Written exam, 23 January 2006

Alle hjælpemidler (bøger, noter, m.v., men *ikke* lommeregner) er tilladte. Det er desuden tilladt at skrive med blyant, hvis resultatet bliver klart læseligt.

YOUR ASSIGNMENT:

20 questions **Q1-Q20** are posed on the subsequent pages.

Q1-Q8, and **Q11-Q18** are *multiple choice questions*. For each of these, the only correct answer is one of the answers proposed. To answer a specific question, you are requested without further explanation *clearly* to write, for example, **"7B"** as your answer to question Q7.

Q9-Q10 and **Q19-Q20** are ordinary *text questions*.

Each correct answer to a
- multiple choice question gives 4 points
- text question gives 9 points

The maximum score is thus 100 points.

Linear Programming (Q1-Q8)

For an LP-problem $\mathbf{LP}_\alpha$ with two *nonnegative* decision variables x_1,x_2 the feasible region F_α is determined by the constraints

$$\begin{array}{rrll} x_1 & & \leq 8 & (C_1) \\ 3x_1 & + x_2 & \leq 28 & (C_2) \\ -x_1 & + 2x_2 & \leq 14 & (C_3) \\ & x_2 & \leq \alpha & (C_\alpha) \quad \alpha \text{ real} \end{array}$$

Q1: For α=6 and viewed as a figure in the (x_1,x_2)-plane, F_α is

1A) nonconvex 1B) empty 1C) a tetragon (or a 4-gon)

1D) a pentagon (or 5-gon) 1E) a hexagon (or 6-gon)

For given α, let BFS(α) be the number of basic feasible solutions to $\mathbf{LP}_\alpha$.

Q2: Within which interval of α will BFS(α) reach its maximum value?

2A) $\alpha \leq 0$ 2B) $0 < \alpha \leq 4$ 2C) $4 < \alpha \leq 7$

2D) $7 < \alpha \leq 10$ 2E) $10 \leq \alpha$

Delete constraint C_α and let F be the feasible region defined by constraints C_1,C_2,C_3 only together with the requirement that x_1 and x_2 should remain nonnegative. F is a convex pentagon with vertices

V_1: (0,0), V_2: (0,7), V_3: (?,?), V_4: (8,4), V_5: (8,0)

Let **LP(k)** be the *family of LP-problems* defined by

LP(k): $\max \{ x_1 + kx_2 : (x_1,x_2) \in F,\ x_1 \geq 0,\ x_2 \geq 0 \}$, k real and finite

Q3: *Claim:* "**LP(k)** has an optimal solution in which the decision variables x_1 and x_2 are integers".

3A) True? 3B) False?

For given k, let I(k,β) be an *iso-profit line* with respect to **LP(k)**, that is, I(k,β) is the line determined by

$$x_1 + kx_2 = \beta$$

where β is some constant.

Let $z(V_i,k)$ be the value of $x_1 + kx_2$ at vertex V_i, i = 1, ... , 5, and let k^* be the value of k for which $z(V_2,k) = z(V_3,k)$.

Q4: Which one of the following statements is correct?

4A) Vertex V_2 solves LP(k^*)

4B) Vertex V_3 solves LP(k^*)

4C) Both vertices V_2 and V_3 solve LP(k^*)

4D) Vertex V_5 solves LP(k^*)

Q5: For which values of k,β will I(k,β) be the iso-profit line containing V_2,V_3?

5A) k = 2, β = 12 5B) k = 2, β = -12

5C) k = -2, β = -14 5D) k = -2, β = 14

Q6: Let **LPD(1)** be the *dual* of **LP(1)**. What does **LPD(1)** look like?

