Introduction to Optimization, DIKU 2007-08
Wednesday 21 November
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Lecture 4, Network problems

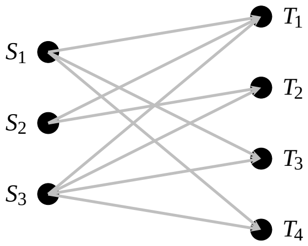
Network problems, transportation model, totally unimodular (property P, interval matrices), max-flow min-cut duality.

Taha section

- 5.1, 5.2, 5.3, 5.4
- 6.3, 6.4
- all examples can be read briefly

Transportation Model (Taha 5.1)

Three suppliers (S_1, S_2, S_3) should provide four customers (T_1, T_2, T_3, T_4) with a particular commodity.



Supplier	S_1	S_2	S_3
Capacity	135	56	93

Customer	T_1	T_2	T_3	T_4
Requirements	62	83	39	91

	Customer			
Supplier	T_1	T_2	T_3	T_4
S_1	132	—	97	103
S_2	85	91	—	—
S_3	106	89	100	98

Transportation Model

Variables: x_{ij} is flow from supplier i to customer j

LP-model

minimize

$$132x_{11}+Mx_{12}+97x_{13}+103x_{14}+85x_{21}+91x_{22}+Mx_{23}+Mx_{24}+106x_{31}+89x_{32}+100x_{33}+98x_{34}$$

subject to

$$\begin{array}{rrrrrrrrrrrrl}
x_{11} & +x_{12} & +x_{13} & +x_{14} & & & & & & & & & \leq 135 \\
 & & & & x_{21} & +x_{22} & +x_{23} & +x_{24} & & & & & \leq 56 \\
 & & & & & & & & x_{31} & +x_{32} & +x_{33} & +x_{34} & \leq 93 \\
x_{11} & & & & +x_{21} & & & & +x_{31} & & & & \geq 62 \\
 & x_{12} & & & & +x_{22} & & & & +x_{32} & & & \geq 83 \\
 & & x_{13} & & & & +x_{23} & & & & +x_{33} & & \geq 39 \\
 & & & x_{14} & & & & +x_{24} & & & & +x_{34} & \geq 91
\end{array}$$

$x_{ij} \geq 0$

What if x_{ij} must be integer

commodities are cars, aircrafts, ...

Totally Unimodular

Definition 1 An $m \times n$ integral matrix A is called *totally unimodular* (TU) if the determinant of each square submatrix of A is equal to $0, 1$ or -1.

Obviously $a_{ij} \in \{0, 1, -1\}$ for all i, j

Recognizing whether A is TU demands an exponential number of steps

Proposition 1 If A is TU then A_B is also TU

Proof: If A is TU then the determinant of each square submatrix of A is equal to 0, 1 or -1. This also holds when restricted to columns in A_B.

Proposition 2 If A is TU then A^{-1} is integral

Proof: From Cramer's rule $A^{-1}_{ij} = C_{ji}/\det(A)$ where C_{ji} is the adjoint matrix

$$C_{ji} = (-1)^{i+j}\det(A_{\text{row } i, \text{ column } j \text{ removed}})$$

Proposition 3 If A is TU and b is integral then any basis solution x_B is integral

Proof:

$$x_B = A_B^{-1}b$$

Totally Unimodular

("*Property P*") Let A be a $(0,1,-1)$ matrix with no more than two nonzero elements in each column. Then A is TU if and only if the rows of A can be divided in two subsets P_1 and P_2 such that if a column contains two nonzero elements, the following statements are true:

1 If both nonzero elements have the same sign, then one is in a row contained in P_1 and the other is in a row contained in P_2.

2 If the two nonzero elements have opposite sign, then both are in rows contained in the same subset.

Example

$$\begin{pmatrix} & & -1 & & & & & \\ & 1 & & 1 & & & 1 & \\ -1 & & & -1 & 1 & & & \\ & & & & & 1 & & \\ & 1 & -1 & & & & & -1 \\ & & & & & -1 & & \end{pmatrix}$$

