# Exercises for "All-pairs shortest paths via fast matrix multiplication"
# Summer school of shortest paths (PATH05) — DIKU

1. Show that Boolean (i.e, or-and) matrix multiplication can be computed in $O(n^{2.38})$ time.

2. Reduce min-plus matrix multiplication to the APSP problem.

3. If $X$ is a matrix of edge weights, let $X^*$ let be the matrix of distances in the corresponding weighted directed graph. (Assume that there are no negative cycles.) Let $AB$ denote the min-plus product of $A$ and $B$, and let $A \vee B = \min\{A, B\}$, the element-wise minimum of the two matrices. Show that if $X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ then

$$X^* = \begin{pmatrix} E & F \\ G & H \end{pmatrix} = \begin{pmatrix} (A \vee BD^*C)^* & EBD^* \\ D^*CE & D^* \vee GBD^* \end{pmatrix} .$$

   Use this relation to show that

$$APSP(n) \leq 2APSP(n/2) + 6MPP(n/2) + O(n^2) ,$$

   where $APSP(n)$ is the cost of solving the APSP problem on graphs with $n$ vertices, and $MPP(n)$ is the cost of computing the min-plus product of two $n \times n$ matrices. Deduce, under reasonable assumptions, that $APSP(n) = O(MPP(n))$.

4. Let $G = (V, E)$ be an unweighted *undirected* graph. Let $G^2 = (V, E^2)$ be the square of $G$, i.e., $(u, v) \in E^2$ if and only if $(u, v) \in E$ or there exists $w \in V$ such that $(u, w), (w, v) \in E$. For two vertices $u, v \in V$, let $\delta(u, v)$ and $\delta^2(u, v)$, respectively, be the distances between $u$ and $v$ in $G$ and $G^2$. Prove the following claims on which the correctness of Seidel's algorithm is based:

   (a) $\delta^2(u, v) = \lceil \frac{\delta(u,v)}{2} \rceil$, for every $u, v \in V$.

   (b) If $\delta(u, v) = 2\delta^2(u, v)$, the for every neighbor $w$ of $v$ in $G$ we have $\delta^2(u, w) \geq \delta^2(u, v)$.

   (c) If $\delta(u, v) = 2\delta^2(u, v) - 1$, the for every neighbor $w$ of $v$ in $G$ we have $\delta^2(u, w) \leq \delta^2(u, v)$, with a strict inequality for at least one neighbor.

   What goes wrong when the graph is *directed*?

5. Let $G = (V, E)$ be a weighted directed graph and let $s$ be a number. We say that a subset $B \subseteq V$ is an $s$-*bridging set* if and only if for every two vertices $u, v \in V$, if every shortest path from $u$ to $v$ uses at least $s$ edges there exists $w \in B$ such that $\delta(u, v) = \delta(u, w) + \delta(w, v)$.

   (a) Show that if $B$ is a random subset of $V$ of size $9n \ln n/s$, then $B$ is an $s$-bridging set, with high probability.

   (b) Suppose that in the $i$-th iteration of the APSP algorithm of Zwick for weighted directed graphs, instead of using a random subset of size $9n \ln n/s$, where $s = (3/2)^{i+1}$, we use an $s/3$-bridging set. Show that the modified algorithm works correctly on unweighted graphs, but may fail on weighted graphs.

   (c) Suitably modify the notion of bridging sets so that it could also be used for weighted graphs.

6. A matrix $W$ is said to be a matrix of *witnesses* for the min-plus product $C = AB$ if and only if for every $i, j$ we have $c_{i,j} = a_{i,w_{i,j}} + b_{w_{i,j},j}$.

   (a) Does the reduction given from min-plus products to algebraic products produce a matrix of witnesses?

   (b) Can you think of an efficient way of computing matrices of witnesses?

   (c) Suppose that for each min-plus product computed by the APSP algorithm we also obtain a corresponding matrix of witnesses. Show how this can be used to obtain a concise representation of all shortest paths.

7. Complete the correctness proof of the $O(Mn^{2.38})$ preprocessing / $O(n)$ query answering algorithm. Then, modify the algorithm so that distances that are obtained using paths composed of at least $s$ edges are reported in $O(n \ln n/s)$ time.