

VA 2007

- Forelæsninger:
 - tirsdage 13-15, Lille UP1,
 - fredage 10-12, i Store UP1
- Øvelser: 3 hold, tirsdage og fredage (1/2 uge forskudt)
- [Hjemmeside](#)

Linear Programming

- Tirsdag, 13/11: **Introduction to LP**, afsnit 29.1 og 29.2
- Fredag, 16/11: **SIMPLEX algorithm**, afsnit 29.3 og 29.5
- Tirsdag, 20/11: **Duality**, afsnit 29.4

Max Flow

- Fredag, 23/11: **Introduction to flow networks, Ford-Fulkerson method**, Sections 26.1 and 26.2
- Tirsdag, 27/11: **Ford-Fulkerson method, Edmonds-Karp algorithm, maximum flow in bipartite graphs**, Sections 26.2 and 26.3.
- Fredag, 30/11: **Push-relabel algorithm**, Section 26.4

Introduction

- Linear Programming by Example
- Geometric Interpretation
- Linear Programming – Brief History
- Standard and Slack Formulations
- SIMPLEX by Example

Diet Problem (after Chvatal)

- Every day Polly needs:
 - 2000 kcal,
 - 55g protein,
 - 800mg calcium.
- She will get other stuff (e.g., iron and vitamins) by taking pills. Not that this could not be included in the model – we just want to keep it simple.
- She wants a diet that will meet the requirements while being neither expensive nor boring.

Value and Price per Serving

Food	Energy (kcal)	Protein (g)	Calcium (mg)	Price per serving
Oatmeal	110	4	2	3
Chicken	205	32	12	24
Eggs	160	13	54	13
Whole milk	160	8	285	9
Cherry pie	420	4	22	20
Pork with beans	260	14	80	19

**10 portions of pork with beans would cover her needs!
And would cost only 190. But ...**

Limits to What Polly Can Stomach

- Oatmeal: at most 4 servings a day.
- Chicken: at most 3 servings a day.
- Eggs: at most 2 servings a day.
- Milk: at most 8 servings a day.
- Cherry pie: at most 2 servings a day.
- Pork with beans: at most 2 servings a day.

8 servings of milk and 2 servings of cherry pie would meet her needs. Boring but she could stomach it. Especially since it would cost 112. Can she find a less expensive diet?

Variables

- X_1 : number of oatmeal servings.
- X_2 : number of chicken servings.
- X_3 : number of eggs servings.
- X_4 : number of milk servings.
- X_5 : number of cherry pie servings.
- X_6 : number of pork and pie servings.

Linear Constraints

$$\begin{array}{rcccccccc} x_1 & & & & & & & & \leq & 4 \\ & x_2 & & & & & & & \leq & 3 \\ & & x_3 & & & & & & \leq & 2 \\ & & & x_4 & & & & & \leq & 8 \\ & & & & x_5 & & & & \leq & 2 \\ & & & & & x_6 & & & \leq & 2 \\ 110x_1 & + & 205x_2 & + & 160x_3 & + & 160x_4 & + & 420x_5 & + & 260x_6 & \geq & 2000 \\ 4x_1 & + & 32x_2 & + & 13x_3 & + & 8x_4 & + & 4x_5 & + & 14x_6 & \geq & 55 \\ 2x_1 & + & 12x_2 & + & 54x_3 & + & 285x_4 & + & 22x_5 & + & 80x_6 & \geq & 800 \\ x_1, & & x_2, & & x_3, & & x_4, & & x_5, & & x_6 & \geq & 0 \end{array}$$

Linear Programming Problem

$$\begin{array}{l}
 \min \quad 3x_1 + 24x_2 + 13x_3 + 9x_4 + 20x_5 + 19x_6 \\
 \text{s.t.} \quad x_1 \leq 4 \\
 \quad \quad \quad x_2 \leq 3 \\
 \quad \quad \quad \quad x_3 \leq 2 \\
 \quad \quad \quad \quad \quad x_4 \leq 8 \\
 \quad \quad \quad \quad \quad \quad x_5 \leq 2 \\
 \quad \quad \quad \quad \quad \quad \quad x_6 \leq 2 \\
 110x_1 + 205x_2 + 160x_3 + 160x_4 + 420x_5 + 260x_6 \geq 2000 \\
 4x_1 + 32x_2 + 13x_3 + 8x_4 + 4x_5 + 14x_6 \geq 55 \\
 2x_1 + 12x_2 + 54x_3 + 285x_4 + 22x_5 + 80x_6 \geq 800 \\
 x_1, x_2, x_3, x_4, x_5, x_6 \geq 0
 \end{array}$$

<http://www-neos.mcs.anl.gov/CaseStudies/dietpy/WebForms/>

Linear Objective Function

$$\begin{array}{ll} \min & 3x_1 + 24x_2 + 13x_3 + 9x_4 + 20x_5 + 19x_6 \\ \text{s.t.} & \text{linear constraints} \end{array}$$

- The value of the **objective function** for a particular set of values for $x_1, x_2, x_3, x_4, x_5, x_6$ is called its **objective value**.
- If a particular set of values for $x_1, x_2, x_3, x_4, x_5, x_6$, satisfies all constraints, it is said to be a **feasible solution** (dansk: tilladt løsning).
- The set of all feasible solutions is called the **feasible region** (dansk: tilladt område). It can be shown to be convex.
- A feasible solution that has the minimum (or maximum) objective value is called an **optimal solution**.