Transportation Model

Variables: x_{ij} is flow from supplier i to customer j

IP-model

minimize

$$132x_{11}+Mx_{12}+97x_{13}+103x_{14}+85x_{21}+91x_{22}+Mx_{23}+Mx_{24}+106x_{31}+89x_{32}+100x_{33}+98x_{34}$$

subject to

$$\begin{array}{cccccccccccccc} x_{11} & +x_{12} & +x_{13} & +x_{14} & & & & & & & & & \leq & 135 \\ & & & & x_{21} & +x_{22} & +x_{23} & +x_{24} & & & & & \leq & 56 \\ & & & & & & & & x_{31} & +x_{32} & +x_{33} & +x_{34} & \leq & 93 \\ x_{11} & & & & +x_{21} & & & & +x_{31} & & & & \geq & 62 \\ & x_{12} & & & & +x_{22} & & & & +x_{32} & & & \geq & 83 \\ & & x_{13} & & & & +x_{23} & & & & +x_{33} & & \geq & 39 \\ & & & x_{14} & & & & +x_{24} & & & & +x_{34} & \geq & 91 \end{array}$$

$x_{ij} \in \mathbb{N}$

Model is TU, hence we always get integer solutions

Applications of TU

- network matrices
- interval matrices
- binary logic matrices

Definition 2 An $m \times n$ matrix A (with only 0-1 elements) is called an *interval matrix* if in each column the 1's appear consecutively; that is $a_{ij} = a_{kj} = 1$ and $k > i+1$ then $a_{\ell j}$ for all ℓ with $i < \ell < k$.

Example

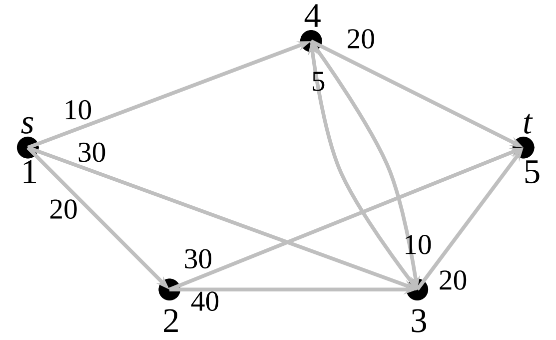
$$\begin{pmatrix} 1 & & & & & 1 & \\ 1 & 1 & & 1 & & 1 & \\ 1 & 1 & & 1 & & & 1 \\ & 1 & 1 & & & & 1 \\ & 1 & & & 1 & & \\ & 1 & & & 1 & & \end{pmatrix}$$

Maximum Flow problem

Weighted, oriented graph $G(V,E,c)$
Flow from u to v is x_{uv}

$$\begin{array}{lll} \max & \sum_{v \in V} x_{sv} & \\ \text{s.t.} & x_{uv} \leq c_{uv} & \text{for all } (u,v) \in E \\ & x_{uv} = -x_{vu} & \text{for all } (u,v) \in E \\ & \sum_{v \in V} x_{uv} = 0 & \text{for all } u \in V \setminus \{s,t\} \end{array}$$

Example (figure 6.29 Taha)



Optimal solution $z = 60$

$$x_{12} = 20,\ x_{13} = 30,\ x_{14} = 10,\ x_{25} = 20,$$
$$x_{34} = 10,\ x_{35} = 20,\ x_{45} = 20,$$

Dual

$$\bar{y}_1 = \bar{y}_3 = 1, \qquad y_{12} = y_{14} = y_{34} = y_{35} = 1$$

Maximum Flow problem

Primal problem

$$
\begin{array}{lll}
\max & x_{12}+x_{13}+x_{14} & \\
\text{s.t.} & x_{12}-x_{23}-x_{25}=0 & & (\bar{y}_2) \\
& x_{13}+x_{23}-x_{34}-x_{35}=0 & & (\bar{y}_3) \\
& x_{14}+x_{34}-x_{43}-x_{45}=0 & & (\bar{y}_4) \\
& x_{uv} \le c_{uv} & (u,v)\in E & (y_{uv}) \\
& x_{uv} \ge 0 & (u,v)\in E &
\end{array}
$$

(skew symmetry handled implicit)
Dual variables $\bar{y}_u$ for flow conservation, y_{uv} for capacity

