Auxiliary Linear Program

- L: LP in standard form:

$$
\begin{aligned}
max \quad & \sum_{j=1}^{n} c_j x_j \\
s.t. \quad & \sum_{j=1}^{n} a_{ij} x_j \leq b_i \quad for \quad i = 1, 2, \ldots, m \\
& x_j \geq 0 \quad for \quad j = 1, 2, \ldots, n
\end{aligned}
$$

- L_{aux}: Auxiliary LP:

$$
\begin{aligned}
max \quad & -x_0 \\
s.t. \quad & \sum_{j=1}^{n} a_{ij} x_j - x_0 \leq b_i \quad for \quad i = 1, 2, \ldots, m \\
& x_j \geq 0 \quad for \quad j = 0, 1, 2, \ldots, n
\end{aligned}
$$

- L_{aux} is bounded and feasible.

L is feasible ⇔ Optimal objective value of L_{aux} is 0

⇒ Let $(s_1,s_2,...,s_n)$ be a feasible solution for L.

Let s_0=0. Then $(s_0,s_1,s_2,...,s_n)$ is a feasible solution for L_{aux} with the objective value 0.

Since $x_0 \geq 0$ in L_{aux} and the objective is to maximize $-x_0$ (or minimize x_0), $(s_0,s_1,s_2,...,s_n)$ is an optimal solution for L_{aux} with the objective value 0.

⇐ Let $(s_0,s_1,s_2,...,s_n)$ be the optimal solution for L_{aux} with the objective value 0.

Then s_0=0 and $(s_1,s_2,...,s_n)$ is a feasible solution for L.

If L is infeasible $\Leftrightarrow$ L_{aux} has a negative optimal solution

$\Rightarrow$ Suppose that L is infeasible.

- Optimal objective value of L_{aux} is not 0.
- L_{aux} is bounded: Let

$$s_0 = |min_{i=1}^{m}\{b_i\}|$$

(s_0, 0, 0,..., 0) is a feasible solution for L_{aux} with negative objective value.

$\Leftarrow$ Obvious

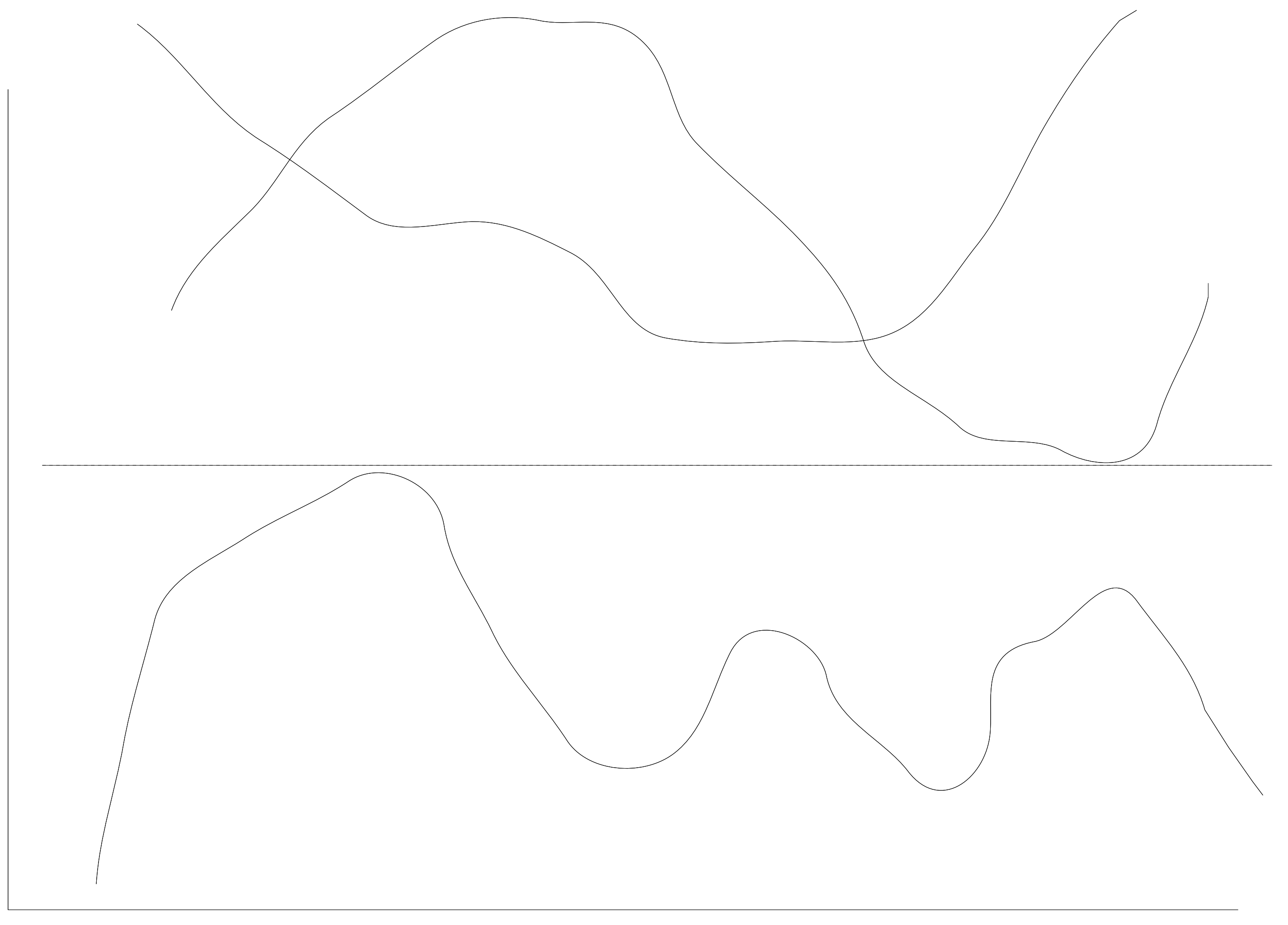
If L is feasible then L_{aux} returns a basic solution for L