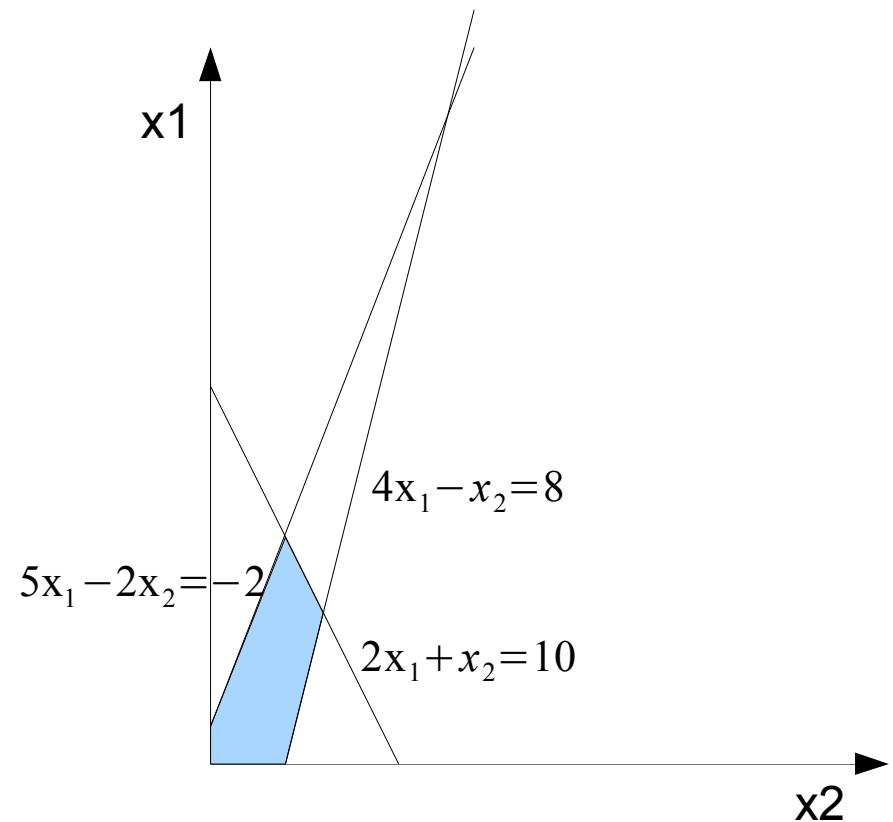
General LP Problem

$$\begin{array}{ll} \min & \sum_{j=1}^n c_j x_j \\ \text{s.t.} & m \text{ linear constraints} \end{array}$$

- Minimization or maximization of a **linear** objective function with n real-valued variables.
- An optimal solution must satisfy m **linear** constraints (inequalities or equalities).
- Strict inequalities are not allowed.
- "programming" in "linear programming" does not refer to any code. It was chosen before computer programming was born.

