EXERCISES. PART 1.

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1 Applications and Modelling

1.1 Questions from "Network Flows"

Exercises from Ahuja et al.'s book "Network Flows" marked with an asterisk are more highly recommended, and those marked with two asterisks are very strongly recommended.

1. Exercise 4.2 from Ahuja et al.'s book "Network Flows".

   Solution: See photocopies from the book's Solution Manual

2. *Exercise 4.3 from Ahuja et al.'s book "Network Flows". Note that in answering this question, you may assume that all books with same height are stored on same height shelf, i.e. that no splitting of height classes is allowed.

   Solution: See photocopies from the book's Solution Manual

3. *Exercise 4.5 from Ahuja et al.'s book "Network Flows".

   Solution: See photocopies from the book's Solution Manual

4. *Exercise 4.6 from Ahuja et al.'s book "Network Flows".

   Solution: See photocopies from the book's Solution Manual

5. **Exercise 4.7 from Ahuja et al.'s book "Network Flows". Note that in answering this question, you may assume that a concentrator can only be located at one of the nodes. Use c_q to denote the cost of establishing a concentrator at node q, and c_{kq} to denote the cost of homing node k onto a concentrator located at node q.

   Solution: See photocopies from the book's Solution Manual

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6. Exercise 4.8 from Ahuja et al.'s book "Network Flows".

   Solution: See photocopies from the book's Solution Manual

7. **Exercise 4.9 from Ahuja et al.'s book "Network Flows". What is the complexity of the solution approach suggested in this question?

   Solution: See photocopies from the book's Solution Manual. The complexity of the approach is pseudopolynomial. Since it requires solution of a shortest path problem, it is polynomial in the number of nodes (and arcs) in the network; here the number of nodes is polynomial in items of input data, in particular the number of personnel required. This is pseudopolynomial in the length of the input data, which is given by the log of the data items.

1.2 Systems of Difference Constraints

8. Find a feasible solution or determine that no feasible solution exists for the following system of difference constraints.

$$\begin{aligned} x_1 - x_2 &\leq 1, & x_1 - x_4 &\leq -4,\\ x_2 - x_3 &\leq 2, & x_2 - x_5 &\leq 7,\\ x_2 - x_6 &\leq 5, & x_3 - x_6 &\leq 10,\\ x_4 - x_2 &\leq 2, & x_5 - x_1 &\leq -1,\\ x_5 - x_4 &\leq 3, & x_6 - x_3 &\leq -8 \end{aligned}$$

   Solution: After constructing the augmented constraint graph and finding the shortest path from node 0 to all other nodes, one finds there are no negative length cycles, and gets the following labels on nodes.

Node	0	1	2	3	4	5	6
Label	0	-5	-3	0	-1	-6	-8

   Setting x_i to be the value of the label on node i for each $i = 1, \ldots, 6$ gives a feasible solution to the system of difference constraints.

9. Find a feasible solution or determine that no feasible solution exists for the following system of

difference constraints.

$$\begin{aligned} x_1 - x_2 &\le 4, & x_1 - x_5 &\le 5, \\ x_2 - x_4 &\le -6, & x_3 - x_2 &\le 1, \\ x_4 - x_1 &\le 3, & x_4 - x_3 &\le 5, \\ x_4 - x_5 &\le 10, & x_5 - x_3 &\le -7, \\ x_5 - x_4 &\le -8 \end{aligned}$$

Solution: After forming the augmented constraint graph and seeking the shortest path from node 0 to all other nodes, using an algorithm with negative length cycle detection, one finds there is a negative length cycle $(2, 3, 5, 4, 2)$ with length $1 - 7 + 10 - 6 = -2$. Thus the system is infeasible.

10. Can any shortest-path length from the new node, node 0, in the augmented constraint graph, be positive? Explain.

    Solution: Since there is an arc of length 0 from node 0 to every other node, the label on every node (representing the length of the shortest path found so far from node 0 to that node) is set to 0 in the first step. Since it is only modified if a shorter path is found, of necessity such a path must have length *less than* 0, and so cannot be positive; the answer is "no".

11. Suppose that in addition to a system of difference constraints, we want to handle equality constraints of the form $x_i = x_j + b_{ij}$. Explain how the augmented constraint graph and shortest path approach can be adapted to solve this variety of constraint system.

    Solution: Such an equality constraint can be written as two difference constraints: $x_i - x_j \le b_{ij}$ and $x_i - x_j \ge b_{ij}$, or equivalently, $x_j - x_i \le -b_{ij}$.

12. Suppose a system of constraints in variables $x_1, x_2, \ldots, x_n$ contains difference constraints, and single variable bounds of the form $x_i \le u_i$ or $x_i \ge l_i$. Explain how the augmented constraint graph and shortest path approach can be adapted to solve this variety of constraint system.

