Introduction to Optimization, DIKU 2007-08
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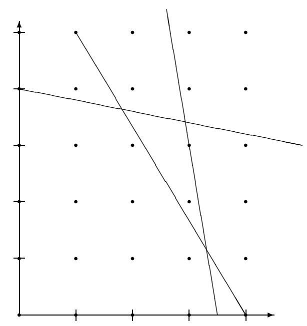
Program of the day:

- Convex Hull, Totally Unimodular (Wolsey sec. 3.1-3.2)
- Cutting planes — a method to obtain tighter bounds and faster convergence to integer solutions (Wolsey chap. 8)
- Application: branch-and-cut algorithms

Convex hull

Smallest convex polyhedral which contains all integer points



feasible solutions	$\{x \in \mathbb{N}^n : Ax \leq b\}$
linear relaxation	$\{x \in \mathbb{R}^n : Ax \leq b\}$
convex hull	$\text{conv}\{x \in \mathbb{R}^n : Ax \leq b\}$

If constraints of an IP-model define the convex hull, then we can solve the problem efficiently.

Totally Unimodularity

Definition 1 An $m \times n$ integral matrix A is called *totally unimodular* (TU) if the determinant of each square submatrix of A is equal to 0, 1 or -1.

Proposition 1 If A is TU and b is an integral vector, then the polyhedral defined by

$$\{x \in \mathbb{R}^n : Ax \leq b\}$$

is integeral (i.e. all corner points are integral), or empty.

Solving IP models

Some IP naturally lead to integer solutions

- Totally unimodular (TU) matrices

Otherwise, reformulation

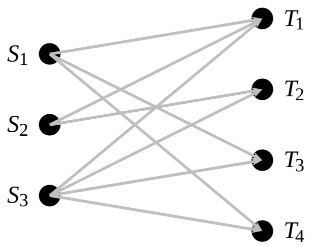
- Reformulation of constraints to TU
- Tightening M, m
- Fixation of variables

Finally, enumrative methods

- Branch-and-bound methods
- Cutting plane methods

Example of TU

Three suppliers (S_1, S_2, S_3) should provide four customers (T_1, T_2, T_3, T_4) with a particular commodity.



Variables: x_{ij} is flow from supplier i to customer j

minimize

$132x_{11}+Mx_{12}+97x_{13}+103x_{14}+85x_{21}+91x_{22}+Mx_{23}+Mx_{24}+106x_{31}+89x_{32}+100x_{33}+98x_{34}$

subject to

$$
\begin{array}{cccccccccccccc}
x_{11} & +x_{12} & +x_{13} & +x_{14} & & & & & & & & & \leq & 135 \\
 & & & & x_{21} & +x_{22} & +x_{23} & +x_{24} & & & & & \leq & 56 \\
 & & & & & & & & x_{31} & +x_{32} & +x_{33} & +x_{34} & \leq & 93 \\
x_{11} & & & & +x_{21} & & & & +x_{31} & & & & \geq & 62 \\
 & x_{12} & & & & +x_{22} & & & & +x_{32} & & & \geq & 83 \\
 & & x_{13} & & & & +x_{23} & & & & +x_{33} & & \geq & 39 \\
 & & & x_{14} & & & & +x_{24} & & & & +x_{34} & \geq & 91
\end{array}
$$

$x_{ij} \in \mathbb{N}$

Good and bad formulations

ii) Reformulate to convex hull

$$(\delta_1 = 1 \lor \delta_2 = 1 \lor \ldots \lor \delta_n = 1) \Rightarrow \delta = 1$$

Can be written

$$(\delta_1 + \delta_2 + \ldots + \delta_n) > 0 \Rightarrow \delta = 1$$

LP-model

$$(\delta_1 + \delta_2 + \ldots + \delta_n) - n\delta \leq 0$$

Better formulation

$$
\begin{array}{l}
\delta_1 = 1 \Rightarrow \delta = 1 \\
\delta_2 = 1 \Rightarrow \delta = 1 \\
\vdots \\
\delta_n = 1 \Rightarrow \delta = 1
\end{array}
$$

LP-model

$$
\begin{array}{l}
\delta_1 - \delta \leq 0 \\
\delta_2 - \delta \leq 0 \\
\vdots \\
\delta_n - \delta \leq 0
\end{array}
$$

Has “property P”

Good and bad formulations

To model $x > 0 \Rightarrow \delta = 1$

$$x - M_1\delta \leq 0 \quad (1)$$
$$x - M_2\delta \leq 0 \quad (2)$$

where $M_1 < M_2$.

