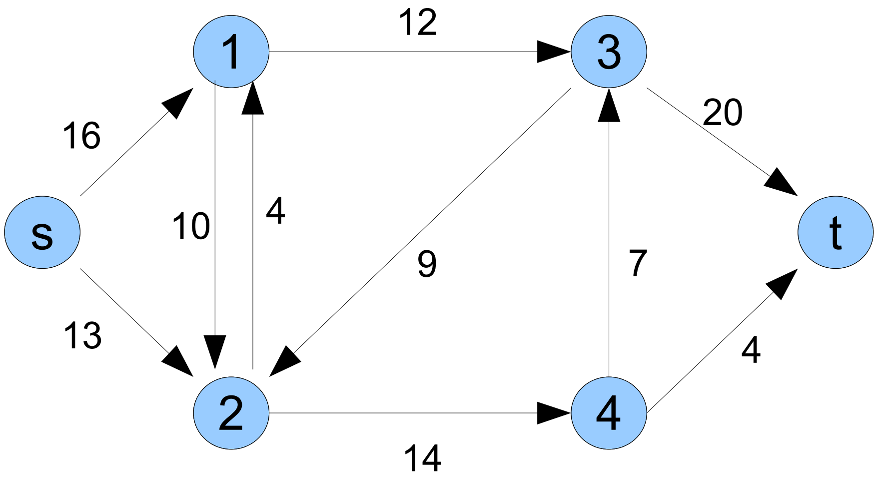
Maximum Flow

- Flow neworks, flow, and max flow problem
- Residual networks, Ford-Fulkerson algorithm
- Cuts and max-flow min-cut theorem
- Edmonds-Karp algorithm
- Matching in bipartite graphs – application of max flow
- Preflow
- Push-relabel algorithm by example
- Correctness of the push-relabel algorithm
- Complexity of push-relabel algorithm

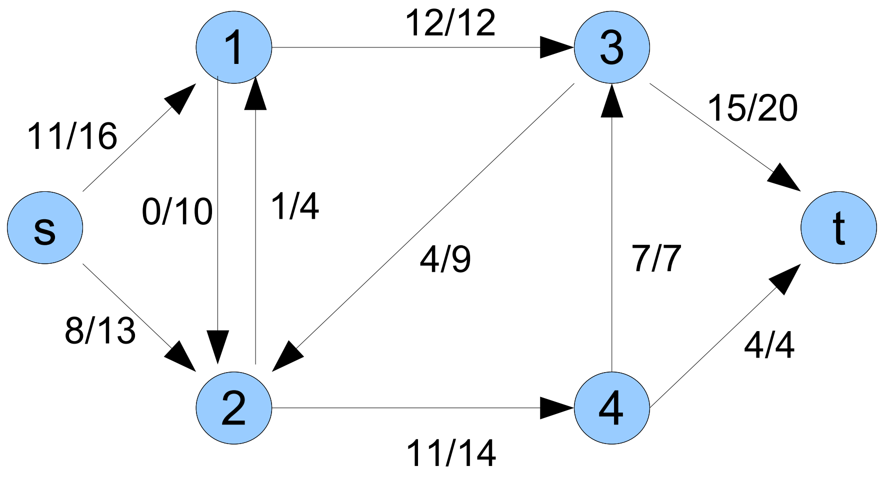
Flow Networks

- Directed graph $G = (V,E)$.
- Edge capacities $c(u,v) \geq 0$ for all $u, v \in V$.
- Source vertex s.
- Sink vertex t.
- Every vertex is reachable from s. Every vertex can reach t.