- We create L_{aux} only if the first basic solution to L is infeasible. This is the case when some *b*-values in L are negative. The same *b*-values reapper in L_{aux} and therefore its first basic solution is infeasible.
- Assume that $b_l < 0$ is the smallest *b*-value in L. We have to show that after the first pivoting (where x_0 enters the basis and x_l leaves the basis, all *b*-values become nonnegative.
- This is a straightforward algebraic manipulation, see p. 815-6.
- Optimal solution to L_{aux} has objective value 0. When x_0 is removed and nonbasic variables in the objective function for L are substituted, a slack form feasible for L is obtained.

SIMPLEX – Open Issues

- How to decide that LP is feasible? **SOLVED**
- What to do if the initial basic solution is infeasible? **SOLVED**
- How to select entering and leaving variables? **SOLVED**
- How to decide that LP is unbounded? **SOLVED**
- Does SIMPLEX terminate? **SOLVED**
- Does it terminate with an optimal solution?

Duality



Upper Bounds on Maximization LP

$$
\begin{array}{llllllllll}
max & 4x_1 & + & x_2 & + & 5x_3 & + & 3x_4 & & \\
s.t. & x_1 & - & x_2 & - & x_3 & + & 3x_4 & \leq & 1 \\
 & 5x_1 & + & x_2 & + & 3x_3 & + & 8x_4 & \leq & 55 \\
 & -x_1 & + & 2x_2 & + & 3x_3 & - & 5x_4 & \leq & 3 \\
 & x_1, & & x_2, & & x_3, & & x_4 & \geq & 0
\end{array}
$$

- Multiply second constraint by 5/3:

$$\frac{25}{3}x_1 + \frac{5}{3}x_2 + 5x_3 + \frac{40}{3}x_4 \leq \frac{275}{3}$$

$$4x_1 + x_2 + 5x_3 + 3x_4 \leq \frac{25}{3}x_1 + \frac{5}{3}x_2 + 5x_3 + \frac{40}{3}x_4 \leq \frac{275}{3}$$

Upper Bounds on Maximization LP

$$\begin{array}{lcccccccccc} max & 4x_1 & + & x_2 & + & 5x_3 & + & 3x_4 & & \\ s.t. & x_1 & - & x_2 & - & x_3 & + & 3x_4 & \leq & 1 \\ & 5x_1 & + & x_2 & + & 3x_3 & + & 8x_4 & \leq & 55 \\ & -x_1 & + & 2x_2 & + & 3x_3 & - & 5x_4 & \leq & 3 \\ & x_1, & & x_2, & & x_3, & & x_4 & \geq & 0 \end{array}$$