Dual problem

$$
\begin{array}{lll}
\min & c_{12}y_{12}+c_{13}y_{13}+\dots & \\
\text{s.t.} & y_{12}+\bar{y}_2 \ge 1 & \\
& y_{13}+\bar{y}_3 \ge 1 & \\
& y_{14}+\bar{y}_4 \ge 1 & \\
& y_{23}+\bar{y}_3-\bar{y}_2 \ge 0 & \\
& y_{34}+\bar{y}_4-\bar{y}_3 \ge 0 & \\
& y_{43}+\bar{y}_3-\bar{y}_4 \ge 0 & \\
& y_{25}-\bar{y}_2 \ge 0 & \\
& y_{35}-\bar{y}_3 \ge 0 & \\
& y_{45}-\bar{y}_4 \ge 0 & \\
& \bar{y}_u \in \mathbb{R} & u \in V\setminus\{s,t\} \\
& y_{uv} \ge 0 & (u,v)\in E
\end{array}
$$

Maximum Flow problem

$$
\begin{array}{lll}
\min & c_{12}y_{12}+c_{13}y_{13}+\dots & \\
\text{s.t.} & y_{12}+\bar{y}_2-1 \ge 0 & \\
& y_{13}+\bar{y}_3-1 \ge 0 & \\
& y_{14}+\bar{y}_4-1 \ge 0 & \\
& y_{23}+\bar{y}_3-\bar{y}_2 \ge 0 & \\
& y_{34}+\bar{y}_4-\bar{y}_3 \ge 0 & \\
& y_{43}+\bar{y}_3-\bar{y}_4 \ge 0 & \\
& y_{25}+0-\bar{y}_2 \ge 0 & \\
& y_{35}+0-\bar{y}_3 \ge 0 & \\
& y_{45}+0-\bar{y}_4 \ge 0 & \\
& \bar{y}_u \in \mathbb{R} & u \in V\setminus\{s,t\} \\
& y_{uv} \ge 0 & (u,v)\in E
\end{array}
$$

can be rewritten

$$
\begin{array}{lll}
\min & c_{12}y_{12}+c_{13}y_{13}+\dots & \\
\text{s.t.} & y_{12}+\bar{y}_2-\bar{y}_1 \ge 0 & \\
& y_{13}+\bar{y}_3-\bar{y}_1 \ge 0 & \\
& y_{14}+\bar{y}_4-\bar{y}_1 \ge 0 & \\
& y_{23}+\bar{y}_3-\bar{y}_2 \ge 0 & \\
& y_{34}+\bar{y}_4-\bar{y}_3 \ge 0 & \\
& y_{43}+\bar{y}_3-\bar{y}_4 \ge 0 & \\
& y_{25}+\bar{y}_5-\bar{y}_2 \ge 0 & \\
& y_{35}+\bar{y}_5-\bar{y}_3 \ge 0 & \\
& y_{45}+\bar{y}_5-\bar{y}_4 \ge 0 & \\
& \bar{y}_s = 1, \bar{y}_t = 0 & \\
& \bar{y}_u \in \mathbb{R} & u \in V\setminus\{s,t\} \\
& y_{uv} \ge 0 & (u,v)\in E
\end{array}
$$

Maximum Flow problem

Primal problem

$$
\begin{array}{lll}
\max & \sum_{v\in V} x_{sv} & \\
\text{s.t.} & \sum_{v\in V} x_{vu} - \sum_{v\in V} x_{uv} = 0 & \text{for all } u \in V\setminus\{s,t\} \\
& 0 \le x_{uv} \le c_{uv} & \text{for all } (u,v)\in E
\end{array}
$$

dual problem

$$
\begin{array}{lll}
\min & \sum_{u\in V}\sum_{v\in V} c_{uv}y_{uv} & \\
\text{s.t.} & \bar{y}_v - \bar{y}_u + y_{uv} \ge 0 & \text{for all } (u,v)\in E \\
& \bar{y}_s = 1 & \\
& \bar{y}_t = 0 & \\
& y_{uv} \ge 0 & \text{for all } (u,v)\in E
\end{array}
$$

Maximum Flow problem, TU

Example 6.4-3 in Taha (page 273)
Primal problem

	x_{12}	x_{13}	x_{14}	x_{23}	x_{25}	x_{34}	x_{35}	x_{43}	x_{45}	
$z=$	1	1	1	0	0	0	0	0	0	
node 2	1	0	0	-1	-1	0	0	0	0	$=0$
node 3	0	1	0	1	0	-1	-1	1	0	$=0$
node 4	0	0	1	0	0	1	0	-1	-1	$=0$
capacity	20	30	10	40	30	10	20	5	20	

Primal problem is TU
Dual problem is TU

Upper bounds on dual variables

y_{uv} is dual variable corresponding to capacity constraint

$$x_{uv} \le c_{uv}$$

How much can objective increase by increasing RHS by 1?
i.e. $y_{uv} \le 1$.