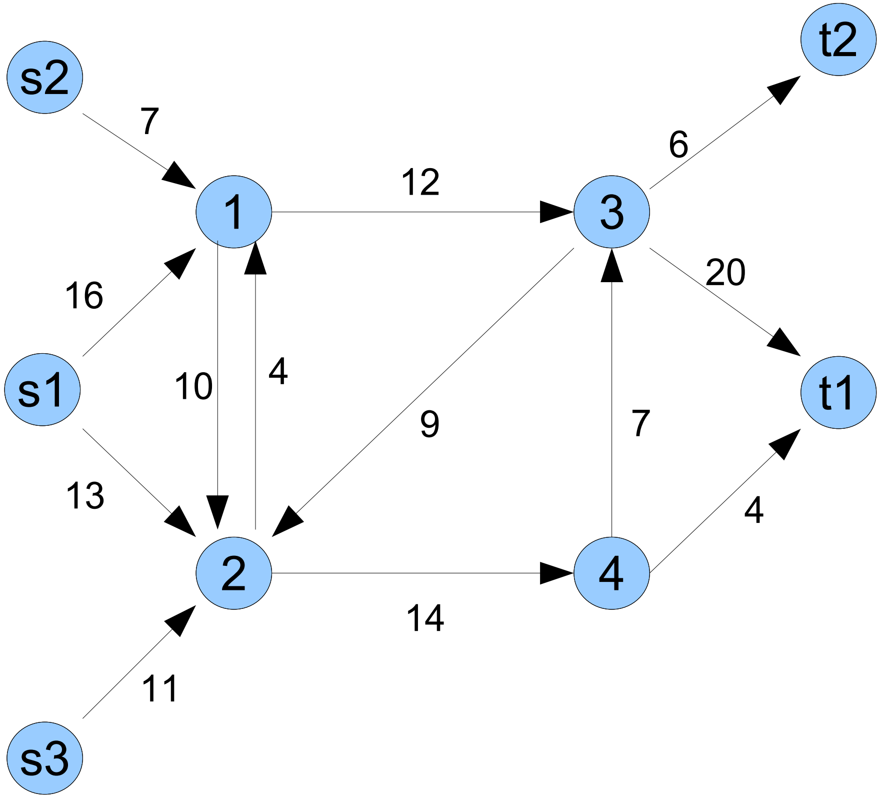
Flow

- Flow; Real-valued function f: $V \times V \rightarrow R$ satisfying;
  - Capacity constraint: $f(u,v) \leq c(u,v)$ for all $u, v \in V$
  - Skew symmetry: $f(u,v) = -f(v,u)$ for all $u, v \in V$
  - Flow conservation: $\sum_{v\in V} f(u,v)=0$ for all $u \in V - \{ s,t \}$
- Flow value $|f|=\sum_{v\in V} f(s,v)$



- Problem: Maximize flow value subject to constraints.

Multiple Sorces and/or Sinks



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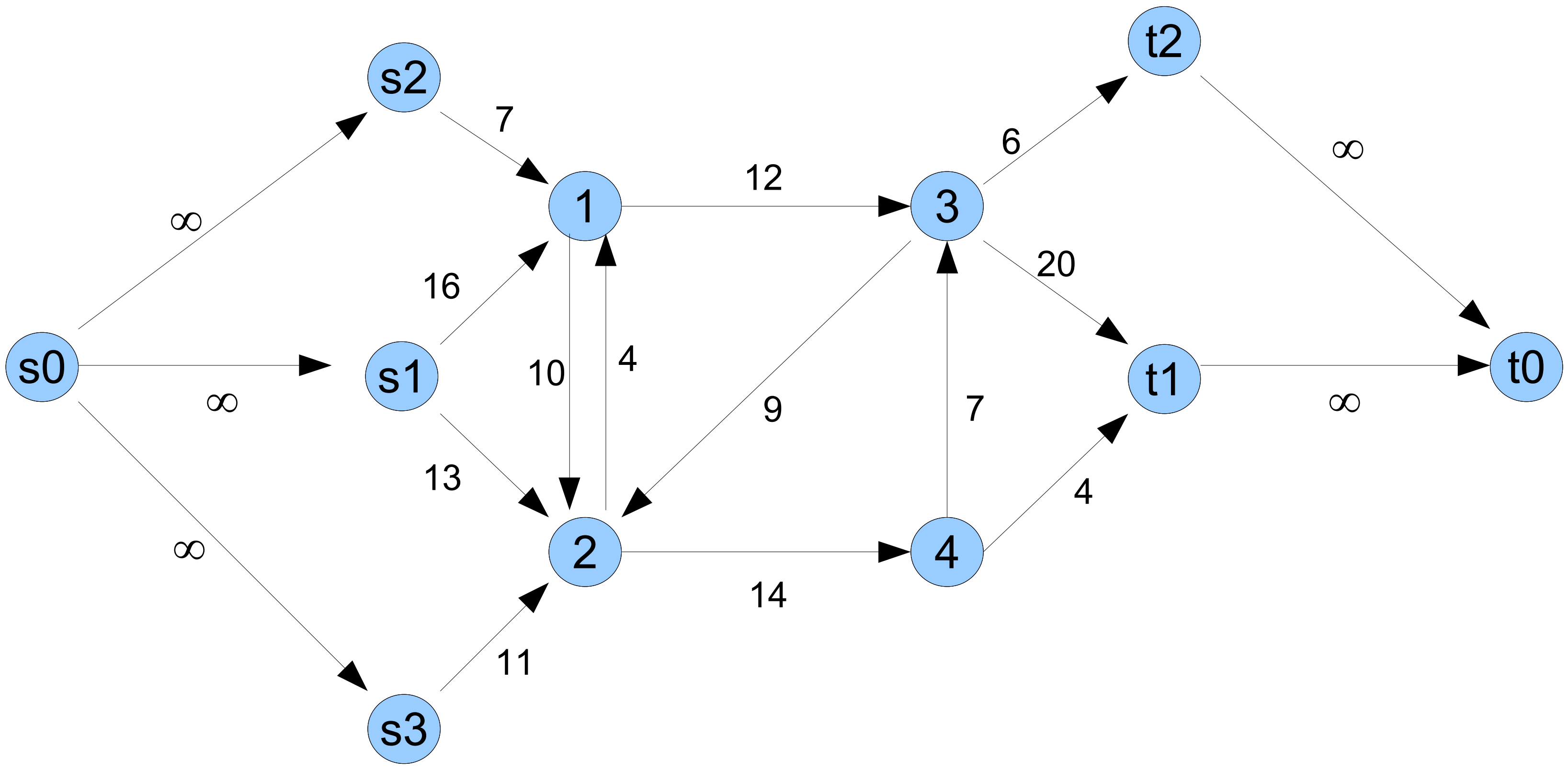