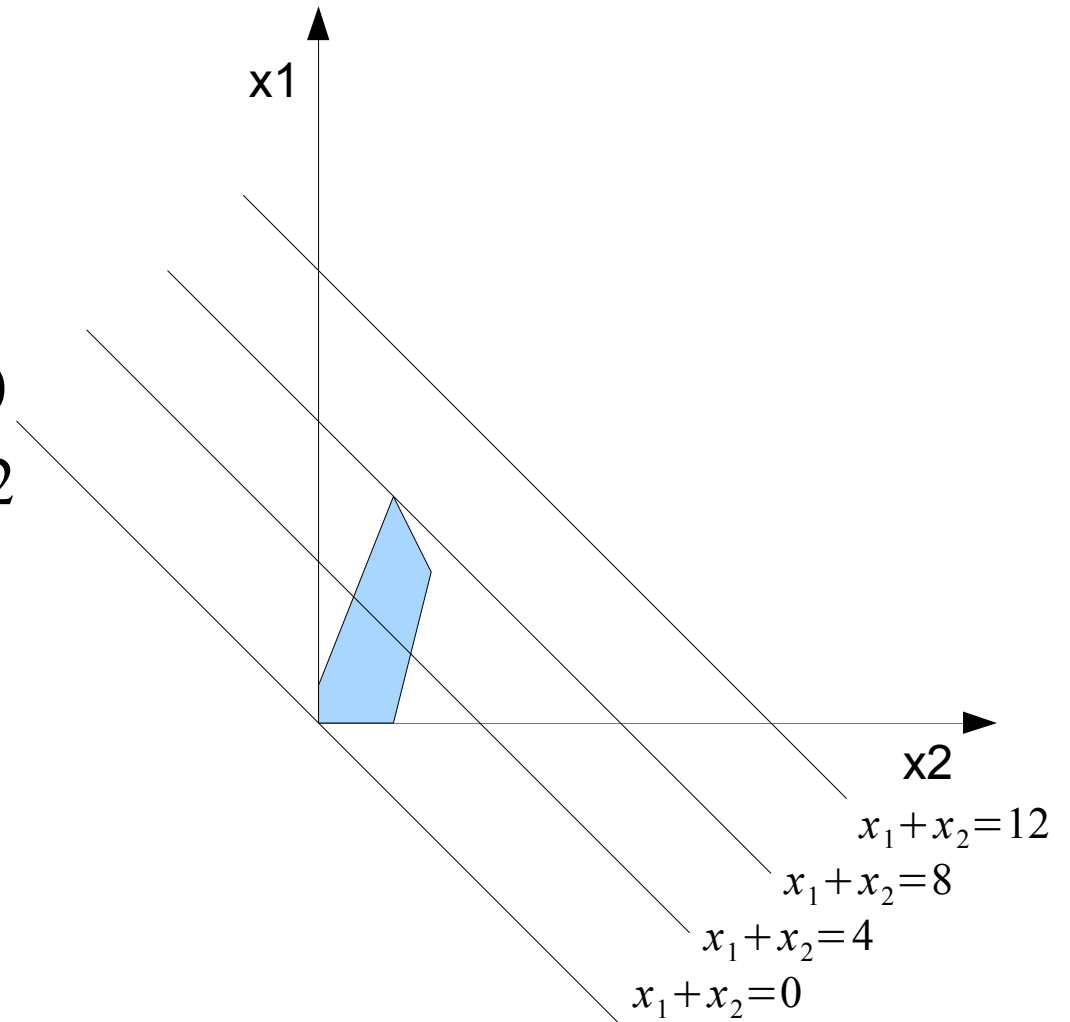
Geometric Interpretation

$$\begin{array}{llllll} \text{max} & x_1 & + & x_2 & & \\ \text{s.t.} & 4x_1 & - & x_2 & \leq & 8 \\ & 2x_1 & + & x_2 & \leq & 10 \\ & 5x_1 & - & 2x_2 & \geq & -2 \\ & x_1, & & x_2 & \geq & 0 \end{array}$$



Geometric Interpretation

$$\begin{array}{llll} \max & x_1 & + & x_2 \\ \text{s.t.} & 4x_1 & - & x_2 \leq 8 \\ & 2x_1 & + & x_2 \leq 10 \\ & 5x_1 & - & 2x_2 \geq -2 \\ & x_1, & & x_2 \geq 0 \end{array}$$



Special Cases of LP

- LP may have no feasible solution (in case of conflicting constraints).
- LP may have feasible solutions but no optimal solution (in case of unboundedness).
- LP may have more than one optimal solution.

Geometric Interpretation in R^3

- 3 variables.
- Each constraint defines a half-space in R^3 . The set of feasible solutions is the intersection of these half-spaces. It is convex. Can be unbounded or empty.
- The set of points in which the objective function has the same value z is a plane.
- The value of the objective function increases as the plane is translated in one normal direction and it decreases as it is translated in the other normal direction.
- If the set of feasible solutions is bounded (in the direction of the normal) and not empty, then there is an optimal solution in an extreme vertex of the convex set of feasible solutions.

Geometric Interpretation in R^d

- d variables.
- Each constraint defines a half-space in R^d . The set of feasible solutions is the intersection of these half-spaces, called **simplex**. It is convex. Can be unbounded or empty.
- The set of points in which the objective function has the same value z is a **hyperplane**.
- The value of the objective function increases or decreases as the hyperplane is translated.
- If the set of feasible solutions is bounded and not empty, then there is an optimal solution in an extreme vertex of the simplex.

General Idea Behind SIMPLEX Algorithm

- SIMPLEX starts with a feasible solution corresponding to some vertex of the simplex. We will show how to find such a vertex (or decide that the feasible region is empty).
- SIMPLEX keeps "jumping" from a vertex of the simplex to a new vertex if the new vertex offers a feasible solution that is better (or at least not worse). We will show how SIMPLEX "jumps". We will show how to avoid "cycling" when SIMPLEX jumps through feasible solutions with the same objective value.
- When no more "jumps" are possible, we will show that SIMPLEX is in an optimal vertex (or the LP is unbounded).
- We will show that the number of jumps is finite (at most equal to the number of simplex vertices).