$\bar{y}_u$ is dual variable corresponding to flow conservation

$$\sum_{v\in V} x_{uv} = 0$$

How much can objective increase by increasing RHS by 1?
i.e. $\bar{y}_u \le 1$

Maximum Flow problem, dual problem

Equivalent dual problem

$$\begin{array}{lll} \min & \sum_{u\in V}\sum_{v\in V} c_{uv}y_{uv} & \\ \text{s.t.} & \bar{y}_v - \bar{y}_u + y_{uv} \geq 0 & (u,v)\in E \\ & \bar{y}_s = 1 & \\ & \bar{y}_t = 0 & \\ & y_{uv} \in \{0,1\} & (u,v)\in E \\ & \bar{y}_u \in \{0,1\} & u \in V \end{array}$$

Define cut

Let $(y^*, \bar{y}^*)$ be an optimal solution to dual problem

$$S = \{u \in V \mid \bar{y}^*_u = 1\}, T = V \setminus S$$

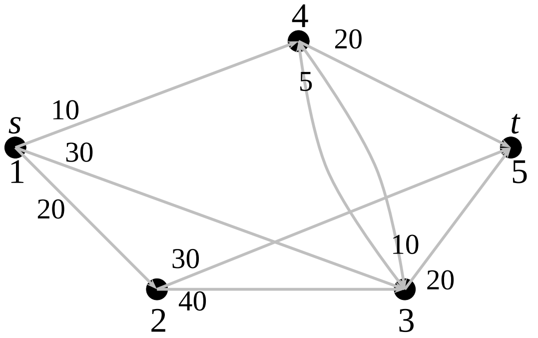
For edge (u,v) constraint $\bar{y}_v - \bar{y}_u + y_{uv} \geq 0$ means

$$\bar{y}_u = 1, \bar{y}_v = 0 \Leftrightarrow y_{uv} = 1$$

$$u \in S, v \in T \Leftrightarrow y_{uv} = 1$$

(S,T) is a cut
Dual problem searches for minimal cut
Strong duality theorem gives max-flow-min-cut relation

Example



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Dual

$$\bar{y}_1 = \bar{y}_3 = 1 \qquad y_{12} = y_{14} = y_{34} = y_{35} = 1$$

Assignment Problem

- n workers
- n jobs
- cost of assigning worker i to job j is c_{ij}

$$\begin{array}{lll} \text{minimize} & \displaystyle\sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}x_{ij} & \\ \text{subject to} & \displaystyle\sum_{j=1}^{n} x_{ij} = 1 & i = 1,\ldots,n \\ & \displaystyle\sum_{i=1}^{n} x_{ij} = 1 & j = 1,\ldots,n \\ & x_{ij} \in \{0,1\} & i,j = 1,\ldots,n \end{array}$$

Example

$$c_{ij} = \begin{pmatrix} 1 & 9 & 4 & 2 & 6 \\ 2 & 8 & 6 & 4 & 6 \\ 3 & 4 & 7 & 6 & 5 \\ 2 & 8 & 4 & 5 & 7 \\ 3 & 6 & 5 & 4 & 6 \end{pmatrix}$$

Assignment Problem

The constraints look as:

$$\begin{array}{cccccccccc} x_{11} & +x_{12} & +x_{13} & & & & & & & = 1 \\ & & & x_{21} & +x_{22} & +x_{23} & & & & = 1 \\ & & & & & & x_{31} & +x_{32} & +x_{33} & = 1 \\ x_{11} & & & +x_{21} & & & +x_{31} & & & = 1 \\ & x_{21} & & & +x_{22} & & & +x_{32} & & = 1 \\ & & x_{31} & & & +x_{32} & & & +x_{33} & = 1 \end{array}$$

(“*Property P*”) Let A be a $(0,1,-1)$ matrix with no more than two nonzero elements in each column. Then A is TU if and only if the rows of A can be divided in two subsets P_1 and P_2 such that if a column contains two nonzero elements, the following statements are true:

1 If both nonzero elements have the same sign, then one is in a row contained in P_1 and the other is in a row contained in P_2.

