Height is non-decreasing

- Height of a vertex $u$ changes only when relabeling.
- When relabeling $u$, $h[u] \leq h[v]$ for all $(u,v) \in E_f$
- $h[u] = 1 + \min \{ h[v] : (u,v) \in E_f \}$ and $h[u]$ is in fact increasing.
Height property is preserved

• By induction on the number of push and relabel operations
• Valid after the initialization.
• After RELABEL($u$)
  – Residual edge $(u,v)$: relabing ensures that $h(u) \leq h(v)+1$.
  – Residual edge $(w,u)$: $h(w) \leq h(u) + 1$ before implies $h(w) < h(u) + 1$ after.
• After PUSH($u,v$)
  – Removal of $(u,v)$ from the residual network removes the height constraint.
  – Addition of $(v,u)$ to the residual network: $h(v) = h(u) - 1 < h(u)$.
\( f \) is a preflow \( \implies \) no path from \( s \) to \( t \) in \( G_f \)

- Assume that there is a path \( p = \{ s=v_0, v_1, \ldots, v_k=t \} \) from \( s \) to \( t \) in the residual network \( G_f \), \( k < |V| \).
- Since \( h \) is a height function, \( h(v_i) \leq h(v_{i+1}) + 1 \) for all \( i=0,1,\ldots,k-1 \)
- Combining these inequalities, \( h(s) \leq h(t) + k \).
- But \( h(t) = 0 \) implying that \( h(s) \leq k < |V| \), a contradiction.
Correctness

- Initialization defines a preflow $f$.
- Pushing and relabeling, given a preflow, leaves a preflow.
- At termination no vertex other than $s$ and $t$ is overflowing, we could otherwise push some flow from this vertex or we could relabel it. Hence $f$ is a flow.
- Since $f$ is also a preflow, there is no path from $s$ to $t$ in $G_f$.
- By max-flow min-cut theorem, $f$ is a maximum flow.
Termination

- Bound on the number of relabel operations: $O(2|V|^2)$
- Bound on the number of saturating pushes: $O(2|V| |E|)$
- Bound on the number of nonsaturating pushes: $O(4|V|^2(|V|+|E|))$
Lemma 26.20

- **CLAIM:** If $u$ is overflowing, then there is a simple path from $u$ to $s$ in $G_f$.
- Let $U = \{ v :$ there is a simple path from $u$ to $v$ in $G_f \}$. 
- Note that $u \in U$. Assume that $s$ is not in $U$.
- Let $v \in U$ and $w \in V-U$. We claim that $f(w,v) \leq 0$. Suppose that $f(w,v) > 0$. Then $f(v,w) < 0$. This implies that $c_f(v,w) = c(v,w) - f(v,w) > 0$. This implies that there is a simple path from $u$ to $w$ in $G_f$, a contradiction.

$$e(U) = f(V, U) = f(V - U, U) + f(U, U) = f(V - U, U) \leq 0$$

- Excesses are nonnegative. This implies that $e(v) = 0$. In particular, $e(u) = 0$, contradicting the assumption that $u$ was overflowing.
Bounding #Relabel Operations

- CLAIM: Height at any vertex is at most $2|V| - 1$.
- $h[s] = |V|$, $h[t] = 0$, cannot increase since neither $s$ nor $t$ is ever overflowing (per definition).
- Let $u$ be any vertex other than $s$ and $t$. Initially, $h[u] = 0$.
- When $u$ is relabeled, it is overflowing. There is a simple path $v_0, v_1, ..., v_k$, $v_0 = u$, $v_k = s$, $k \leq |V| - 1$, from $u$ to $s$ in the residual network $G_f$.
- $h$ is a height function, and therefore $h[v_i] \leq h[v_{i+1}] + 1$ for $i=0, 1, ..., k-1$.
- Hence, $h[u] = h[v_0] \leq h[v_k] + k \leq h[s] + |V| - 1 = 2|V| - 1$.
- The total number of relabel operations: $O(2|V|^2)$
Bounding #Saturating Pushes

- Suppose that a saturated push from \( u \) to \( v \) occurred. At that time, \( h[v] = h[u] - 1 \).

- Next push from \( u \) to \( v \) cannot occur unless some flow is pushed from \( v \) to \( u \). This can only happen if \( h[u] = h[v] - 1 \), that is when \( h[v] = h[u] + 1 \).

- Since \( h[v] \) never decreases, \( h[v] \) increases by at least 2 between two saturated pushes from \( u \) to \( v \).

- Since \( h[u] \) never decreases, \( h[u] \) increases by at least 2 between two saturated pushes from \( v \) to \( u \).

- Heights start at 0 and increase to at most \( 2|V| - 1 \)

- The number of saturated pushes between \( u \) and \( v \) (in both directions) cannot occur more than \( 2|V| \) times.

- The total number of saturated pushes is \( O(2|V||E|) \).
Bounding #Nonsaturating Pushes

- CLAIM: Number of nonsaturating pushes is $O(4|V|^2(|V|+|E|))$.
- Let
  \[ \Phi = \sum_{v: e(v) > 0} h[v] \]
- $\Phi$ is initially 0 and remains nonnegative all the time. It can change after relabeling, saturating and nonsaturating push.
  - $\Phi$ can increase by at most $2|V|-1$ after each relabeling.
  - $\Phi$ can increase by at most $2|V|-1$ after each saturating push (heights are unchanged but $v$ can become overflowing).
  - Show that a nonsaturating push decreases $\Phi$ by at least 1.
Bounding #Nonsaturating Pushes

- Before a nonsaturating push, \( u \) was overflowing.
- It is not overflowing after a nonsaturating push.
- \( v \) is overflowing after any push unless it is \( s \) or \( t \).
  - \( \Phi \) therefore has decreased by exactly \( h[u] \),
  - \( \Phi \) has increased by 0 (if \( v \) was \( s \) or \( t \) or if it was overflowing before the push) or by \( h[v] \) (if \( v \) was not overflowing before the push).
  - Since \( h[u] - h[v] = 1 \), it follows that \( \Phi \) decreased by at least 1 during a nonsaturating push.
  - \( \Phi \geq 0 \) and \( \Phi \leq 2|V|(2|V|^2 + 2|V||E|) = 4|V|^2(|V| + |E|) \)
- Number of nonsaturating pushes is at most \( 4|V|^2(|V| + |E|) \).