Flow for Sets of Vertices

- X and Y are subsets of vertices in G=(V,E)

$$f(X,Y)=\sum_{x\in X}\sum_{y\in Y} f(x,y)$$

$$f(X,X)=0, \forall X\subseteq V$$

$$f(X,Y)=-f(Y,X), \forall X,Y\subseteq V$$

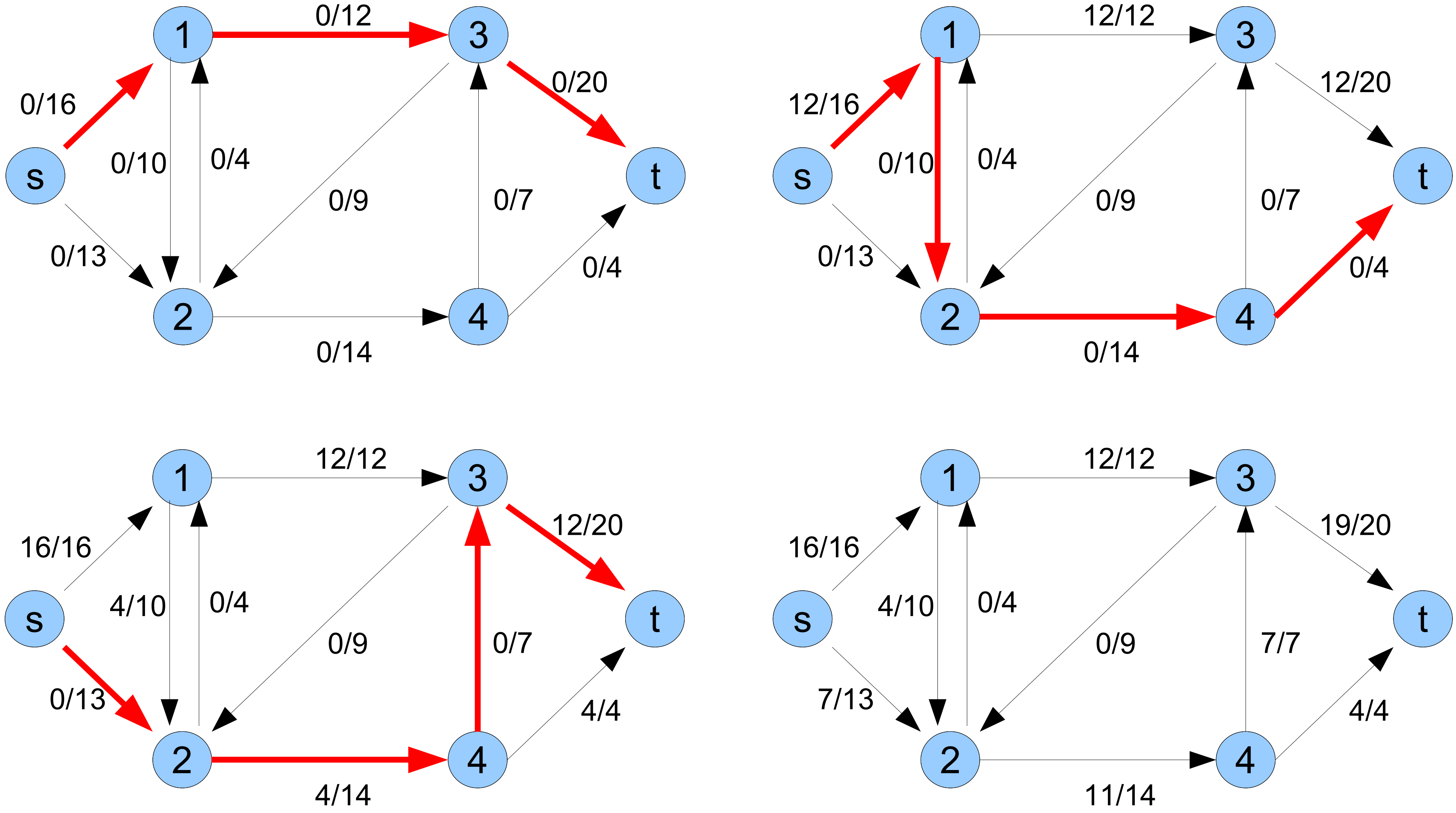
$$f(X\cup Y,Z)=f(X,Z)+f(Y,Z), \forall X,Y,Z\in V, X\cap Y=\emptyset$$