    Solution: Recall that we can add a constant to every variable in a feasible solution to a system of difference constraints, and still have a solution. We will add a new variable x_{n+1}, and whatever its value is, we will take that and add its negative to all other variables in the solution found by the augmented constraint graph and shortest path approach, to generate our final solution. Thus if $x_1, \ldots, x_n, x_{n+1}$ is the shortest path solution, our final solution will be $x'_i = x_i - x_{n+1}$. To satisfy the upper bound constraint we need $x'_i \le u_i$, which is equivalent to asking that $x_i - x_{n+1} \le u_i$. Note that this is a difference constraint! To satisfy the lower bound constraint we need $x'_i \ge l_i$, which is equivalent to asking that $x_i - x_{n+1} \geq l_i$, or equivalently that $x_{n+1} - x_i \leq -l_i$. Note that this, too, is a difference constraint! Thus we can include a new node $n+1$ in the augmented constraint graph, with an arc from $n+1$ to node i of length u_i for every variable x_i with an upper bound constraint, and an arc from node i to node $n+1$ of length $-l_i$ for every variable x_i with a lower bound constraint. By adding the value of $-x_{n+1}$ to the value of every other variable found by the shortest path approach, we obtain a solution feasible to the original system.

2 Optimization and Duality

13. Write down the LP dual of the LP formulation of the shortest path problem, over digraph $G = (V, A)$, with start node s, end node t, and lengths c_{ij} for all $(i, j) \in A$, using dual variables u_i for each node $i \in V$.

    Solution: The LP dual is as follows.

$$\begin{array}{lll} \max & u_t - u_s & \\ \text{s.t.} & u_j - u_i \leq c_{ij}, & \forall (i,j) \in A \end{array}$$

In what follows, you may assume that the shortest path problem is feasible, i.e. that there exists path from s to t in the digraph. Furthermore, for simplicity, you may assume that all nodes in the graph are reachable from node s.

(a) Explain why it is that we can, without loss of generality, add a constraint $u_s = 0$ to the LP dual.

Solution: The feasible set for the LP dual is a system of difference constraints, thus for any solution u, the solution u' formed by taking $u'_i = u_i + \Delta$ for some constant Δ, for all $i \in V$, is also feasible. Furthermore, u and u' have the same objective value: $u'_t - u'_s = (u_t + \Delta) - (u_s + \Delta) = u_t - u_s$. So for any optimal solution u to the LP dual, we may take $\Delta = -u_s$, and form u' also optimal, satisfying $u'_s = 0$.

(b) Prove that if there is a negative length cycle in the network reachable from s, then the (primal) shortest path LP is unbounded below.

Solution: Let Δ be the indicator vector for arcs in the negative length cycle. Then obviously Δ satisfies $\sum_{j\,:\,(j,i)\in A} \Delta_{ji} - \sum_{j\,:\,(i,j)\in A} \Delta_{ij} = 0$ for all $i \in V$. Since by assumption the problem is feasible, there exists a path P from s to t in the digraph. Let x be the indicator vector for the arcs in this path. Then obviously x is feasible for the primal shortest path LP, and it must be that $x+\alpha\Delta$ is feasible for the shortest path LP for any $\alpha \in \mathbb{R}$, as

$$\begin{aligned}
& \sum_{j\,:\,(j,i)\in A} (x_{ji}+\alpha\Delta_{ji}) - \sum_{j\,:\,(i,j)\in A} (x_{ij}-\alpha\Delta_{ij}) \\
= & \sum_{j\,:\,(j,i)\in A} x_{ji} - \sum_{j\,:\,(i,j)\in A} x_{ij} + \alpha\Big(\sum_{j\,:\,(j,i)\in A} \Delta_{ji} - \sum_{j\,:\,(i,j)\in A} \Delta_{ij}\Big) \\
= & \sum_{j\,:\,(j,i)\in A} x_{ji} - \sum_{j\,:\,(i,j)\in A} x_{ij}.
\end{aligned}$$

Now for any $M \in \mathbb{R}$, there exists α such that $c(x+\alpha\Delta) < M$. For example, take $\alpha = \frac{M-cx}{c\Delta}+1$, which is well defined since Δ indicates a negative length cycle means $c\Delta < 0$. Then

$$\alpha > \frac{M-cx}{c\Delta} \quad \Leftrightarrow \quad \alpha c\Delta < M - cx \quad \Leftrightarrow \quad cx + \alpha c\Delta < M \quad \Leftrightarrow \quad c(x+\alpha\Delta) < M.$$

This shows that there exist feasible solutions for the LP with arbitrarily small objective values; the LP is unbounded below.

