General LP Problem

$$\begin{aligned} &min \quad \sum_{j=1}^{n} c_j x_j \\ &s.t. \quad m \text{ linear constraints} \end{aligned}$$

- Minimization or maximization of a **linear** objective function with *n* real-valued variables.
- An optimal solution must satisfy *m* **linear** constraints (inequalities or equalities).
- Strict inequalities are not allowed.
- ”programming” in ”linear programming” does not refer to any code. It was chosen before computer programming was born.

LP in Standard Form

- *n* variables, *m* constraints

$$
\begin{aligned}
max \quad & \sum_{j=1}^{n} c_j x_j \\
s.t. \quad & \sum_{j=1}^{n} a_{ij} x_j \quad \leq \quad b_i \quad for \quad i = 1, 2, \ldots, m \\
& \qquad\quad x_j \quad \geq \quad 0 \quad for \quad j = 1, 2, \ldots, n
\end{aligned}
$$

LP in Slack Form

$$\begin{array}{llll} max & \sum_{j=1}^{n} c_j x_j & & \\ s.t. & \sum_{j=1}^{n} a_{ij} x_j + x_{n+i} & = b_i & for \quad i=1,2,\ldots,m \\ & x_j & \geq 0 & for \quad j=1,2,\ldots,n+m \end{array}$$

$$\begin{array}{llll} max & z & = \sum_{j=1}^{n} c_j x_j & \\ s.t. & x_{n+i} & = b_i - \sum_{j=1}^{n} a_{ij} x_j & for \quad i=1,2,\ldots,m \\ & x_j & \geq 0 & for \quad j=1,2,\ldots,n+m \end{array}$$

$$\begin{array}{lll} z & = 0 + \sum_{j=1}^{n} c_j x_j & \\ x_{n+i} & = b_i - \sum_{j=1}^{n} a_{ij} x_j & for \quad i=1,2,\ldots,m \end{array}$$

Pivoting in General

- PIVOT(*N*,*B*,*A*,*b*,*c*,*v*,*l*,*e*)
  - Compute the coefficients of the bounding constraint so that the **entering** basic variable x_e is expressed as a linear combination of the other variables.

$$b_e = b_l / a_{le} \qquad a_{ej} = a_{lj} / a_{le}, \forall j \in N \setminus e \qquad a_{el} = 1 / a_{le}$$

  - Compute the coefficients of the remaining constraints and the objective function (by substituting x_e by the right-hand side of the rewritten binding equation).

$$b_i = b_i - a_{ie} b_e, \forall i \in B \setminus l \quad a_{ij} = a_{ij} - a_{ie} a_{ej}, \forall j \in N \setminus e \quad a_{il} = -a_{ie} a_{el}$$

$$v = v + c_e b_e \qquad c_j = c_j - c_e a_{ej}, \forall j \in N \setminus e \qquad c_l = -c_e a_{el}$$

  - Compute new sets of basic and nonbasic variables (remove x_e from *N* and add it to *B*, remove x_l from *B* and add it to *N*).

SIMPLEX – Example Continued

- LP in slack form:

$$
\begin{array}{lllllllll}
z & = & 0 & + & 3x_1 & + & x_2 & + & 2x_3 \\
x_4 & = & 30 & - & x_1 & - & x_2 & - & 3x_3 \\
x_5 & = & 24 & - & 2x_1 & - & 2x_2 & - & 5x_3 \\
x_6 & = & 36 & - & 4x_1 & - & x_2 & - & 2x_3
\end{array}
$$

- Set all **nonbasic** variables (right-hand side) to 0.
- Compute values of **basic** variables: x_4=30, x_5=24, x_6=36.
- Compute the objective value z (= 0).
- This gives the feasible basic solution (0,0,0,30,24,36).
- It is feasible; not always the case – we were lucky.

SIMPLEX – Open Issues

- How to decide that LP is feasible?
- What to do if the initial basic solution is infeasible?
- How to select entering and leaving variables?
- How to decide that LP is unbounded?
- Does SIMPLEX terminate?
- Does it terminate with an optimal solution?

Termination

- SIMPLEX computes a feasible basic solution during each iteration.
- When does SIMPLEX terminate?
  - When all coefficients in the objective function are negative.
  - When it becomes obvious that LP is unbounded.