$$f(Z,X\cup Y)=f(Z,X)+f(Z,Y), \forall X,Y,Z\in V, X\cap Y=\emptyset$$

- Proofs: Exercise 26.1-4

Ford-Fulkerson Method

- initialize flow *f* to 0-flow.
- while there is a flow augmenting path do augment *f*.



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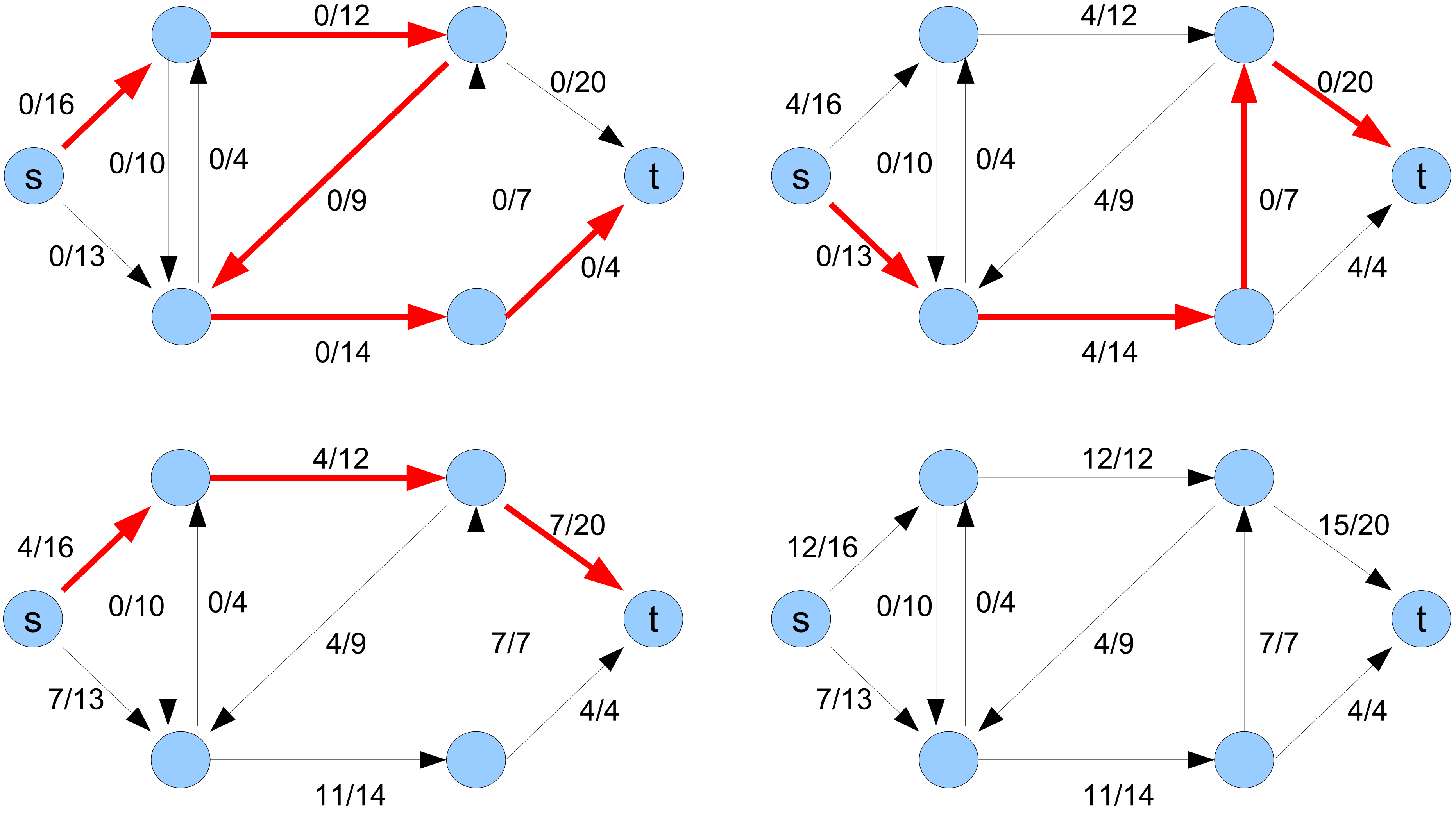
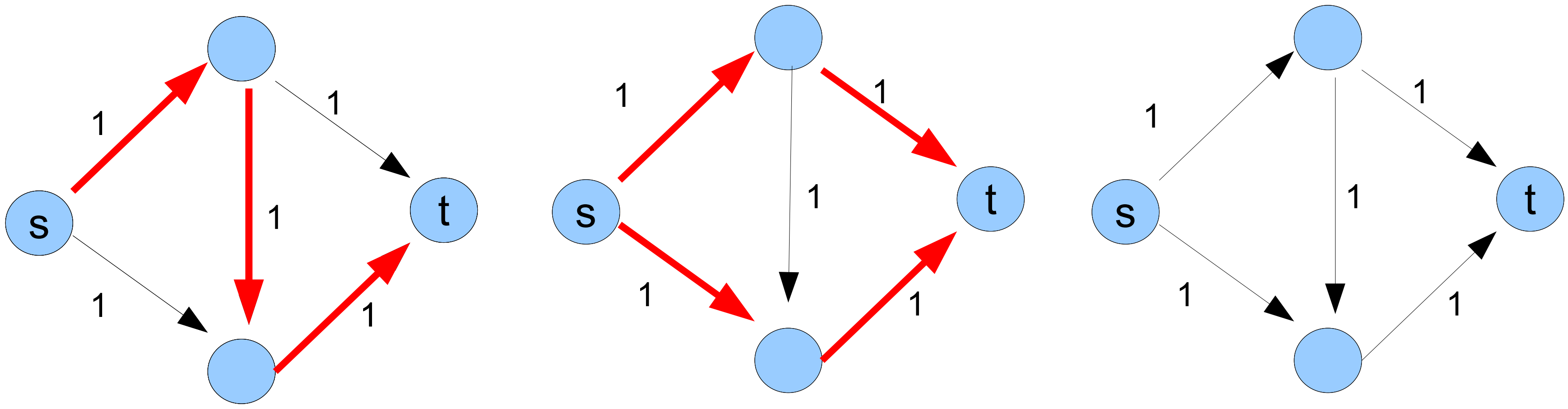
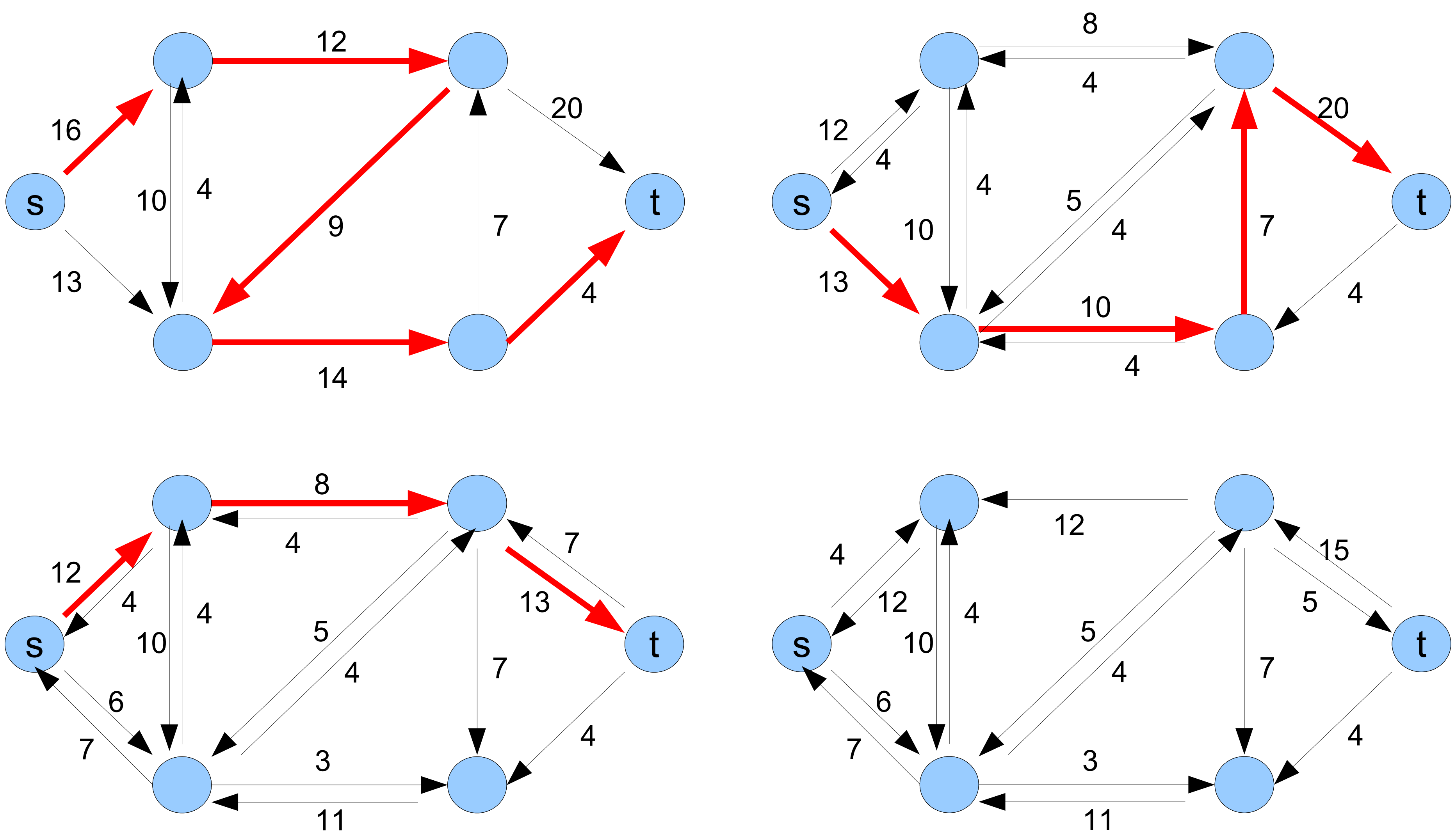