Then (1) defines a smaller subset than (2) in LP model

- Solutions to (1) are also solutions to (2)
  Consider (x, δ) which is a solution to (1)

$$x \leq M_1\delta \leq M_2\delta \quad \Rightarrow \quad x - M_2\delta \leq 0$$

- Solutions to (2) exists which are not solutions to (1)
  Consider (x, δ) where $\delta = \frac{x}{M_2}$ and $x > 0$

$$x - M_2\delta \leq 0$$
$$x - M_1\delta = x - M_1\frac{x}{M_2} = x\left(1 - \frac{M_1}{M_2}\right) > 0$$

Cutting Planes – introduction

- branch-and-bound: divide and conquer.
- cutting plane: add inequalities which separate fractional solution from solution space.

Development

- 50’s cutting plane (Gomory: simplex, no $\mathcal{NP}$-hardness)
- 70’s tighten formulation in preprocessing
- 80-90’s branch-and-cut (Padberg, Rinaldi)

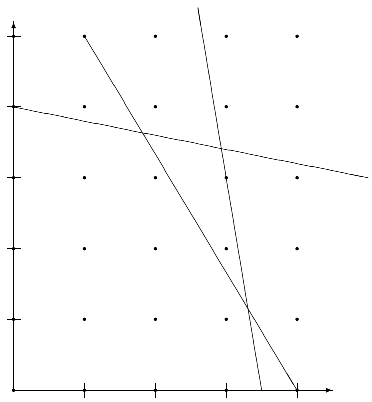
Preprocessing $\rightarrow$ part of solution process

Definitions

- valid inequality
- cuts: valid inequalities which separates LP-solution
- facets: inequalities defining convex hull

Cuts and facets are redundant for IP formulation
Tighten formulation for LP relaxation

Cuts and facets



Example

Preprocessing, integer variables

$$
\begin{array}{ll}
\text{maximize} & \dots \\
\text{subject to} & \begin{array}{rcrcrcrcr}
7x_1 & + & 3x_2 & - & 4x_3 & - & 2x_4 & \leq & 1 \\
-2x_1 & + & 7x_2 & + & 3x_3 & + & 4x_4 & \leq & 6 \\
 & - & 2x_2 & - & 3x_3 & - & 6x_4 & \leq & -5 \\
3x_1 & & & - & 2x_3 & & & \geq & -1 \\
x & \in & \mathbb{B}^4 & & & & & &
\end{array}
\end{array}
$$

Generating logical inequalities

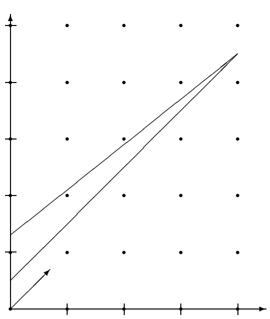
From constraint 1 we see that

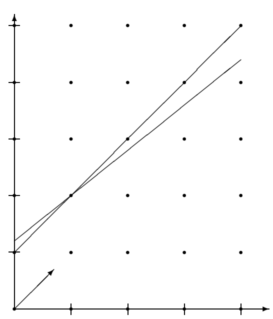
- if $x_1 = 1$ and $x_2 = 1$ then infeasible, thus

$$x_1 + x_2 \leq 1$$

Example

$$
\begin{array}{ll}
\text{maximize} & x_1 + x_2 \\
\text{subject to} & -2x_1 + \;\; 2x_2 \geq 1 \\
 & -8x_1 + 10x_2 \leq 13 \\
 & x_1, x_2 \geq 0, \text{integer}
\end{array}
$$





Tightening formulation

$$
\begin{array}{rcl}
-2x_1 + 2x_2 & \geq & 1 \\
-x_1 + x_2 & \geq & 1/2 \\
-x_1 + x_2 & \geq & 1
\end{array}
\qquad
\begin{array}{rcl}
-8x_1 + 10x_2 & \leq & 13 \\
-4x_1 + 5x_2 & \leq & 13/2 \\
-4x_1 + 5x_2 & \leq & 6
\end{array}
$$