History of LP

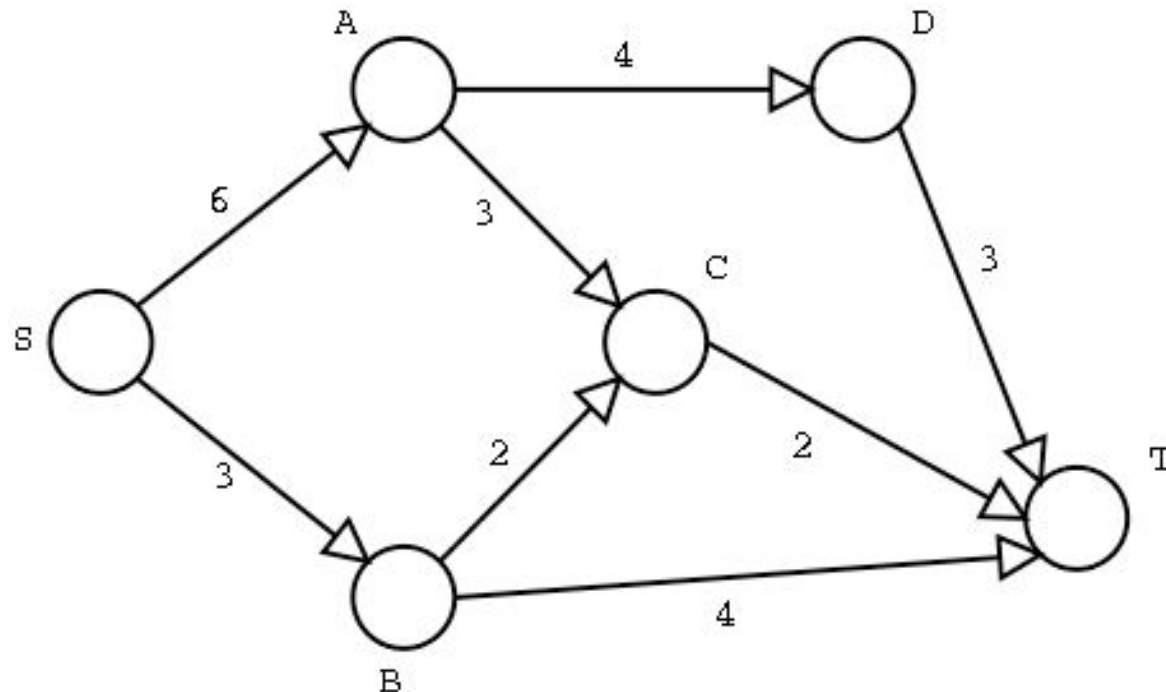
- L.V. Kantorovich pointed out in 1939 the importance of restricted classes of LPs.
- T.C. Koopmans realized in 1947 the importance of LP for the analysis of classical economic theories.
- G.B. Dantzig designed in 1947 the simplex method to solve LP for U.S. Air Force. Not a polynomial algorithm!
- Many applications followed over the years. Work of Kantorovich as rediscovered in 1950's.
- In 1975 Kantorovich and Koopmans got the Nobel prize.
- L.G. Khachian discovered first polynomial algorithm in 1979. Terribly slow.
- N. Karmarkar discovered second polynomial algorithm in 1984. Practical.

Applications of LP

- Scheduling problems: airline wishes to schedule its flight crews on all flights while using as few crew members as possible.
- Manufacturing problems: How much of each type of product should be produced subject to technical and financial constraints.
- Location problems: Locating drills to maximize the amount of oil that will be extracted under given budget constraints.
- Many network and graph problems can be formulated as LP; dimensioning telecommunication and distribution networks.
- Allocation of financial assets to maximize profit or minimize risk.
- Integer linear programming problems.

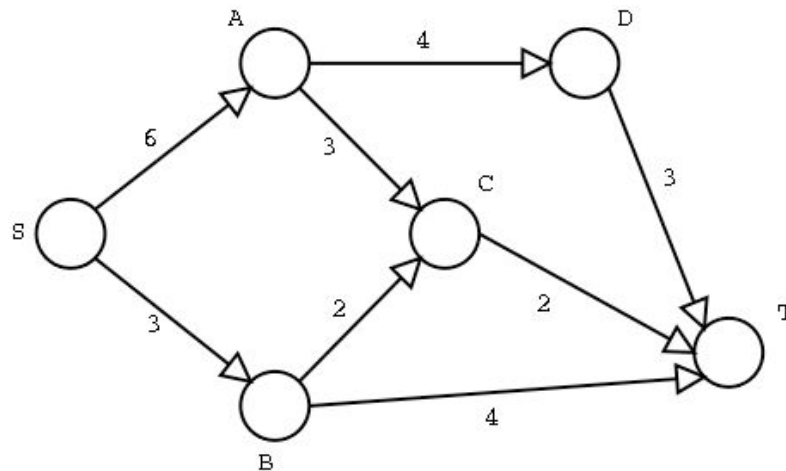
Maximum Flow

- **Given:** A directed graph $G = (V, E)$ where each pair of vertices u and v has a real-valued, nonnegative capacity $c(u, v)$ ($= 0$ if (u, v) is not an edge in E), a source vertex s , and a destination vertex t .
- **Find:** A **maximum flow** $f: V \times V \rightarrow \mathbf{R}$ from s to t



Flow

- A **flow** from s to t in G is a real-valued function $f: V \times V \rightarrow \mathbf{R}$ satisfying the following **flow constraints**:
 - Capacity constraints: $f(u,v) \leq c(u,v), \forall u, v \in V.$
 - Skew symmetry: $f(u,v) = -f(v,u), \forall u, v \in V.$
 - Flow conservation: $\sum_{u \in V} f(u,v) = \sum_{w \in V} f(v,w), \forall v \in V - \{s, t\}$
- Flow **value** $|f|$ is defined as
$$\sum_{v \in V} f(s, v)$$
- Maximize $|f|$ subject to the flow constraints.