Something is Wrong!

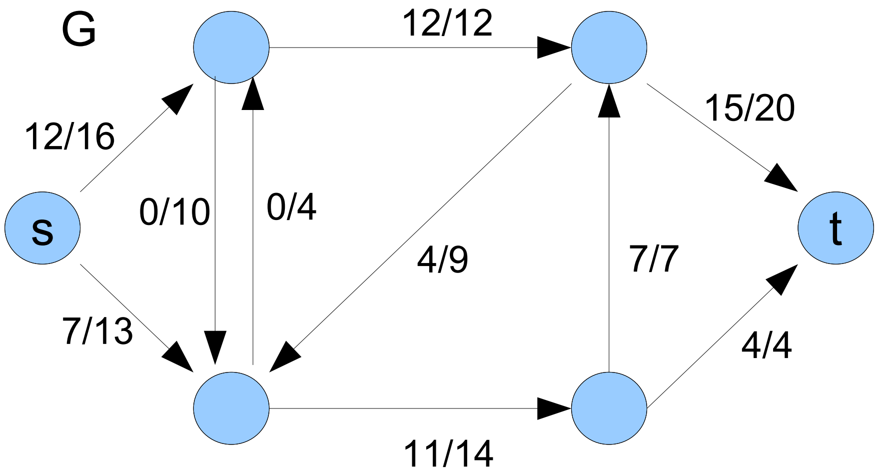


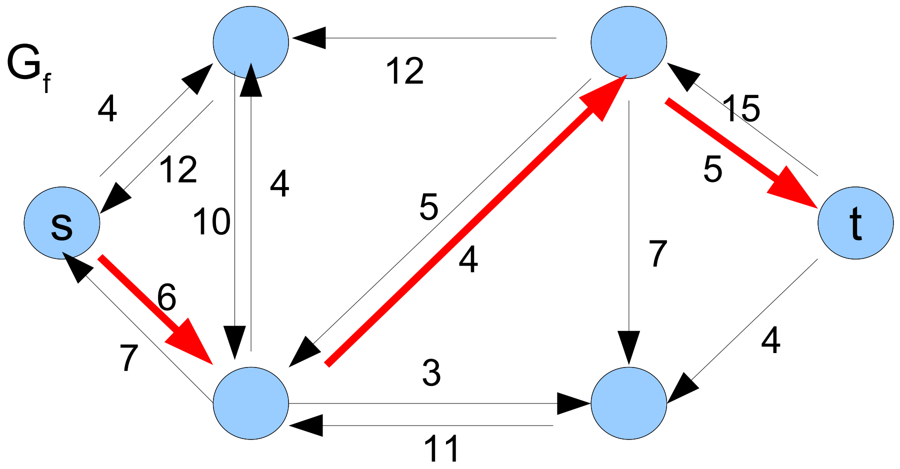
Residual Networks



Flows in residual and original networks

- Let f be a flow in network $G=(V,E)$
- Let f' be a flow in the residual network G_f of G induced by the flow f.
- Define $(f + f')(u,v) = f(u,v) + f'(u,v)$ for all u, v.
- $f + f'$ is a flow in G with value $| f+f' | = | f | + | f' |$.