- Add the second constraint to the third constraint:

$$4x_1 + 3x_2 + 6x_3 + 3x_4 \leq 58$$

$$4x_1 + x_2 + 5x_3 + 3x_4 \leq 4x_1 + 3x_2 + 6x_3 + 3x_4 \leq 58$$

Upper Bounds on Maximization LP

$$
\begin{array}{lcccccccccc}
max & 4x_1 & + & x_2 & + & 5x_3 & + & 3x_4 & & \\
s.t. & x_1 & - & x_2 & - & x_3 & + & 3x_4 & \leq & 1 \\
 & 5x_1 & + & x_2 & + & 3x_3 & + & 8x_4 & \leq & 55 \\
 & -x_1 & + & 2x_2 & + & 3x_3 & - & 5x_4 & \leq & 3 \\
 & x_1, & & x_2, & & x_3, & & x_4 & \geq & 0
\end{array}
$$

- Construct a linear combination of the constraints using nonnegative multipliers y_1, y_2, and y_3:

$$y_1(x_1 - x2 - x_3 + 3x_4) + y_2(5x_1 + x_2 + 3x_3 + 8x_4) + y_3(-x_1 + 2x_2 + 3x_3 - 5x_4) \leq y_1 + 55y_2 + 3y_3$$

$$(y_1 + 5y_2 - y_3)x_1 + (-y_1 + y_2 + 2y_3)x_2 + (-y_1 + 3y_2 + 3y_3)x_3 + (3y_1 + 8y_2 - 5y_3)x_4 \leq y_1 + 55y_2 + 3y_3$$

- Left-hand side will be an upper bound for the LP if the coefficients at each x_j are at least as big as the corresponding coefficients in the objective function

$$y_1 + 5y_2 - y_3 \geq 4 \qquad -y_1 + y_2 + 2y_3 \geq 1 \qquad 3y_1 + 8y_2 - 5y_3 \geq 5 \qquad -y_1 + 3y_2 + 3y_3 \geq 3$$

- Any set of nonnegative multipliers y_i satisfying these inequalities also satisfies $4x_1 + x_2 + 5x_3 + 3x_4 \leq y_1 + 55y_2 + 3y_3$
- Good upper bound: minimize right-hand side s.t. constraints.

Good Upper Bound

$$
\begin{array}{lccccccc}
min & y_1 & + & 55y_2 & + & 3y_3 & & \\
s.t. & y_1 & + & 5y_2 & - & y_3 & \geq & 4 \\
 & -y_1 & + & y_2 & + & 2y_3 & \geq & 1 \\
 & -y_1 & + & 3y_2 & + & 3y_3 & \geq & 5 \\
 & 3y_1 & + & 8y_2 & - & 5y_3 & \geq & 3 \\
 & y_1, & & y_2, & & y_3 & \geq & 0
\end{array}
$$

$$
\begin{array}{lccccccccc}
max & 4x_1 & + & x_2 & + & 5x_3 & + & 3x_4 & & \\
s.t. & x_1 & - & x_2 & - & x_3 & + & 3x_4 & \leq & 1 \\
 & 5x_1 & + & x_2 & + & 3x_3 & + & 8x_4 & \leq & 55 \\
 & -x_1 & + & 2x_2 & + & 3x_3 & - & 5x_4 & \leq & 3 \\
 & x_1, & & x_2, & & x_3, & & x_4 & \geq & 0
\end{array}
$$

Duality

- The identification of a dual problem is almost always coupled with the discovery of a polynomial-time algorithm.
- Duality provides a proof that a solution is optimal.