SIMPLEX: LP is Unbounded

- Can x_2 be increased without violating feasibility? By how much?

$$
\begin{aligned}
z &= 111/4 + x_2/16 - x_5/8 - 11x_6/16 \\
x_4 &= 69/4 + 3x_2/16 + 5x_5/8 - x_6/16 \\
x_3 &= 3/2 + 3x_2/8 - x_5/4 + x_6/8 \\
x_1 &= 33/4 + x_2/16 + x_5/8 - 5x_6/16
\end{aligned}
$$

- If x_2 is increased then x_4 also increases.
- If x_2 is increased then x_3 also increases.
- If x_2 is increased then x_1 also increases.
- No constraint is binding; LP is unbounded.

Termination

- The number of basic solutions is finite: Number of basic variables is *m.* They are selected from among *m*+*n* variables. This can be done in

$$\binom{m+n}{m} = \frac{(n+m)!}{n!\,m!}$$

  ways

- Each basic solution has exactly one objective value. If the objective value increases at each iteration, we will eventually end up with a solution where the coefficients of the objective function are all negative (or we will realize that the LP is unbounded).
- Is it possible that the objective value does not change?

Degeneracy

$$\begin{aligned} z &= 0 + x_1 + x_2 + x_3 \\ x_4 &= 8 - x_1 - x_2 \\ x_5 &= x_2 - x_3 \end{aligned}$$

$$\begin{aligned} z &= 8 + x_3 - x_4 \\ x_1 &= 8 - x_2 - x_4 \\ x_5 &= x_2 - x_3 \end{aligned}$$

$$\begin{aligned} z &= 8 + x_2 - x_4 - x_5 \\ x_1 &= 8 - x_2 - x_4 \\ x_3 &= x_2 - x_5 \end{aligned}$$

Degeneracy

$$
\begin{aligned}
z &= 8 + x_2 - x_4 - x_5 \\
x_1 &= 8 - x_2 - x_4 \\
x_3 &= x_2 - x_5
\end{aligned}
$$

$$
\begin{aligned}
z &= 16 - x_1 - 2x_4 - x_5 \\
x_2 &= 8 - x_1 - x_4 \\
x_3 &= 8 - x_1 - x_4 - x_5
\end{aligned}
$$

Cycling

- Is it possible to get the same slack form more than once? Than SIMPLEX has a problem.
- It is in fact possible. To show this we use a specific rules for selecting entering and leaving variables at each iteration.
- The entering variable will always be a nonbasic variable that has the largest positive coefficient in the z-row.
- If two or more basic variables compete for leaving the basis, then the candidate with the smallest subscript will be made to leave.

Cycling - Example

$$
\begin{aligned}
z &= 0 + 10x_1 - 57x_2 - 9x_3 - 24x_4 \\
x_5 &= 0 - 0.5x_1 + 5.5x_2 + 2.5x_3 - 9x_4 \\
x_6 &= 0 - 0.5x_1 + 1.5x_2 + 0.5x_3 - x_4 \\
x_7 &= 1 - x_1
\end{aligned}
$$

$$
\begin{aligned}
z &= 0 + 53x_2 + 41x_3 - 204x_4 - 20x_5 \\
x_1 &= 0 + 11x_2 + 5x_3 - 18x_4 - 2x_5 \\
x_6 &= 0 - 4x_2 - 2x_3 + 8x_4 + x_5 \\
x_7 &= 1 - 11x_2 - 5x_3 + 18x_4 + 2x_5
\end{aligned}
$$

$$
\begin{aligned}
z &= 0 + 14.5x_3 - 98x_4 - 6.75x_5 - 13.25x_6 \\
x_1 &= 0 - 0.5x_3 + 4x_4 + 0.75x_5 - 2.75x_6 \\
x_2 &= 0 - 0.5x_3 + 2x_4 + 0.25x_5 - 0.25x_6 \\
x_7 &= 1 + 0.5x_3 - 4x_4 - 0.75x_5 - 13.25x_6
\end{aligned}
$$

Cycling - Example

$$
\begin{aligned}
z &= 0 + 18x_4 + 15x_5 - 93x_6 - 29x_1\\
x_2 &= 0 - 2x_4 - 0.5x_5 + 2.5x_6 + x_1\\
x_3 &= 0 + 8x_4 + 1.5x_5 - 5.5x_6 - 2x_1\\
x_7 &= 1 - x_1
\end{aligned}
$$

$$
\begin{aligned}
z &= 0 + 10.5x_5 - 70.5x_6 - 20x_1 - 9x_2\\
x_3 &= 0 - 0.5x_5 + 4.5x_6 + 2x_1 - 4x_2\\
x_4 &= 0 - 0.25x_5 + 1.25x_6 + 0.5x_1 - 0.5x_2\\
x_7 &= 1 - x_1
\end{aligned}
$$