Skew Symmetry of f + f'

- Prove that $(f + f')(u,v) = -(f + f')(v,u)$

$$\begin{aligned}
(f+f')(u,v) &= f(u,v)+f'(u,v) \\
&= -f(v,u)-f'(v,u) \\
&= -(f(v,u)+f'(v,u)) \\
&= -(f+f')(v,u)
\end{aligned}$$

Capacity Constraint

- Prove that $(f + f')(u,v) \leq c(u,v)$

$$\begin{aligned}(f+f')(u,v) &= f(u,v)+f'(u,v)\\ &\leq f(u,v)+(c(u,v)-f(u,v))\\ &= c(u,v)\end{aligned}$$

Flow Conservation

- Prove that $\sum_{v \in V} (f + f')(u, v) = 0$

$$\begin{aligned} \sum_{v \in V} (f + f')(u, v) &= \sum_{v \in V} (f(u, v) + f'(u, v)) \\ &= \sum_{v \in V} f(u, v) + \sum_{v \in V} f'(u, v) \\ &= 0 + 0 \end{aligned}$$

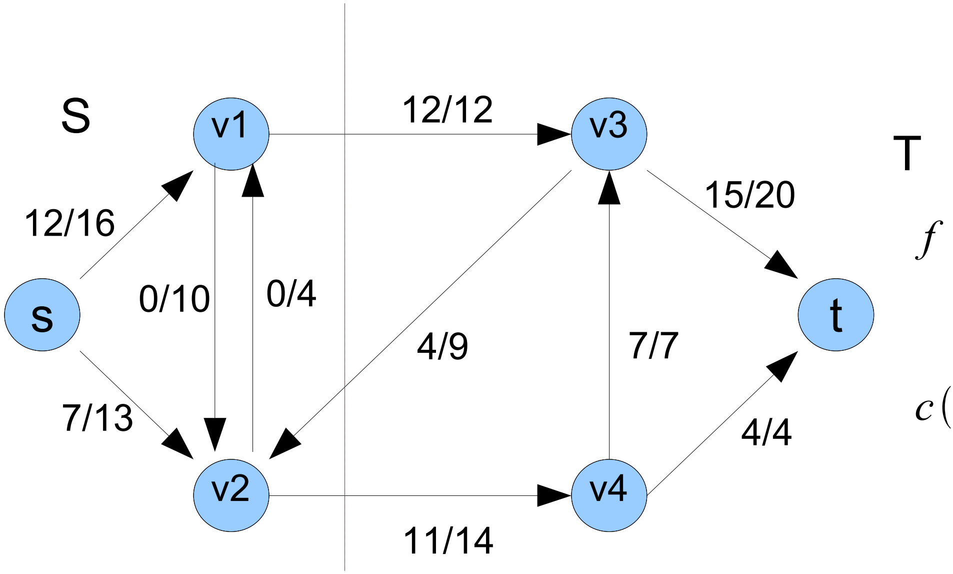
Flow Value

- Prove that $| f + f' | = | f | + | f' |$

$$
\begin{aligned}
|f + f'| &= \sum_{v \in V} (f + f')(s, v) \\
&= \sum_{v \in V} (f(s, v) + f'(s, v)) \\
&= \sum_{v \in V} f(s, v) + \sum_{v \in V} f'(s, v) \\
&= |f| + |f'|
\end{aligned}
$$

Cut of Flow Network

- Let $G=(V,E)$, $S \subseteq V$, $T=V-S$, $s \in S$, $t \in T$.
- Partition (S,T) of V is then called a **cut**.
- **Net flow** across the cut (S,T) for a flow f is defined to be $f(S,T)$.
- **Capacity** $c(S,T)$ of the cut (S,T) is defined to be $\sum_{u \in S,\, v \in T} c(u,v)$
- **Minimum cut** of a flow network is a cut of minimum capacity (over all cuts of the network).



$$f(S,T)=f(v_1,v_3)+f(v_2,v_3)+f(v_2,v_4)=12+(-4)+11=19$$

$$c(S,T)=c(v_1,v_3)+c(v_2,v_4)=12+14=26$$

Net Flow Across Cut

- $f(S,T) = |f|$
- Proof.