General LP Problem

$$\begin{array}{ll} \min & \sum_{j=1}^n c_j x_j \\ \text{s.t.} & m \text{ linear constraints} \end{array}$$

- Minimization or maximization of a **linear** objective function with n real-valued variables.
- An optimal solution must satisfy m **linear** constraints (inequalities or equalities).
- Strict inequalities are not allowed.
- "programming" in "linear programming" does not refer to any code. It was chosen before computer programming was born.

Equivalent LPs

- Two maximization LPs L and L' are **equivalent** iff for each feasible solution \mathbf{x} to L with the objective value z , there is a corresponding feasible solution \mathbf{x}' to L' with the same objective value z , and vice versa.
- Similarly for two minimization LPs.
- A minimization LP L and a maximization LP L' are **equivalent** iff for each feasible solution \mathbf{x} to L with the objective value z , there is a corresponding feasible solution \mathbf{x}' to L' with the objective value $-z$.

LP in Standard Form

- Maximization of a linear function.
- n non-negative real-valued variables.
- m linear inequalities ("less than or equal to").

$$\begin{aligned} \max \quad & \sum_{j=1}^n c_j x_j \\ \text{s.t.} \quad & \sum_{j=1}^n a_{ij} x_j \leq b_i \quad \text{for } i=1, 2, \dots, m \\ & x_j \geq 0 \quad \text{for } j=1, 2, \dots, n \end{aligned}$$

Converting LP into Standard Form

$$\begin{array}{llllll} \textit{min} & -2x_1 & + & 3x_2 & & \\ \textit{s.t.} & x_1 & + & x_2 & = & 7 \\ & x_1 & - & 2x_2 & \leq & 4 \\ & & & x_1 & \geq & 0 \end{array}$$

- Minimization LP is converted to an equivalent maximization problem by negating the coefficients of the objective function.

$$\begin{array}{llllll} \textit{max} & 2x_1 & - & 3x_2 & & \\ \textit{s.t.} & x_1 & + & x_2 & = & 7 \\ & x_1 & - & 2x_2 & \leq & 4 \\ & & & x_1 & \geq & 0 \end{array}$$

Converting LP into Standard Form

$$\begin{array}{llllll} \text{max} & 2x_1 & - & 3x_2 & & \\ \text{s.t.} & x_1 & + & x_2 & = & 7 \\ & x_1 & - & 2x_2 & \leq & 4 \\ & x_1 & & & \geq & 0 \end{array}$$

- Every variable x_j without the non-negativity constraint is replaced by two non-negative variables x'_j and x''_j and each occurrence of x_j is replaced by $x'_j - x''_j$.

$$\begin{array}{llllllll} \text{max} & 2x_1 & - & 3x'_2 & + & 3x''_2 & & \\ \text{s.t.} & x_1 & + & x'_2 & - & x''_2 & = & 7 \\ & x_1 & - & 2x'_2 & + & 2x''_2 & \leq & 4 \\ & x_1, & & x'_2, & & x''_2 & \geq & 0 \end{array}$$

Converting LP into Standard Form

$$\begin{array}{rcllcl}
 \textit{max} & 2x_1 & - & 3x'_2 & + & 3x''_2 \\
 \textit{s.t.} & x_1 & + & x'_2 & - & x''_2 & = & 7 \\
 & x_1 & - & 2x'_2 & + & 2x''_2 & \leq & 4 \\
 & x_1, & & x'_2, & & x''_2 & \geq & 0
 \end{array}$$

- Each equality constraint is replaced by a pair of "opposite" inequality constraints.

$$\begin{array}{rcllcl}
 \textit{max} & 2x_1 & - & 3x'_2 & + & 3x''_2 \\
 \textit{s.t.} & x_1 & + & x'_2 & - & x''_2 & \leq & 7 \\
 & x_1 & + & x'_2 & - & x''_2 & \geq & 7 \\
 & x_1 & - & 2x'_2 & + & 2x''_2 & \leq & 4 \\
 & x_1, & & x'_2, & & x''_2 & \geq & 0
 \end{array}$$

Converting LP into Standard Form

$$\begin{array}{llllll} \max & 2x_1 & - & 3x_2' & + & 3x_2'' \\ \text{s.t.} & x_1 & + & x_2' & - & x_2'' & \leq & 7 \\ & x_1 & + & x_2' & - & x_2'' & \geq & 7 \\ & x_1 & - & 2x_2' & + & 2x_2'' & \leq & 4 \\ & x_1, & & x_2', & & x_2'' & \geq & 0 \end{array}$$

- Inequalities are "turned around" by multiplying both sides by -1.

$$\begin{array}{llllll} \max & 2x_1 & - & 3x_2' & + & 3x_2'' \\ \text{s.t.} & x_1 & + & x_2' & - & x_2'' & \leq & 7 \\ & -x_1 & - & x_2' & + & x_2'' & \leq & -7 \\ & x_1 & - & 2x_2' & + & 2x_2'' & \leq & 4 \\ & x_1, & & x_2', & & x_2'' & \geq & 0 \end{array}$$

Converting LP into a Standard Form

$$\begin{array}{llllll} \max & 2x_1 & - & 3x_2' & + & 3x_2'' \\ \text{s.t.} & x_1 & + & x_2' & - & x_2'' & \leq & 7 \\ & -x_1 & - & x_2' & + & x_2'' & \leq & -7 \\ & x_1 & - & 2x_2' & + & 2x_2'' & \leq & 4 \\ & x_1, & & x_2', & & x_2'' & \geq & 0 \end{array}$$