Cycling - Example

$$
\begin{aligned}
z &= 0 + 24x_6 + 22x_1 - 93x_2 - 21x_3 \\
x_4 &= 0 - x_6 - 0.5x_1 + 1.5x_2 + 0.5x_3 \\
x_5 &= 0 + 9x_6 + 4x_1 - 8x_2 - 2x_3 \\
x_7 &= 1 - x_1
\end{aligned}
$$

$$
\begin{aligned}
z &= 0 + 10x_1 - 57x_2 - 9x_3 - 24x_4 \\
x_5 &= 0 - 0.5x_1 + 5.5x_2 + 2.5x_3 - 9x_4 \\
x_6 &= 0 - 0.5x_1 + 1.5x_2 + 0.5x_3 - x_4 \\
x_7 &= 1 - x_1
\end{aligned}
$$

Cycling

- **Claim**: If SIMPLEX fails to terminate then it cycles.
- **Proof:** Suppose that SIMPLEX does not cycle but it fails to terminate. So it must generate infinite number of different slack forms.
- However, the number of different bases is finite. If we can show that a slack form for a given basis is unique then we have a contradiction.

$$z = v + \sum_{j \notin B} c_j x_j \qquad z = v^* + \sum_{j \notin B} c^*_j x_j$$

$$x_i = b_i - \sum_{j \notin B} a_{ij} x_j \quad for \quad i \in B \qquad x_i = b^*_i - \sum_{j \notin B} a^*_{ij} x_j \quad for \quad i \in B$$

- Let $k \notin B$ and let t be any number. Set $x_k=t$ and set all other nonbasic variables to 0.
- Then $x_i=b_i-a_{ik}t$ for all $i \in B$, $z=v+c_kt$.
- This is a solution to the ”LEFT” system. Since ”LEFT” and ”RIGHT” are equivalent, it is also a solution to the ”RIGHT” system. Hence,
- $b_i-a_{ik}t = b^*_i-a^*_{ik}t$ for all $i \in B$
- $v+c_kt = v^*+c^*_kt$
- These equalities must hold for all choices of t. Therefore $b_i = b^*_i$ $a_{ik}=a^*_{ik}$ for all $i \in B$, and $v=v^*$, $c_k=c^*_k$.
- Since k was chosen arbitrarily, the two slack forms are identical.

Avoiding Cycling

- Perturb input slightly so that it is impossible to have two basic solutions with the same objective value.
- Always choose the entering and leaving variables with the smallest indicies.

Perturbation Method

$$1 \gg e_1 \gg e_2 \gg \ldots \gg e_{m-1} \gg e_m > 0$$

$$\begin{aligned}
z &= 0 + 10x_1 - 57x_2 - 9x_3 - 24x_4 \\
x_5 &= e_1 - 0.5x_1 + 5.5x_2 + 2.5x_3 - 9x_4 \\
x_6 &= e_2 - 0.5x_1 + 1.5x_2 + 0.5x_3 - x_4 \\
x_7 &= 1 + e_3 - x_1
\end{aligned}$$

$$\begin{aligned}
z &= 20e_2 - 27x_2 + x_3 - 44x_4 - 20x_6 \\
x_5 &= e_1 - e_2 + 4x_2 + 2x_3 - 8x_4 + x_6 \\
x_1 &= 2e_2 + 3x_2 + x_3 - 2x_4 - 2x_6 \\
x_7 &= 1 - 2e_2 + e_3 - 3x_2 - x_3 - 44x_4 - 20x_6
\end{aligned}$$

$$\begin{aligned}
z &= 1 + 18e_2 + e_3 - 30x_2 - 42x_4 - 18x_6 - x_7 \\
x_5 &= 2 + e_1 - 5e_2 + 2e_3 - 2x_2 - 4x_4 + 5x_6 - 2x_7 \\
x_1 &= 1 + e_3 - x_7 \\
x_3 &= 1 - 2e_2 + e_3 - 3x_2 + 2x_4 + 2x_6 - x_7
\end{aligned}$$