Motivation

Integer programming problem (IP)

$$\max\{cx : x \in X\}$$

where $X = \{x : Ax \leq b, x \in \mathbb{Z}^n_+\}$. Reformulate to

$$\max\{cx : x \in \text{conv}(X)\}$$

For any c, an optimal solution to LP is also optimal to IP

Valid inequalities (def. 8.1)

Consider the problem:

$$
\begin{array}{ll}
\text{maximize} & f(x) \\
\text{subject to} & x \in X
\end{array}
$$

An inequality

$$\pi x \leq \pi_0$$

is a *valid inequality* for $X \subseteq \mathbb{R}^n$ if

$$\pi x \leq \pi_0 \;\; \text{for all} \;\; x \in X$$

Characterization of valid inequalities (sec. 8.3.2)

Consider the problem:

$$\begin{array}{ll} \text{maximize} & f(x) \\ \text{subject to} & x \in X \end{array}$$

where

$$X = \{y \in \mathbb{Z} : y \leq b\}$$

then the inequality

$$y \leq \lfloor b \rfloor$$

is valid for X

- Simple observation
- Complete characterization

Overview of cuts

- Chvatal cuts
- Gomory cuts (Modular cuts)
- Chvatal-Gomory cuts
- Disjunctive cuts
- Cover inequalities
- Clique inequalities
- Problem specific cuts

Notice

- Cuts and facets are independent of objective function
- A tight formulation can be used for any objective

Example of Facets

The problem

$$\begin{array}{lrcl} \text{minimize} & 2x_1 + 7x_2 + 2x_3 & & \\ \text{subject to} & x_1 + 4x_2 + x_3 & \geq & 10 \\ & 4x_1 + 2x_2 + 2x_3 & \geq & 13 \\ & x_1 + x_2 - x_3 & \geq & 0 \\ & x_1, x_2, x_3 & \geq & 0,\ \text{integer} \end{array}$$

has the facets

$$\begin{array}{rcl} x_1 + 4x_2 + x_3 & \geq & 10 \\ 2x_1 + x_2 + x_3 & \geq & 7 \\ x_1 + x_2 - x_3 & \geq & 0 \\ x_1 + 3x_2 + x_3 & \geq & 9 \\ 2x_1 + 4x_2 + x_3 & \geq & 13 \\ x_1 + x_2 + x_3 & \geq & 5 \\ x_1 + 2x_2 & \geq & 5 \\ 2x_1 + x_2 & \geq & 4 \\ x_1 & \geq & 0,\ \text{integer} \\ x_2 & \geq & 0,\ \text{integer} \\ x_3 & \geq & 0,\ \text{integer} \end{array}$$

Using the new formulation we obtain an integer optimal solution by solving the LP-relaxed problem. (For any objective function).

Chvátal Cuts

Valid inequalities for a pure IP-model (minimization)

1. Add constraints, using suitable multipliers
2. Divide through by a common coefficient factor
3. Round up right-hand-side to the next integer

Example

$$\begin{array}{lrcll} \text{minimize} & 2x_1 + 7x_2 + 2x_3 & & & \\ \text{subject to} & x_1 + 4x_2 + x_3 & \geq & 10 & (1) \\ & 4x_1 + 2x_2 + 2x_3 & \geq & 13 & (2) \\ & x_1 + x_2 - x_3 & \geq & 0 & (3) \\ & x_1 & \geq & 0 & (4) \\ & x_2 & \geq & 0 & (5) \\ & x_3 & \geq & 0 & (6) \\ & x_1, x_2, x_3 \ \text{integer} & & & \end{array}$$

1 times (2) is

$$4x_1 + 2x_2 + 2x_3 \ \geq \ 13$$

divide by two

$$2x_1 + x_2 + x_3 \ \geq \ 6\tfrac{1}{2}$$

left hand side is integral, thus round up right-hand

$$2x_1 + x_2 + x_3 \ \geq \ 7$$

Example (continued)

$$\begin{array}{lrcll}
\text{minimize} & 2x_1+7x_2+2x_3 & & & \\
\text{subject to} & x_1+4x_2+x_3 & \geq & 10 & (1)\\
 & 4x_1+2x_2+2x_3 & \geq & 13 & (2)\\
 & x_1+x_2-x_3 & \geq & 0 & (3)\\
 & x_1 & \geq & 0 & (4)\\
 & x_2 & \geq & 0 & (5)\\
 & x_3 & \geq & 0 & (6)\\
 & x_1,x_2,x_3 \text{ integer} & & &
\end{array}$$