$$\begin{aligned}
f(S,T) &= f(S,V) - f(S,S) \\
&= f(S,V) \\
&= f(s,V) + f(S-s,V) \\
&= f(s,V) \\
&= |f|
\end{aligned}$$

Upper Bound on Flow Value

- Let (*S*,*T*) be a cut of minimum capacity in a flow network *G* = (*V*,*E*), and let *f* be a flow in *G*. Then

$$
\begin{aligned}
|f| &= f(S,T) \\
&= \sum_{u \in S} \sum_{v \in T} f(u,v) \\
&\le \sum_{u \in S} \sum_{v \in T} c(u,v) \\
&= c(S,T)
\end{aligned}
$$

Max-Flow Min-Cut Theorem

- Let f be a flow in a flow network $G = (V,E)$ with source s and sink t. The following statements are equivalent:
  1. f is a maximum flow in G.
  2. The residual network G_f has no augmenting paths.
  3. $| f | = c(S,T)$ for some cut (S,T) in G.

(1) ===> (2)

- Suppose that f is a maximum flow in G while G_f has an augmenting path P.
- Hence there is a non-zero flow f' in G_f (along the edges of P), zero elsewhere.
- Hence $f + f'$ is a flow in G with value strictly greater than $| f |$.
- Contradiction.

(2) ===> (3)

- Suppose that G_f has no augmenting path.
- Let S = { $v \in V$: there is a path from s to v in G_f }
- Let $T = V - S$.
- (S,T) is a cut.
- For each (u,v), $u \in S$, $v \in T$, we have $f(u,v) = c(u,v)$; otherwise $(u,$v) would be an edge in G_f.
- Hence, $| f | = f(S,T) = c(S,T)$.

(3) ===> 1

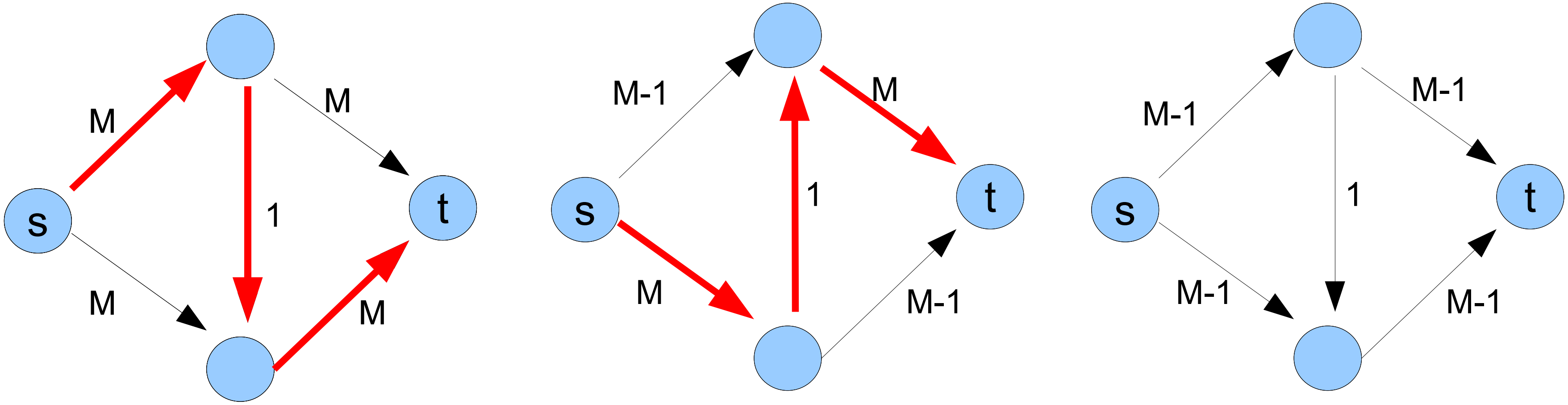
- We have shown that $|f| \leq c(S,T)$ for all cuts (S,T).
- $|f| = c(S,T)$ implies therefore that f is a maximum flow.

Ford-Fulkerson Algorithm

- initialize flow *f* to 0-flow.
- construct the residual network G_f (trivial for 0-flow).
- while there is a flow augmenting path in G_f do
  - augment *f* by pushing as much as possible through the augmenting path.
  - construct the residual network for the increased flow.

Ford-Fulkerson - Complexity

- Searching for an augmenting path: $O(V+E') = O(E)$, use for example depth-first search.
- If capacities are integral, there can be as many as | f* | iterations



Edmonds-Karp Algorithm

- Use breadth-first search!!!
- This variant of Ford-Fulkerson algorithm runs in $O(nm^2)$.