Small Indicies

$$
\begin{aligned}
z &= 0 + 10x_1 - 57x_2 - 9x_3 - 24x_4 \\
x_5 &= 0 - 0.5x_1 + 5.5x_2 + 2.5x_3 - 9x_4 \\
x_6 &= 0 - 0.5x_1 + 1.5x_2 + 0.5x_3 - x_4 \\
x_7 &= 1 - x_1
\end{aligned}
$$

$$
\begin{aligned}
z &= 0 + 53x_2 + 41x_3 - 204x_4 - 20x_5 \\
x_1 &= 0 + 11x_2 + 5x_3 - 18x_4 - 2x_5 \\
x_6 &= 0 - 4x_2 - 2x_3 + 8x_4 + x_5 \\
x_7 &= 1 - 11x_2 - 5x_3 + 18x_4 + 2x_5
\end{aligned}
$$

$$
\begin{aligned}
z &= 0 + 14.5x_3 - 98x_4 - 6.75x_5 - 13.25x_6 \\
x_1 &= 0 - 0.5x_3 + 4x_4 + 0.75x_5 - 2.75x_6 \\
x_2 &= 0 - 0.5x_3 + 2x_4 + 0.25x_5 - 0.25x_6 \\
x_7 &= 1 + 0.5x_3 - 4x_4 - 0.75x_5 - 13.25x_6
\end{aligned}
$$

Small Indicies

$$z = 0 + 18x_4 + 15x_5 - 93x_6 - 29x_1$$
$$x_2 = 0 - 2x_4 - 0.5x_5 + 2.5x_6 + x_1$$
$$x_3 = 0 + 8x_4 + 1.5x_5 - 5.5x_6 - 2x_1$$
$$x_7 = 1 - x_1$$

$$z = 0 + 10.5x_5 - 70.5x_6 - 20x_1 - 9x_2$$
$$x_3 = 0 - 0.5x_5 + 4.5x_6 + 2x_1 - 4x_2$$
$$x_4 = 0 - 0.25x_5 + 1.25x_6 + 0.5x_1 - 0.5x_2$$
$$x_7 = 1 - x_1$$

Small Indicies

$$
\begin{aligned}
z &= 0 + 24x_6 + 22x_1 - 93x_2 - 21x_3 \\
x_5 &= 0 + 9x_6 + 4x_1 - 8x_2 - 2x_3 \\
x_4 &= 0 - x_6 - 0.5x_1 + 1.5x_2 + 0.5x_3 \\
x_7 &= 1 - x_1
\end{aligned}
$$

$$
\begin{aligned}
z &= 0 - 20x_6 - 27x_2 + x_3 - 44x_4 \\
x_5 &= 0 + x_6 + 4x_2 + 2x_3 - 8x_4 \\
x_1 &= 0 - 2x_6 + 3x_2 + x_3 - 2x_4 \\
x_7 &= 1 + 2x_6 - 3x_2 - x_3 + 2x_4
\end{aligned}
$$

$$
\begin{aligned}
z &= 1 - 18x_6 - 30x_2 - 42x_4 - x_7 \\
x_5 &= 2 + 5x_6 - 2x_2 - 4x_4 - 2x_7 \\
x_1 &= 1 - x_7 \\
x_3 &= 1 + 2x_6 - 3x_2 + 2x_4 - x_7
\end{aligned}
$$

Infeasible First Basic Solution

$$
\begin{array}{llllll}
max & 2x_1 & - & x_2 & & \\
s.t. & 2x_1 & - & x_2 & \leq & 2 \\
 & x_1 & - & 5x_2 & \leq & -4
\end{array}
$$

$$
\begin{array}{lllllll}
z & = & 0 & + & 2x_1 & - & x_2 \\
x_3 & = & 2 & - & 2x_1 & + & x_2 \\
x_4 & = & -4 & - & x_1 & + & 5x_2
\end{array}
$$

- x_1 and x_2 would be set to 0 in the first basic solution.
- This solution is infeasible since x_4 = -4.

Auxiliary LP

- We will define a related auxiliary LP.
- This auxiliary LP is feasible and bounded.
- Optimal value of this auxiliary LP will indicate if the LP is feasible.
- If LP is feasible, then the slack form of this auxiliary LP will yield a feasible basic solution to the LP (and the corresponding slack form).

Auxiliary Linear Program

- L: LP in standard form:

$$\begin{aligned} max \quad & \sum_{j=1}^{n} c_j x_j \\ s.t. \quad & \sum_{j=1}^{n} a_{ij} x_j \leq b_i \quad for \quad i = 1, 2, \ldots, m \\ & x_j \geq 0 \quad for \quad j = 1, 2, \ldots, n \end{aligned}$$

- L_{aux}: Auxiliary LP:

$$\begin{aligned} max \quad & -x_0 \\ s.t. \quad & \sum_{j=1}^{n} a_{ij} x_j - x_0 \leq b_i \quad for \quad i = 1, 2, \ldots, m \\ & x_j \geq 0 \quad for \quad j = 0, 1, 2, \ldots, n \end{aligned}$$

- L_{aux} is bounded and feasible.