Facets

$$\begin{array}{rcll}
x_1+4x_2+x_3 & \geq & 10 & (a)\\
2x_1+x_2+x_3 & \geq & 7 & (b)\\
x_1+x_2-x_3 & \geq & 0 & (c)\\
x_1+3x_2+x_3 & \geq & 9 & (d)\\
2x_1+4x_2+x_3 & \geq & 13 & (e)\\
x_1+x_2+x_3 & \geq & 5 & (f)\\
x_1+2x_2 & \geq & 5 & (g)\\
2x_1+x_2 & \geq & 4 & (h)\\
x_1,x_2,x_3 & \geq & 0,\ \text{integer} &
\end{array}$$

Obtained as

(d) : 5 × (1), 1 × (b), 1 × (6), divide 7
(f) : 1 × (1), 3 × (b), 3 × (6), divide 7
(g) : 4 × (1), 1 × (b), 5 × (3), divide 11
(h) : 1 × (b), 1 × (c), 1 × (4), divide 2
(e) : 3 × (d), 1 × (b), 3 × (g), divide 4

Chvátal Cuts (maximization)

$$\begin{array}{ll}
\text{maximize} & \sum_{j=1}^{n} c_j x_j\\
\text{subject to} & \sum_{j=1}^{n} a_{1j}x_j \leq b_1\\
 & \vdots\\
 & \sum_{j=1}^{n} a_{mj}x_j \leq b_m\\
 & x_j \in \mathbb{Z}_+, \quad j=1,\ldots,n
\end{array}$$

1 Take a linear combination of the constraints

$$\sum_{j=1}^{n}(\sum_{i=1}^{m} u_i a_{ij})x_j \leq (\sum_{i=1}^{m} u_i b_i)$$

in short

$$\sum_{j=1}^{n} a'_j x_j \leq b'$$

2 Divide through by a common factor $d|a'_j,\ j=1,\ldots,n$

$$\sum_{j=1}^{n} \frac{a'_j}{d} x_j \leq \frac{b'}{d}$$

3 Since all a'_j/d are integers round down b'

$$\sum_{j=1}^{n} \frac{a'_j}{d} x_j \leq \lfloor \frac{b'}{d} \rfloor$$

Chvatal-Gomory (maximization)

$$\begin{array}{ll}
\text{maximize} & \sum_{j=1}^{n} c_j x_j\\
\text{subject to} & \sum_{j=1}^{n} a_{1j}x_j \leq b_1\\
 & \vdots\\
 & \sum_{j=1}^{n} a_{mj}x_j \leq b_m\\
 & x_j \in \mathbb{Z}_+, \quad j=1,\ldots,n
\end{array}$$

1 Take a linear combination of the constraints

$$\sum_{j=1}^{n}(\sum_{i=1}^{m} u_i a_{ij})x_j \leq (\sum_{i=1}^{m} u_i b_i)$$

in short

$$\sum_{j=1}^{n} a'_j x_j \leq b'$$

2 Since $x \geq 0$ implies $\sum_{j=1}^{n} a'_j x_j \geq \sum_{j=1}^{n} \lfloor a'_j \rfloor x_j$ we have

$$\sum_{j=1}^{n} \lfloor a'_j \rfloor x_j \leq b'$$

3 Since $x_j \in \mathbb{Z}_+$ implies $\lfloor a'_j \rfloor x_j \in \mathbb{Z}$ we get

$$\sum_{j=1}^{n} \lfloor a'_j \rfloor x_j \leq \lfloor b' \rfloor$$

Chvatal-Gomory (Theorem 8.4)

$$X = \{x : Ax \leq b, x \in \mathbb{Z}_+^n\}$$

Every valid inequality for X can be obtained by applying the Chvatal-Gomory procedure a finite number of times.