- Renaming the variables

$$\begin{array}{llllll} \max & 2x_1 & - & 3x_2 & + & 3x_3 \\ \text{s.t.} & x_1 & + & x_2 & - & x_3 & \leq & 7 \\ & -x_1 & - & x_2 & + & x_3 & \leq & -7 \\ & x_1 & - & 2x_2 & + & 2x_3 & \leq & 4 \\ & x_1, & & x_2, & & x_3 & \geq & 0 \end{array}$$

LP in Standard Form

- n variables, m constraints

$$\begin{aligned} \max \quad & \sum_{j=1}^n c_j x_j \\ \text{s.t.} \quad & \sum_{j=1}^n a_{ij} x_j \leq b_i \quad \text{for } i=1,2,\dots,m \\ & x_j \geq 0 \quad \text{for } j=1,2,\dots,n \end{aligned}$$

Slack Variables (Overskudsvariable)

- Consider one of the constraints, for example

$$2x_1 + 3x_2 + x_3 \leq 5$$

- For every feasible solution x_1, x_2, x_3 , the value of the left-hand side is at most the value of the right-hand side.
- Often there can be a **slack** between these two values.
- Denote the slack by x_4 .
- By requiring that $x_4 \geq 0$, we can replace the inequality by the equality

$$2x_1 + 3x_2 + x_3 + x_4 = 5$$

Slack Variables

$$\begin{aligned} \max \quad & \sum_{j=1}^n c_j x_j \\ \text{s.t.} \quad & \sum_{j=1}^n a_{ij} x_j \leq b_i \quad \text{for } i=1,2,\dots,m \\ & x_j \geq 0 \quad \text{for } j=1,2,\dots,n \end{aligned}$$

$$\begin{aligned} \max \quad & \sum_{j=1}^n c_j x_j \\ \text{s.t.} \quad & \sum_{j=1}^n a_{ij} x_j + x_{n+i} = b_i \quad \text{for } i=1,2,\dots,m \\ & x_j \geq 0 \quad \text{for } j=1,2,\dots,n+m \end{aligned}$$

LP in Slack Form

$$\begin{aligned} \max \quad & \sum_{j=1}^n c_j x_j \\ \text{s.t.} \quad & \sum_{j=1}^n a_{ij} x_j + x_{n+i} = b_i \quad \text{for } i=1,2,\dots,m \\ & x_j \geq 0 \quad \text{for } j=1,2,\dots,n+m \end{aligned}$$

$$\begin{aligned} \max \quad z &= \sum_{j=1}^n c_j x_j \\ \text{s.t.} \quad x_{n+i} &= b_i - \sum_{j=1}^n a_{ij} x_j \quad \text{for } i=1,2,\dots,m \\ x_j &\geq 0 \quad \text{for } j=1,2,\dots,n+m \end{aligned}$$

$$\begin{aligned} z &= 0 + \sum_{j=1}^n c_j x_j \\ x_{n+i} &= b_i - \sum_{j=1}^n a_{ij} x_j \quad \text{for } i=1,2,\dots,m \end{aligned}$$

Standard to Slack Form - Example

$$\begin{array}{rcllcl}
 \text{max} & 2x_1 & - & 3x_2 & + & 3x_3 & & \\
 \text{s.t.} & x_1 & + & x_2 & - & x_3 & \leq & 7 \\
 & -x_1 & - & x_2 & + & x_3 & \leq & -7 \\
 & x_1 & - & 2x_2 & + & 2x_3 & \leq & 4
 \end{array}$$

$$\begin{array}{rcllclclcl}
 \text{max} & 2x_1 & - & 3x_2 & + & 3x_3 & & & \\
 \text{s.t.} & x_1 & + & x_2 & - & x_3 & + & x_4 & = & 7 \\
 & -x_1 & - & x_2 & + & x_3 & & + & x_5 & = & -7 \\
 & x_1 & - & 2x_2 & + & 2x_3 & & & + & x_6 & = & 4
 \end{array}$$

$$\begin{array}{rcllclclcl}
 z & = & 0 & + & 2x_1 & - & 3x_2 & + & 3x_3 & & \\
 x_4 & = & 7 & - & x_1 & - & x_2 & + & x_3 & & \\
 x_5 & = & -7 & + & x_1 & + & x_2 & - & x_3 & & \\
 x_6 & = & 4 & - & x_1 & + & 2x_2 & - & 2x_3 & &
 \end{array}$$

Basic Solutions

- Any solution of LP in standard form yields a solution of LP in the corresponding slack form (with the same objective value) and vice versa.
- Setting right-hand side variables of the slack form to 0 yields a **basic solution**.
- Left-hand side variables are called **basic**. Right-hand side variables are called **nonbasic**.
- The basic variables are said to constitute a **basis**.
- Note that a basic solution does not need to be feasible.