INITIALIZE_SIMPLEX

- Let l be the index of the minimum b_l.
- If $b_l \geq 0$, let $N=\{1,2,...,n\}$, $B=\{n+1,n+2,...,n+m\}$ and return $(N,B,A,b,c,0)$.
- Otherwise form L_{aux} in its slack form.
- $(N,B,A,b,c,v) \leftarrow$ PIVOT$(N,B,A,b,c,v,l,0)$. This basic solution is feasible for L_{aux}.
- Keep pivoting until SIMPLEX terminates. If the returned objective value is 0, then use the returned slack form (with x_0 removed) and restored objective function as a starting point for original LP problem. If the returned objective value is negative, then the original LP problem is infeasible.

INITIALIZE_SIMPLEX - Example

L:

$$\begin{aligned} max \quad & 2x_1 - x_2 \\ s.t. \quad & 2x_1 - x_2 \leq 2 \\ & x_1 - 5x_2 \leq -4 \end{aligned}$$

L_{aux}

$$\begin{aligned} max \quad & -x_0 \\ s.t. \quad & 2x_1 - x_2 - x_0 \leq 2 \\ & x_1 - 5x_2 - x_0 \leq -4 \end{aligned}$$

- If the value of the optimal solution to L_{aux} is 0, then if we disregard x_0, we get a feasible solution to L. It is basic (at most 2 variables are positive, others are 0).
- If the value of the optimal solution to L_{aux} is negative, then at least one of the constraints of L is never satisfied. L is infeasible.

INITIALIZE_SIMPLEX - Example

$$\begin{aligned}
z &= 0 & & & & & - x_0 \\
x_3 &= 2 & - 2x_1 & + x_2 & + x_0 \\
x_4 &= -4 & - x_1 & + 5x_2 & + x_0
\end{aligned}$$

$$\begin{aligned}
z &= & & & & - (x_4 + 4 + x_1 - 5x_2) \\
x_3 &= 2 & - 2x_1 & + x_2 & + (x_4 + 4 + x_1 - 5x_2) \\
x_0 &= 4 & + x_1 & - 5x_2 & + x_4
\end{aligned}$$

$$\begin{aligned}
z &= -4 & - x_1 & + 5x_2 & - x_4 \\
x_3 &= 6 & - x_1 & - 4x_2 & + x_4 \\
x_0 &= 4 & + x_1 & - 5x_2 & + x_4
\end{aligned}$$

feasible!!

INITIALIZE_SIMPLEX - Example

$$
\begin{aligned}
z &= -4 - x_1 + 5x_2 - x_4 \\
x_3 &= 6 - x_1 - 4x_2 + x_4 \\
x_0 &= 4 + x_1 - 5x_2 + x_4
\end{aligned}
$$

$$
\begin{aligned}
z &= -4 - x_1 + 5(4/5 + x_1/5 + x_4/5 - x_0/5) - x_4 \\
x_3 &= 6 - x_1 - 4(4/5 + x_1/5 + x_4/5 - x_0/5) + x_4 \\
x_2 &= 4/5 + x_1/5 - x_0/5 + x_4/5
\end{aligned}
$$

INITIALIZE_SIMPLEX - Example

$$\begin{array}{lllllllll} z & = & 0 & & & - & x_0 & & \\ x_3 & = & 14/5 & - & 9x_1/5 & + & 4x_0/5 & + & x_4/5 \\ x_2 & = & 4/5 & + & x_1/5 & - & x_0/5 & + & x_4/5 \end{array}$$

$$\begin{array}{lllllllll} z & = & 0 & + & 2x_1 & - & \left(4/5 + x_1/5 - x_0/5 + x_4/5\right) & & \\ x_3 & = & 14/5 & - & 9x_1/5 & + & 4x_0/5 & + & x_4/5 \\ x_2 & = & 4/5 & + & x_1/5 & - & x_0/5 & + & x_4/5 \end{array}$$

$$\begin{array}{lllllll} z & = & -4/5 & + & 9x_1/4 & - & x_4/5 \\ x_3 & = & 14/5 & - & 9x_1/5 & + & x_4/5 \\ x_2 & = & 4/5 & + & x_1/5 & + & x_4/5 \end{array}$$