LP in Standard Form and Its Dual

$$
\begin{array}{lccccccc}
max & 3x_1 & + & x_2 & + & 2x_3 & & \\
s.t. & x_1 & + & x_2 & + & 3x_3 & \leq & 30 \\
 & 2x_1 & + & 2x_2 & + & 5x_3 & \leq & 24 \\
 & 4x_1 & + & x_2 & + & 2x_3 & \leq & 36 \\
 & x_1 & , & x_2 & , & x_3 & \geq & 0
\end{array}
$$

$$
\begin{array}{lccccccc}
min & 30y_1 & + & 24y_2 & + & 36y_3 & & \\
s.t. & y_1 & + & 2y_2 & + & 4y_3 & \geq & 3 \\
 & y_1 & + & 2y_2 & + & y_3 & \geq & 1 \\
 & 3y_1 & + & 5y_2 & + & 2y_3 & \geq & 2 \\
 & y_1 & , & y_2 & , & y_3 & \geq & 0
\end{array}
$$

LP in Standard Form and Its Dual

$$\begin{aligned} max \quad & \sum_{j=1}^{n} c_j x_j \\ s.t. \quad & \sum_{j=1}^{n} a_{ij} x_j \leq b_i \quad & for \quad i=1,2,\ldots,m \\ & x_j \geq 0 \quad & for \quad j=1,2,\ldots,n \end{aligned}$$

$$\begin{aligned} min \quad & \sum_{i=1}^{m} b_i y_i \\ s.t. \quad & \sum_{i=1}^{m} a_{ij} y_i \geq c_j \quad & for \quad j=1,2,\ldots,n \\ & y_i \geq 0 \quad & for \quad i=1,2,\ldots,m \end{aligned}$$

Weak Duality

- ***x****: feasible solution to the primal LP.
- ***y****: feasible solution to the dual LP.
- Claim $\sum_{j=1}^{n} c_j x_j^* \le \sum_{i=1}^{m} b_i y_i^*$
- Proof:

$$
\begin{aligned}
\sum_{j=1}^{n} c_j x_j^* &\le \sum_{j=1}^{n} \left(\sum_{i=1}^{m} a_{ij} y_i^*\right) x_j^* && \text{from the dual} \\
&= \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij} x_j^*\right) y_i^* \\
&\le \sum_{i=1}^{m} b_i y_i^* && \text{from the primal}
\end{aligned}
$$

Importance of Weak Duality

- ***x****: feasible solution to the primal LP.
- ***y****: feasible solution to the dual LP.
- If

$$\sum_{j=1}^{n} c_j x_j^* = \sum_{i=1}^{m} b_i y_i^*$$

then ***x**** and ***y**** are optimal solutions to the primal and to the dual LPs, respectively.

Final Feasible Basic Solution and Corresponding Dual Solution

$$
\begin{array}{lllllllll}
max & z & = & 28 & - & x_3/6 & - & x_5/6 & - & 2x_6/3 \\
s.t. & x_4 & = & 18 & - & x_3/2 & + & x_5/2 & + & 0x_6 \\
 & x_2 & = & 4 & - & 8x_3/3 & - & 2x_5/3 & + & x_6/3 \\
 & x_1 & = & 8 & + & x_3/6 & + & x_5/6 & - & x_6/3
\end{array}
$$

Basic variables: x_1=8, x_2=4, x_4=18

Objective value z = 28

$$
y_i = \begin{cases} -c'_{n+i} & \text{if } (n+i) \in N \\ 0 & \text{otherwise} \end{cases}
$$

y_1 = 0 (since x_4 is basic), y_2 = 1/6, y_3 = 2/3

$$
\begin{array}{lllllllll}
min & 30y_1 & + & 24y_2 & + & 36y_3 & & \\
s.t. & y_1 & + & 2y_2 & + & 4y_3 & \geq & 3 \\
 & y_1 & + & 2y_2 & + & y_3 & \geq & 1 \\
 & 3y_1 & + & 5y_2 & + & 2y_3 & \geq & 2 \\
 & y_1 & , & y_2 & , & y_3 & \geq & 0
\end{array}
$$

Feasible Solution to the Dual

$$
\begin{array}{llllllll}
\min & 30y_1 & + & 24y_2 & + & 36y_3 & & \\
s.t. & y_1 & + & 2y_2 & + & 4y_3 & \geq & 3 \\
 & y_1 & + & 2y_2 & + & y_3 & \geq & 1 \\
 & 3y_1 & + & 5y_2 & + & 2y_3 & \geq & 2 \\
 & y_1 & , & y_2 & , & y_3 & \geq & 0
\end{array}
$$

