	3.81	2.14	2.37	1.31
5	3.40	2.10	2.85	1.66	2.42	1.39	1.59	0.89
6	2.36	1.46	1.88	1.17	1.55	0.97	1.05	0.67
7	1.76	1.12	1.41	0.86	1.21	0.74	0.78	0.42
8	1.36	0.85	1.11	0.66	0.94	0.53	0.61	0.33
9	1.04	0.66	0.80	0.49	0.73	0.43	0.49	0.28
10	0.88	0.52	0.70	0.40	0.58	0.31	0.40	0.21
16	0.32	0.20	0.26	0.16	0.22	0.14	0.16	0.10
32	0.08	0.05	0.07	0.04	0.06	0.04	0.04	0.02

	$n=20$		$n=50$		$n=100$	
λ	$T_{\max}$	$\alpha=0$	$T_{\max}$	$\alpha=0$	$T_{\max}$	$\alpha=0$
2	7.47	3.19	5.03	2.12	3.46	1.51
3	3.01	1.47	1.70	0.78	1.23	0.61
4	1.72	0.93	1.02	0.51	0.72	0.39
5	1.09	0.58	0.72	0.39	0.47	0.25
6	0.73	0.39	0.44	0.21	0.30	0.16
7	0.57	0.32	0.35	0.19	0.24	0.14
8	0.44	0.24	0.25	0.13	0.18	0.09
9	0.30	0.16	0.20	0.11	0.15	0.08
10	0.26	0.13	0.18	0.09	0.11	0.05
16	0.11	0.06	0.07	0.03	0.04	0.02
32	0.03	0.01	0.02	0.01	0.01	0.01

Table 2: MST improvement in percent in relation to two values: The first one is the length of the worst case MST ($T_{\max}$) which is the longest MST found among 600 uniformly distributed values of α. The second one is the length of the MST when there is no rotation at all ($\alpha = 0$). All values are averages on 100 random instances.

	n=2	n=3	n=4	n=5	n=10	n=20	n=50	n=100
Random	21.21	11.46	8.19	7.00	3.17	2.17	1.28	0.92
VLSI	14.61	7.01	5.96	7.62	4.88	2.90	-	-

Table 3: RSMT improvement in percent obtained by rotating as compared to not rotating at all. Random: Randomly generated instances (averages over 100 instances). VLSI: VLSI design instances (averages over 100 instances).