SIMPLEX – Example Continued

- LP in slack form:

$$\begin{aligned}z &= 0 + 3x_1 + x_2 + 2x_3 \\x_4 &= 30 - x_1 - x_2 - 3x_3 \\x_5 &= 24 - 2x_1 - 2x_2 - 5x_3 \\x_6 &= 36 - 4x_1 - x_2 - 2x_3\end{aligned}$$

- Set all **nonbasic** variables (right-hand side) to 0.
- Compute values of **basic** variables: $x_4=30$, $x_5=24$, $x_6=36$.
- Compute the objective value z ($= 0$).
- This gives the feasible basic solution $(0,0,0,30,24,36)$.
- It is feasible; not always the case – we were lucky.

SIMPLEX: 1. Pivoting

- Can x_1 be increased without violating feasibility?

$$z = 0 + 3x_1 + x_2 + 2x_3$$

$$x_4 = 30 - x_1 - x_2 - 3x_3$$

$$x_5 = 24 - 2x_1 - 2x_2 - 5x_3$$

$$x_6 = 36 - 4x_1 - x_2 - 2x_3$$

- If x_1 is increased to 1, then $x_4=29$, $x_5=22$, $x_6=32$ while $z=3$.
(1,0,0,29,22,32) is a feasible solution.
- If x_1 is increased to 2, then $x_4=28$, $x_5=20$, $x_6=28$ while $z=6$.
(2,0,0,28,20,28) is a feasible solution.
- If x_1 is increased to 3, then $x_4=27$, $x_5=18$, $x_6=24$ while $z=9$.
(3,0,0,27,18,24) is a feasible solution.

SIMPLEX: 1. Pivoting

- Can x_1 be increased without violating feasibility? By how much?

$$\begin{aligned}z &= 0 + 3x_1 + x_2 + 2x_3 \\x_4 &= 30 - x_1 - x_2 - 3x_3 \\x_5 &= 24 - 2x_1 - 2x_2 - 5x_3 \\x_6 &= 36 - 4x_1 - x_2 - 2x_3\end{aligned}$$

- If x_1 is increased beyond 30 then x_4 becomes negative.
- If x_1 is increased beyond 12 then x_5 becomes negative.
- If x_1 is increased beyond 9 then x_6 becomes negative.
- Constraint defining x_6 is **binding**.

SIMPLEX: 1. Pivoting

- So x_1 can be increased to 9 without losing feasibility. The feasible solution is $(9,0,0,21,6,0)$ and $z = 27$.
- We will now rewrite the slack form to an equivalent slack form with x_1, x_4, x_5 as basic variables and with $(9,0,0,21,6,0)$ being its feasible basic solution.
- This rewriting is called **pivoting**.
- Binding constraint defining x_6 is rewritten so that it has x_1 on its left-hand side.
- All occurrences of x_1 in other constraints and in the objective function are replaced by the right-hand side of the binding constraint.

SIMPLEX: 1. Pivoting

$$\begin{aligned}
 z &= 0 + 3(9 - x_2/4 - x_3/2 - x_6/4) + x_2 + 2x_3 \\
 x_4 &= 30 - (9 - x_2/4 - x_3/2 - x_6/4) - x_2 - 3x_3 \\
 x_5 &= 24 - 2(9 - x_2/4 - x_3/2 - x_6/4) - 2x_2 - 5x_3 \\
 x_1 &= 9 - x_2/4 - x_3/2 - x_6/4
 \end{aligned}$$

$$\begin{aligned}
 z &= 27 + x_2/4 + x_3/2 - 3x_6/4 \\
 x_4 &= 21 - 3x_2/4 - 5x_3/2 + x_6/4 \\
 x_5 &= 6 - 3x_2/2 - 4x_3 + x_6/2 \\
 x_1 &= 9 - x_2/4 - x_3/2 - x_6/4
 \end{aligned}$$

New basic variables: $x_1=9$, $x_4=21$, $x_5=6$

New objective value $z = 27$

New feasible basic solution: (9,0,0,21,6,0)

SIMPLEX: 2. Pivoting

- Can x_3 be increased without violating feasibility? By how much?

$$\begin{aligned}z &= 27 + x_2/4 + x_3/2 - 3x_6/4 \\x_4 &= 21 - 3x_2/4 - 5x_3/2 + x_6/4 \\x_5 &= 6 - 3x_2/2 - 4x_3 + x_6/2 \\x_1 &= 9 - x_2/4 - x_3/2 - x_6/4\end{aligned}$$

- If x_3 is increased beyond $42/5$ then $x_4 < 0$.
- If x_3 is increased beyond $3/2$ then $x_5 < 0$.
- If x_3 is increased beyond $9/2$ then $x_1 < 0$.
- Constraint defining x_5 is binding.