$y_1 = 0$ (since x_4 is basic)
$y_2 = 1/6$
$y_3 = 2/3$

Objective value: $30 \times 0 + 24 \times 1/6 + 36 \times 2/3 = 28$

$1 \times 0 + 2 \times 1/6 + 4 \times 2/3 = 3$
$1 \times 0 + 2 \times 1/6 + 1 \times 2/3 = 1$
$3 \times 0 + 5 \times 1/6 + 2 \times 2/3 = 13/6$

Duality Theorem

- Suppose that SIMPLEX terminates with a feasible basic solution $\boldsymbol{x}^* = (x_1^*, x_2^*, \ldots, x_n^*)$ with:
  - N and B denoting the nonbasic and basic variables for the final slack form
  - c' denoting the coefficients of the objective function in the final slack form.
- Let $\boldsymbol{y}^* = (y_1^*, y_2^*, \ldots, y_m^*)$ be defined by

$$y_i^* = \begin{cases} -c'_{n+i} & \text{if } (n+i) \in N \\ 0 & \text{otherwise} \end{cases}$$

- Then $\boldsymbol{x}^*$ is an optimal solution to the primal LP, $\boldsymbol{y}^*$ is an optimal solution to the dual LP and

$$\sum_{j=1}^{n} c_j x_j^* = \sum_{i=1}^{m} b_i y_i^*$$

Proof of Duality Theorem

- We have to show that
  - y^* is feasible solution for the dual, and
  - $\sum_{j=1}^{n} c_j x_j^* = \sum_{i=1}^{m} b_i y_i^*$

Proof of Duality Theorem

- Objective function of the final slack form of the primal is:

$$v^* + \sum_{j \in N} c_j^* x_j = v^* + \sum_{j \in N} c_j^* x_j + \sum_{j \in B} 0\, x_j = v^* + \sum_{j=1}^{n+m} c_j^* x_j$$

- Optimal value of the primal objective function: $v^* = \sum_{j=1}^{n} c_j x_j^*$

$$\sum_{j=1}^{n} c_j x_j = v^* + \sum_{j=1}^{n} c_j^* x_j + \sum_{j=n+1}^{n+m} c_j^* x_j = v^* + \sum_{j=1}^{n} c_j^* x_j - \sum_{i=1}^{m} y_i^* (b_i - \sum_{j=1}^{n} a_{ij} x_j)$$

$$= (v^* - \sum_{i=1}^{m} b_i y_i^*) + \sum_{j=1}^{n} (c_j^* + \sum_{i=1}^{m} a_{ij} y_i^*) x_j$$

- This must hold for every choice of x_1, x_2, ..., x_n. Hence

$v^* = \sum_{i=1}^{m} b_i y_i^*$ and $c_j = c_j^* + \sum_{i=1}^{m} a_{ij} y_i^*, \forall j = 1, 2, \ldots, n$

- Since $c_j^* \leq 0, \forall k = 1, 2, \ldots, n+m$, we get

$\sum_{i=1}^{m} a_{ij} y_i^* \geq c_j, \forall j = 1, 2, \ldots, n$ and $y_i^* \geq 0, \forall i = 1, 2, \ldots, m$

Primal Dual Combinations

		Dual		
		Optimal	**Infeasible**	**Unbounded**
	Optimal	*Possible*	*Impossible*	*Impossible*
Primal	**Infeasible**	*Impossible*	*Possible*	*Possible*
	Unbounded	*Impossible*	*Possible*	*Impossible*

Both Primal and Dual Infeasible

$$
\begin{array}{llcllcr}
max & 2x_1 & - & x_2 & & \\
s.t. & x_1 & - & x_2 & \leq & 1 \\
 & -x_1 & + & x_2 & \leq & -2 \\
 & x_1, & & x_2 & \geq & 0
\end{array}
$$

$$
\begin{array}{llcllcr}
min & y_1 & - & 2y_2 & & \\
s.t. & y_1 & - & y_2 & \geq & 2 \\
 & -y_1 & + & y_2 & \geq & -1 \\
 & y_1, & & y_2 & \geq & 0
\end{array}
$$

Practical Implications

- If $m >> n$ then the number of constraint in the dual will be much smaller than in the primal.
- Number of pivots in SIMPLEX is usually less than $1.5m$ and only rarely is higher than $3m$.
- Number of pivots increases very slowly with n.
- Solving dual will in such cases be more efficient.