Alternatively, observe that the network is actually the constraint graph for the dual problem. Let $(i_0, i_1, \ldots, i_{k-1})$ denote the negative length cycle. Suppose that the LP dual is feasible, with feasible solution u. Then u must satisfy the dual constraints for this cycle:

$$u_{i_{(j+1) \bmod k}} - u_{i_j} \leq c_{i_j i_{(j+1) \bmod k}}$$

for all $j = 0, \ldots, k-1$. Adding up these constraints over all $j = 0, \ldots, k-1$ yields zero on the left-hand side, since every variable appears as many times subtracted as it is added, and yields the length of the cycle (the c values summed over all arcs in the cycle), on the right-hand side. Since the cycle has negative length, we can deduce that zero is less than or equal to a negative number, which is obviously impossible. Thus it must be that the LP dual is infeasible, and hence by LP duality theory the primal shortest path LP is unbounded below.

(c) Explain why it is that if the (primal) shortest path LP is unbounded below, then there must exist a negative length cycle in the network.

Solution: Suppose the (primal) shortest path LP is unbounded below, and let r be a ray of the LP feasible polyhedron, with $cr < 0$. Now r a ray of the LP feasible polyhedron means that $r \geq 0$ and

$$\sum_{j\,:\,(j,i)\in A} r_{ji} - \sum_{j\,:\,(i,j)\in A} r_{ij} = 0,$$

for all $i \in V$, i.e. r induces a non-negative circulation in G. Now any such circulation can be decomposed into a sum of positive flows on cycles. Suppose that there are K such cycles, let

Δ_k be the indicator vector for the arcs in the kth cycle, and let $\alpha_k > 0$ denote the flow on the kth cycle, so we have that

$$r = \sum_{k=1}^{K} \alpha_k \Delta_k.$$

Now it must be that for some $k \in \{1, \ldots, K\}$, $c\Delta_k < 0$, since $cr < 0$ and $\alpha_j > 0$ for all $j = 1, \ldots, K$. Thus the kth cycle is a negative length cycle; such a cycle must exist.

(d) Suppose that there are no negative length cycles in the network. Show that if u_i is taken to be the length of the shortest path from node s to node i for each $i \in V$, then u must be feasible for the LP dual.

Solution: If there are no negative length cycles in the network, then the primal shortest path LPs for paths from s to i for any node $i \in V$ cannot be unbounded below, (by part (13c)), and so must have an optimal solution. Thus the shortest path from node s to node i for each $i \in V$ exists. Let u_i be the length of this path. The Bellman-Ford optimality condition for shortest paths ensures that for each $(i, j) \in A$,

$$u_j \leq u_i + c_{ij} \qquad \Leftrightarrow \qquad u_j - u_i \leq c_{ij}.$$

These are precisely the dual LP constraints, hence u is feasible for the LP dual.

(e) Prove that if the network has no negative length cycle, and the shortest path tree is unique, then the solution to the (primal) shortest path LP is unique, and is the indicator vector for the shortest path from s to t.

Solution: If there are no negative length cycles in the network, then the shortest path from node s to node i for each $i \in V$ exists. Let u_i be the length of this path. If arc $(i, j) \in A$ is on the shortest path tree, then

$$u_j = u_i + c_{ij},$$

so for the shortest path tree to be unique, it must be that

$$T = \{(i, j) \in A \ : \ u_j = u_i + c_{ij}\}$$

induces a (spanning) tree in G rooted at s. (Since by assumption every node in the network is reachable from s, the tree must span the graph.) Now by LP complementary slackness conditions,

$$\text{if} \quad u_j - u_i < c_{ij} \quad \text{then} \quad x_{ij} = 0.$$

Thus the support graph of x, (the set of arcs on which x can take on non-zero, i.e. positive, values), must be a subgraph of (V, T). Now there is only one, unique, path in (V, T) from s to

t, since T induces a tree, so the unique solution to the equations

$$\sum_{j\,:\,(j,i)\in T} x_{ji} - \sum_{j\,:\,(i,j)\in T} x_{ij} = \begin{cases} -1, & i = s \\ 0, & i \neq s, t \\ 1, & i = t \end{cases} \qquad \forall i \in V$$

is the binary vector indicating the arcs in the unique path from s to t in (V, T). This can be proved more formally by considering the above equations in order of decreasing depth in the tree. The equations for all leaf nodes j take the form $x_{i_T(j),j} = d$, where $i_T(j)$ is the unique predecessor of node j in the tree T, and $d = 1$ if $j = t$, $d = 0$ otherwise. This determines the values of the x variables on all arcs entering leaf nodes. Inductive arguments yield the desired result.