SIMPLEX: 2. Pivoting

$$\begin{aligned}
 z &= 27 + x_2/4 + \frac{1}{2}(3/2 - 3x_2/8 + x_6/8 - x_5/4) - 3x_6/4 \\
 x_4 &= 21 - 3x_2/4 - \frac{5}{2}(3/2 - 3x_2/8 + x_6/8 - x_5/4) + x_6/4 \\
 x_3 &= 3/2 - 3x_2/8 - x_5/4 + x_6/8 \\
 x_1 &= 9 - x_2/4 - \frac{1}{2}(3/2 - 3x_2/8 + x_6/8 - x_5/4) - x_6/4
 \end{aligned}$$

$$\begin{aligned}
 z &= 111/4 + x_2/16 - x_5/8 - 11x_6/16 \\
 x_4 &= 69/4 + 3x_2/16 + 5x_5/8 - x_6/16 \\
 x_3 &= 3/2 - 3x_2/8 - x_5/4 + x_6/8 \\
 x_1 &= 33/4 - x_2/16 + x_5/8 - 5x_6/16
 \end{aligned}$$

New basic variables: $x_1=33/4$, $x_3=3/2$, $x_4=69/4$

New objective value $z = 27.75$

New feasible basic solution: $(33/4, 0, 3/2, 69/4, 0, 0)$

SIMPLEX: 3. Pivoting

- Can x_2 be increased without violating feasibility? By how much?

$$\begin{aligned}z &= 111/4 + x_2/16 - x_5/8 - 11x_6/16 \\x_4 &= 69/4 + 3x_2/16 + 5x_5/8 - x_6/16 \\x_3 &= 3/2 - 3x_2/8 - x_5/4 + x_6/8 \\x_1 &= 33/4 - x_2/16 + x_5/8 - 5x_6/16\end{aligned}$$

- If x_2 is increased then x_4 also increases.
- If x_2 is increased beyond 4 then $x_3 < 0$.
- If x_2 is increased beyond 132 then $x_1 < 0$.
- Constraint defining x_3 is binding.

SIMPLEX: 3.Pivoting

$$\begin{aligned}
 z &= 111/4 + \frac{1}{16}(4 - 8x_3/3 - 2x_5/3 + x_6/3) - x_5/8 - 11x_6/16 \\
 x_4 &= 69/4 + \frac{1}{16}(4 - 8x_3/3 - 2x_5/3 + x_6/3) + 5x_5/8 - x_6/16 \\
 x_2 &= 4 - 8x_3/3 - 2x_5/3 + x_6/3 \\
 x_1 &= 33/4 - \frac{1}{16}(4 - 8x_3/3 - 2x_5/3 + x_6/3) + x_5/8 - 5x_6/16
 \end{aligned}$$

$$\begin{aligned}
 z &= 28 - x_3/6 - x_5/6 - 2x_6/3 \\
 x_4 &= 18 - x_3/2 + x_5/2 + 0x_6 \\
 x_2 &= 4 - 8x_3/3 - 2x_5/3 + x_6/3 \\
 x_1 &= 8 + x_3/6 + x_5/6 - x_6/3
 \end{aligned}$$

New basic variables: $x_1=8$, $x_2=4$, $x_4=18$

New objective value $z = 28$

New feasible basic solution: $(8, 4, 0, 18, 0, 0)$ is optimal

Pivoting in General

- PIVOT(N, B, A, b, c, v, l, e)
 - Compute the coefficients of the bounding constraint so that the **entering** basic variable x_e is expressed as a linear combination of the other variables.

$$b_e = b_l / a_{le} \quad a_{ej} = a_{lj} / a_{le}, \quad \forall j \in N \setminus e \quad a_{el} = 1 / a_{le}$$

- Compute the coefficients of the remaining constraints and the objective function (by substituting x_e by the right-hand side of the rewritten binding equation).

$$b_i = b_i - a_{ie} b_e, \quad \forall i \in B \setminus l \quad a_{ij} = a_{ij} - a_{ie} a_{ej}, \quad \forall j \in N \setminus e \quad a_{il} = -a_{ie} a_{el}$$

$$v = v + c_e b_e \quad c_j = c_j - c_e a_{ej}, \quad \forall j \in N \setminus e \quad c_l = -c_e a_{el}$$

- Compute new sets of basic and nonbasic variables (remove x_e from N and add it to B , remove x_l from B and add it to N).

LP in Standard and Slack Forms

$$\begin{aligned} \max \quad & \sum_{j=1}^n c_j x_j \\ \text{s.t.} \quad & \sum_{j=1}^n a_{ij} x_j \leq b_i \quad \text{for } i=1,2,\dots,m \\ & x_j \geq 0 \quad \text{for } j=1,2,\dots,n \end{aligned}$$

$$\begin{aligned} \max \quad & \sum_{j=1}^n c_j x_j \\ \text{s.t.} \quad & \sum_{j=1}^n a_{ij} x_j + x_{n+i} = b_i \quad \text{for } i=1,2,\dots,m \\ & x_j \geq 0 \quad \text{for } j=1,2,\dots,n+m \end{aligned}$$

SIMPLEX

- SIMPLEX starts with a slack form corresponding to some feasible basic solution and iterates:
 - Select a nonbasic variable x_e with $c_e > 0$. If no such x_e exists, SIMPLEX terminates. We will show that the feasible basic solution is optimal.
 - Select a basic variable x_l that most severely limits the nonbasic variable x_e . Ties are broken arbitrarily. We will show that LP is unbounded if no such x_l exists.
 - Pivot.

SIMPLEX – Open Issues

- How to decide that LP is feasible?
- What to do if the initial basic solution is infeasible?
- How to select entering and leaving variables?
- How to decide that LP is unbounded?
- Does SIMPLEX terminate?
- Does it terminate with an optimal solution?