6A)

$$\begin{aligned} \min \quad & 8y_1 + 14y_2 + 28y_3 \\ & y_1 - y_2 + 3y_3 \geq 1 \\ & 2y_2 + y_3 \geq 1 \\ & y_1 \geq 0,\ y_2 \geq 0,\ y_3 \geq 0 \end{aligned}$$

6B) min $8y_1 + 14y_2 + 28y_3$

$$\begin{aligned} y_1 - y_2 + 3y_3 &\le 1 \\ 2y_2 + y_3 &\le 1 \end{aligned}$$

$y_1 \ge 0,\ y_2 \ge 0,\ y_3 \ge 0$

6C) max $-8y_1 - 14y_2 - 28y_3$

$$\begin{aligned} y_1 - y_2 + 3y_3 &\le 1 \\ 2y_2 + y_3 &\le 1 \end{aligned}$$

$y_1 \ge 0,\ y_2 \ge 0,\ y_3 \ge 0$

Let x_{i+2} be the slack variable associated with constraint C_i of **LP(1)**, i=1,2,3. When **LP(1)** is solved to optimality, the final slack form reads

$z =$	16	$-(2/7)x_4$	$+\gamma x_5$
$x_1 =$	6	$+(1/7)x_4$	$-(2/7)x_5$
$x_2 =$	10	$-(3/7)x_4$	$-(1/7)x_5$
$x_3 =$	δ	$-(1/7)x_4$	$+(2/7)x_5$

Let y_1^0, y_2^0, y_3^0 be the values of y_1,y_2,y_3 in an optimal solution to LPD(1).

Q7: What is the value of y_1^0 ?

7A) -1 7B) 0 7C) 1/7

7D) 2/7 7D) 3/7 7E) something else

Q8: What is the value of y_3^0 ?

8A) -1 8B) 0 8C) 1/7

8D) 2/7 8D) 3/7 8E) something else

Max flow (Q9-Q10)

Consider a flow network with vertex set V = $\{V_1, V_2, \ldots, V_6\}$. *Booze, Inc.* owns a distillery located at vertex V_1 from where a fleet of trucks, destinated for the export harbour at V_6, is leaving daily.

The good sales made up to Christmas prompted the management to organize a grand New Years's party for all the employees with ample supplies of the company's own products. Unfortunately, indoor fireworks were mixed with powerful rockets causing severe damage when lit in the OR analyst's office. Amongst others, parts of the *Max Flow Model (MF-Booze)* went up in smoke.

What is left of the network itself and the *optimal* solution appears from the following table:

Edge (i,j)	Flow f(i,j)	Capacity c(i,j)
V_1, V_2	?	4
V_1, V_3	6	?
V_2, V_4	?	2
V_2, ?	?	4
V_3, V_4	?	8
V_3, ?	?	7
V_4, V_6	?	9
?, ?	?	?

In spite of heavy hangovers the morning after some *Booze* employee recalls that

$|f| = 9$ is the largest number of trucks that the capacitated network admits of and, hence, the value of max flow

min cut (S,T): $S = \{V_1, V_2, V_5\}$, $T = V \setminus S$

All (directed) paths from V_1 to V_6 include *exactly* 3 edges

Q9 ***(text question):***

Reconstruct the full network and the corresponding optimal solution by replacing all the question marks "?" by vertex numbers (first column) or integers (flow, capacity)

Road pricing: Let E be the edge set comprising the 8 edges of the original flow network and let p(i,j) be the *cost* of traversing edge (i,j) by a single truck,

$$p(i,j) = c(i,j), \qquad \text{all } (i,j) \in E$$

where c(i,j) take the values answering **Q9**.

Furthermore, let k be the *reward* (or profit) obtained by *Booze* for each truck reaching its destination.

With $|f^*|$ denoting the value of the flow from V_1 to V_6, i.e. the number of trucks arriving daily at V_6, *Booze** is the problem of maximizing

$$z = k \times |f^*| - \Sigma_E\, p(i,j)f(i,j)$$

subject to the constraints applying for *MF-Booze*.

Q10 ***(text question):***

Solve *Booze** for all values of k. For each relevant interval of k, only the optimizing value of $|f^*|$ need be specified.