Height is non-decreasing

- Height of a vertex u changes only when relabeling.
- When relabeling u, $h[u] \leq h[v]$ for all $(u,v) \in E_f$
- $h[u] = 1 + \min \{ h[v] : (u,v) \in E_f \}$ and $h[u]$ is in fact increasing.

Height property is preserved

- By induction on the number of push and relabel operations
- Valid after the initialization.
- After RELABEL(u)
  - Residual edge (u,v): relabing ensures that $h(u) \leq h(v)+1$.
  - Residual edge (w,u): $h(w) \leq h(u) + 1$ before implies $h(w) < h(u) + 1$ after.
- After PUSH(u,v)
  - Removal of (u,v) from the residual network removes the height constraint.
  - Addition of (v,u) to the residual network: $h(v) = h(u) – 1 < h(u)$.

f is a preflow ==> no path from s to t in G_f

- Assume that there is a path $p = \{ s=v_0, v_1,..., v_k=t \}$ from s to t in the residual network G_f, $k < |V|$.
- Since h is a height function, $h(v_i) \leq h(v_{i+1}) + 1$ for all $i=0,1,...,k-1$
- Combining these inequalities, $h(s) \leq h(t) + k$.
- But $h(t) = 0$ implying that $h(s) \leq k < |V|$, a contradiction.

Correctness

- Initialization defines a preflow *f*.
- Pushing and relabeling, given a preflow, leaves a preflow.
- At termination no vertex other than *s* and *t* is overflowing, we could otherwise push some flow from this vertex or we could relabel it. Hence *f* is a flow.
- Since *f* is also a preflow, there is no path from *s* to *t* in G_f
- By max-flow min-cut theorem, *f* is a maximum flow.

Termination

- Bound on the number of relabel operations: $O(2|V|^2)$
- Bound on the number of saturating pushes: $O(2|V|\ |E|)$
- Bound on the number of nonsaturating pushes: $O(4|V|^2(|V|+|E|))$

Lemma 26.20

- CLAIM: If *u* is overflowing, then there is a simple path from *u* to *s* in G_f.
- Let *U* = { *v* : there is a simple path from *u* to *v* in G_f }.
- Note that *u* ϵ *U.* Assume that *s* is not in *U*.
- Let *v* ϵ *U* and *w* ϵ *V*-*U*. We claim that *f*(*w*,*v*) ≤ 0. Suppose that *f*(*w*,*v*) > 0. Then *f*(*v*,*w*)< 0. This implies that $c_f(v,w) = c(v,w) - f(v,w) > 0$. This implies that there is a simple path from *u* to w in G_f, a contradiction.

$$e(U)=f(V,U)=f(V-U,U)+f(U,U)=f(V-U,U)\leq 0$$

- Excesses are nonnegative. This implies that *e*(*v*) = 0. In particular, *e*(*u*) = 0, contradicting the assumption that *u* was overflowing.

Bounding #Relabel Operations

- CLAIM: Height at any vertex is at most $2|V| - 1$.
- $h[s] = |V|$, $h[t] = 0$, cannot increase since neither s nor t is ever overflowing (per definition).
- Let u be any vertex other than s and t. Initially, $h[u] = 0$.
- When u is relabeled, it is overflowing. There is a simple path v_0, v_1, ..., v_k, $v_0 = u$, $v_k = s$, $k \leq |V| - 1$, from u to s in the residual network G_f.
- h is a height function, and therefore $h[v_i] \leq h[v_{i+1}] + 1$ for i=0, 1, ..., k-1.
- Hence, $h[u] = h[v_0] \leq h[v_k] + k \leq h[s] + |V| - 1 = 2|V| - 1$.
- The total number of relabel operations: $O(2|V|^2)$

Bounding #Saturating Pushes

- Suppose that a saturated push from u to v occurred. At that time, $h[v] = h[u] - 1$.
- Next push from u to v cannot occur unless some flow is pushed from v to u. This can only happen if $h[u] = h[v] - 1$, that is when $h[v] = h[u] + 1$.
- Since $h[v]$ never decreases, $h[v]$ increases by at least 2 between two saturated pushes from u to v.
- Since $h[u]$ never decreases, $h[u]$ increases by at least 2 between two saturated pushes from v to u.
- Heights start at 0 and increase to at most $2|V| - 1$
- The number of saturated pushes between u and v (in both directions) cannot occur more than $2|V|$ times.
- The total number of saturated pushes is $O(2|V||E|)$.

Bounding #Nonsaturating Pushes

- CLAIM: Number of nonsaturating pushes is $O(4|V|^2(|V|+|E|))$.
- Let

$$Phi = \sum_{v:\, e(v)>0} h[v]$$

- Φ is initially 0 and remains nonnegative all the time. It can change after relabeling, saturating and nonsaturating push.
  - Φ can increase by at most $2|V|-1$ after each relabeling.
  - Φ can increase by at most $2|V|-1$ after each saturating push (heights are unchanged but *v* can become overflowing).
  - Show that a nonsaturating push decreases Φ by at least 1.

Bounding #Nonsaturating Pushes

- Before a nonsaturating push, *u* was overflowing.
- It is not overflowing after a nonsaturating push.
- *v* is overflowing after any push unless it is *s* or *t*.
  - Φ therefore has decreased by exactly *h*[*u*],
  - Φ has increased by 0 (if *v* was *s* or *t* or if it was overflowing before the push) or by *h*[*v*] (if *v* was not overflowing before the push).
  - Since $h[u] - h[v] = 1$, it follows that Φ decreased by at least 1 during a nonsaturating push.
  - $\Phi \geq 0$ and $\Phi \leq 2|V|(2|V|^2 + 2|V||E|) = 4|V|^2(|V| + |E|)$
- Number of nonsaturating pushes is at most $4|V|^2(|V| + |E|)$.