Notice

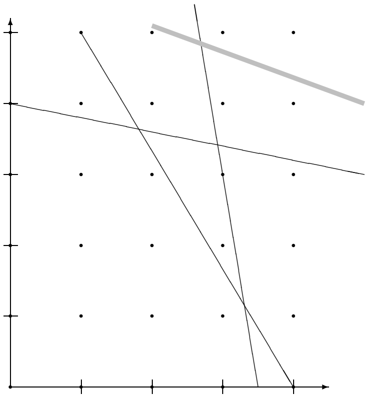
- No stronger inequalities than Chvatal-Gomory exists.
- Even the facet constraints can be generated as Chvatal-Gomory cuts.
- No constructive (polynomial) algorithm for how the linear combination of constraints should be chosen.
- In practice, the derivation of Chvatal-Gomory cuts must rely on specific features of a given application.

Gomory cuts is a systematical way of deriving cutting planes.

Only 0-1 case

All bounded integer variables can be expressed as sum of binary variables.

The set $X = P \cap \mathbb{Z}^n$



$P = \{x \in \mathbb{R}^n : Ax \leq b, 0 \leq x \leq 1\} \neq \emptyset$
$X = P \cap \mathbb{Z}^n$

Nemhauser and Wolsey, Proposition 1.1 page 208:

If inequality $\pi x \leq \pi_0$ is valid for P then it can be obtained as a C-G cut	(*)

- LP-redundant constraints are C-G inequalities
- Theorem 8.4 deals with IP-constraints

Proof (0-1 case)

Assume that

$$\pi x \leq \pi_0 \text{ where } \pi, \pi_0 \text{ integers}$$

is a valid inequality for X. We will show that this inequality can be obtained by using the C-G procedure a finite number of times.

- Step 1: Find a large number $t \in \mathbb{Z}_+$ such that
$$\pi x \leq \pi_0 + t$$
is a valid C-G inequality
- Step 2: Prove that if
$$\pi x \leq \pi_0 + \tau + 1$$
for $\tau \in \mathbb{Z}_+$ is a C-G inequality for X then also
$$\pi x \leq \pi_0 + \tau$$
is a C-G inequality for X.
- Step 3: Use step 2 for $\tau = t-1, \ldots, 0$ each time getting a new C-G inequality

(Proof by induction)

Step 1

The inequality
$$\pi x \leq \pi_0 + t$$
is valid for P for some $t \in \mathbb{Z}_+$.

Proof

We have the inequality

$$x \leq 1$$

derive C-G inequality using multipliers $u = \pi$

$$\pi x \leq \pi 1$$

choosing $t = \pi 1 - \pi_0$ (π, π_0 is integer) we get the form

$$\pi x \leq \pi 1 = \pi_0 + t$$

for some $t \in \mathbb{Z}_+$

Note that $t < \infty$ as $P \subseteq [0,1]^n$ is bounded so $\max\{\pi x \mid x \in P\} < \infty$.

Step 2

Difficult part

a) Prove that if $\pi x \leq \pi_0 + \tau + 1$ with $\tau \in \mathbb{Z}_+$ is a C-G cut then
$$\pi x \leq \pi_0 + \tau + \sum_{j \in N^0} x_j + \sum_{j \in N^1} (1 - x_j)$$
is a C-G inequality for X for every partition (N^0, N^1) of $N = \{1, \ldots, n\}$.

b) Use partitionings $(T^0 \cup \{n\}, T^1)$ and $(T^0, T^1 \cup \{n\})$ to obtain a new inequality for (T^0, T^1).

c) Derive all valid inequalities for partitionings of $N' = \{1, \ldots, n-1\}$

d) Repeating this procedure n times implies that we eliminate the sums on the right side and thus
$$\pi x \leq \pi_0 + \tau$$
is a C-G cut

Time complexity

- part (c) takes $O(2^n)$,
part (d) is performed n times,
in total $O(n2^n)$
- we run Step 2 $O(t)$ times, thus in total $O(tn2^n)$.

Step 2, a)

An inequality is valid for P if it is valid for all extreme points $\{x^1,\ldots,x^m\}$ of P

Assume that $\pi x \le \pi_0 + \tau + 1$ with $\tau \in \mathbb{Z}_+$ is a valid cut. Let (N^0, N^1) be any partitioning of $N = \{1,\ldots,n\}$. Consider an extreme point x^k of P

- x^k integer: then $\pi x^k \le \pi_0$ (since $\pi x \le \pi_0$ valid for X)
- x^k fractional: exists $\varepsilon > 0$ such that

$$\varepsilon^k \le \sum_{j\in N^0} x_j^k + \sum_{j\in N^1} (1 - x_j^k)$$

Choose $\alpha = \min_{x^k \text{ vertex in } P} \varepsilon^k$

Using $M \ge (\tau+1)/\alpha$, we have

$$\tau + 1 \le M\alpha \le M\left(\sum_{j\in N^0} x_j + \sum_{j\in N^1}(1-x_j)\right)$$

adding π_0 at both sides we get valid inequality for P

$$\pi x \le \pi_0 + \tau + 1 \le \pi_0 + M\left(\sum_{j\in N^0} x_j + \sum_{j\in N^1}(1-x_j)\right)$$

Due to (*) the inequality is a C-G cut.

