EXERCISES. PART 1.

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1 Applications and Modelling

1.1 Questions from "Network Flows"

Exercises from Ahuja et al.'s book "Network Flows" marked with an asterisk are more highly recommended, and those marked with two asterisks are very strongly recommended.

1. Exercise 4.2 from Ahuja et al.'s book "Network Flows".

2. *Exercise 4.3 from Ahuja et al.'s book "Network Flows". Note that in answering this question, you may assume that all books with same height are stored on same height shelf, i.e. that no splitting of height classes is allowed.

3. *Exercise 4.5 from Ahuja et al.'s book "Network Flows".

4. *Exercise 4.6 from Ahuja et al.'s book "Network Flows".

5. **Exercise 4.7 from Ahuja et al.'s book "Network Flows". Note that in answering this question, you may assume that a concentrator can only be located at one of the nodes. Use c_q to denote the cost of establishing a concentrator at node q, and c_{kq} to denote the cost of homing node k onto a concentrator located at node q.

6. Exercise 4.8 from Ahuja et al.'s book "Network Flows".

7. **Exercise 4.9 from Ahuja et al.'s book "Network Flows". What is the complexity of the solution approach suggested in this question?

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1.2 Systems of Difference Constraints

8. Find a feasible solution or determine that no feasible solution exists for the following system of difference constraints.

$$
\begin{aligned}
x_1 - x_2 &\leq 1, & x_1 - x_4 &\leq -4, \\
x_2 - x_3 &\leq 2, & x_2 - x_5 &\leq 7, \\
x_2 - x_6 &\leq 5, & x_3 - x_6 &\leq 10, \\
x_4 - x_2 &\leq 2, & x_5 - x_1 &\leq -1, \\
x_5 - x_4 &\leq 3, & x_6 - x_3 &\leq -8
\end{aligned}
$$

9. Find a feasible solution or determine that no feasible solution exists for the following system of difference constraints.

$$
\begin{aligned}
x_1 - x_2 &\leq 4, & x_1 - x_5 &\leq 5, \\
x_2 - x_4 &\leq -6, & x_3 - x_2 &\leq 1, \\
x_4 - x_1 &\leq 3, & x_4 - x_3 &\leq 5, \\
x_4 - x_5 &\leq 10, & x_5 - x_3 &\leq -7, \\
x_5 - x_4 &\leq -8
\end{aligned}
$$

10. Can any shortest-path length from the new node, node 0, in the augmented constraint graph, be positive? Explain.

11. Suppose that in addition to a system of difference constraints, we want to handle equality constraints of the form $x_i = x_j + b_{ij}$. Explain how the augmented constraint graph and shortest path approach can be adapted to solve this variety of constraint system.

12. Suppose a system of constraints in variables $x_1, x_2, \ldots, x_n$ contains difference constraints, and single variable bounds of the form $x_i \leq u_i$ or $x_i \geq l_i$. Explain how the augmented constraint graph and shortest path approach can be adapted to solve this variety of constraint system.

2 Optimization and Duality

13. Write down the LP dual of the LP formulation of the shortest path problem, over digraph $G = (V, A)$, with start node s, end node t, and lengths c_{ij} for all $(i, j) \in A$, using dual variables u_i for each node $i \in V$.

In what follows, you may assume that the shortest path problem is feasible, i.e. that there exists path from s to t in the digraph. Furthermore, for simplicity, you may assume that all nodes in the graph are reachable from node s.

(a) Explain why it is that we can, without loss of generality, add a constraint $u_s = 0$ to the LP dual.

(b) Prove that if there is a negative length cycle in the network reachable from s, then the (primal) shortest path LP is unbounded below.

(c) Explain why it is that if the (primal) shortest path LP is unbounded below, then there must exist a negative length cycle in the network.

(d) Suppose that there are no negative length cycles in the network. Show that if u_i is taken to be the length of the shortest path from node s to node i for each $i \in V$, then u must be feasible for the LP dual.

(e) Prove that if the network has no negative length cycle, and the shortest path tree is unique, then the solution to the (primal) shortest path LP is unique, and is the indicator vector for the shortest path from s to t.