Step 2, a)

We have just shown that the following is a C-G inequality

$$\pi x \le \pi_0 + M\left(\sum_{j\in N^0} x_j + \sum_{j\in N^1}(1-x_j)\right)$$

By assumption we had the C-G inequality

$$\pi x \le \pi_0 + \tau + 1$$

use weights $1/M$ and $(M-1)/M$ for the two inequalities getting C-G inequality

$$\pi x \le \pi_0 + \tau + \sum_{j\in N^0} x_j + \sum_{j\in N^1}(1-x_j)$$

Step 2, b)

Use partitions $(T^0 \cup \{n\}, T^1)$ and $(T^0, T^1 \cup \{n\})$

$$\pi x \le \pi_0 + \tau + \sum_{j\in T^0\cup\{n\}} x_j + \sum_{j\in T^1}(1-x_j)$$

and

$$\pi x \le \pi_0 + \tau + \sum_{j\in T^0} x_j + \sum_{j\in T^1\cup\{n\}}(1-x_j)$$

using multipliers $1/2$ and $1/2$ we get C-G inequality

$$\pi x \le \pi_0 + \tau + \sum_{j\in T^0} x_j + \sum_{j\in T^1}(1-x_j)$$

Chvatal-Gomory rank-k inequalities

Problem

$$\max\{cx \mid Ax \le b, x \in \{0,1\}^n\}$$

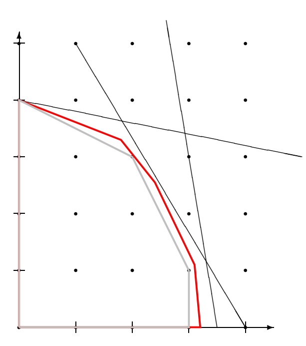
$$I^0 = \{\text{original inequalities } Ax \le b\}$$

Chvatal-Gomory rank-1 inequalities

$$I^1 = \left\{\text{CG inequalities obtained using } I^0\right\}$$

Chvatal-Gomory rank-k inequalities

$$I^k = \left\{\text{CG inequalities obtained using } I^{k-1}\right\}$$



$$P^k = \left\{x \in \mathbb{R}^n \mid ax \le b, \quad (ax \le b \in I^k)\right\}$$

Chvatal-Gomory rank-k inequalities

Convex hull

$$P_{IP} = \text{conv}\left\{Ax \leq b, x \in \{0,1\}^n\right\}$$

The smallest k such that

$$P^k = P_{IP}$$

is called the *Chvatal-Gomory rank* of the problem.

Chvatal-Gomory rank is a measure of complexity of problem

Eisenbrand and Schulz (1999) showed, maximum rank k

$$(1+\varepsilon)n \leq k \leq O(n^2 \log n)$$

Rank of inequality

A valid inequality has CG rank k if it can be generated as a CG inequality based on inequalities of rank $0, 1, \ldots, k-1$, but does not have rank less than k.

- Often considered when new valid inequalities are proposed for a problem.
- Determining an upper bound on the rank is often done by trial-and-error.
- Lower bound proofs are generally even harder.
- Not aware of any existing computational method for testing rank 2.

Separating rank 1 CG inequalities is NP-hard (Eisenbrand 1999).

Branch-and-cut algorithms

Combines best properties from Branch-and-bound and cutting plane.

- Basically a branch-and-bound algorithm
- at each node solve LP-relaxation to find bound
- generate valid inequalities which separate the LP-solution, and which are *valid for the whole problem*
- maintain pool of valid inequalities
- branch when cuts become weak
- convergence ensured by branch-and-bound

Improvements

- Heuristic for generating cut
- Problem specific cuts
- Heuristic for removing cuts

If separation problem is “easy” the cut is not tight