2 If the two nonzero elements have opposite sign, then both are in rows contained in the same subset.

Simplex will return 0-1 solution
However, simplex does not have polynomial running time

Assignment Problem

Dual problem

- u_i dual variable for “row” constraints
- v_j dual variable for “column” constraints

Thus

$$\begin{array}{lll} \text{maximize} & \sum_{i=1}^{n} u_i + \sum_{j=1}^{n} v_j & \\ \text{subject to} & u_i + v_j \le c_{ij} & i, j = 1, \ldots, n \\ & u_i, v_j \in \mathbb{R} & \end{array}$$

Example

$$\begin{array}{cc} \begin{array}{c} 1 \\ 1 \\ 3 \\ 1 \\ 2 \end{array} & \begin{pmatrix} 1 & 9 & 4 & 2 & 6 \\ 2 & 8 & 6 & 4 & 6 \\ 3 & 4 & 7 & 6 & 5 \\ 2 & 8 & 4 & 5 & 7 \\ 3 & 6 & 5 & 4 & 6 \end{pmatrix} \\ & \begin{array}{ccccc} 0 & 0 & 2 & 0 & 1 \end{array} \end{array}$$

feasible, non-optimal dual solution

Hungarian Algorithm

$$\begin{pmatrix} 1 & 9 & 4 & 2 & 6 \\ 2 & 8 & 6 & 4 & 6 \\ 3 & 4 & 7 & 6 & 5 \\ 2 & 8 & 4 & 5 & 7 \\ 3 & 6 & 5 & 4 & 6 \end{pmatrix}$$

row reduction (subtract from row)

$$\begin{array}{cc} \begin{array}{c} 1 \\ 2 \\ 3 \\ 2 \\ 3 \end{array} & \begin{pmatrix} 1 & 9 & 4 & 2 & 6 \\ 2 & 8 & 6 & 4 & 6 \\ 3 & 4 & 7 & 6 & 5 \\ 2 & 8 & 4 & 5 & 7 \\ 3 & 6 & 5 & 4 & 6 \end{pmatrix} \end{array}$$

column reduction (subtract from column)

$$\begin{array}{c} \begin{pmatrix} 0 & 8 & 3 & 1 & 5 \\ 0 & 6 & 4 & 2 & 4 \\ 0 & 1 & 4 & 3 & 2 \\ 0 & 6 & 2 & 3 & 5 \\ 0 & 3 & 2 & 1 & 3 \end{pmatrix} \\ \begin{array}{ccccc} 0 & 1 & 2 & 1 & 2 \end{array} \end{array}$$

Mark independent set S of zeros

$$\begin{pmatrix} \boxed{0} & 7 & 1 & 0 & 3 \\ 0 & 5 & 2 & 1 & 2 \\ 0 & 0 & 2 & 2 & \boxed{0} \\ 0 & 5 & \boxed{0} & 2 & 3 \\ 0 & 2 & 0 & \boxed{0} & 1 \end{pmatrix}$$

not feasible

Hungarian Algorithm, create additional zero

Find minimum number of lines, covering all zero’s

$$\begin{pmatrix} \cancel{0} & 7 & \cancel{1} & \cancel{0} & 3 \\ \cancel{0} & 5 & \cancel{2} & \cancel{1} & 2 \\ \cancel{0} & \cancel{0} & \cancel{2} & \cancel{2} & \cancel{0} \\ \cancel{0} & 5 & \cancel{0} & \cancel{2} & 3 \\ \cancel{0} & 2 & \cancel{0} & \cancel{0} & 1 \end{pmatrix}$$

Choose min element $\Delta = c_{ij}$ not covered by any line.
Add Δ to all covered rows.
Subtract Δ from all *not* covered columns.

$$\begin{array}{cc} \begin{array}{c} \\ \\ +\Delta \\ \\ \\ \end{array} & \begin{pmatrix} \cancel{0} & 7 & \cancel{1} & \cancel{0} & 3 \\ \cancel{0} & 5 & \cancel{2} & \cancel{1} & 2 \\ \cancel{0} & \cancel{0} & \cancel{2} & \cancel{2} & \cancel{0} \\ \cancel{0} & 5 & \cancel{0} & \cancel{2} & 3 \\ \cancel{0} & 2 & \cancel{0} & \cancel{0} & 1 \end{pmatrix} \\ & \begin{array}{ccccc} & -\Delta & & & -\Delta \end{array} \end{array}$$

All c_{ij} are nonnegative after operation
Previous zero’s are unchanged (apart from redundant)
We create a new independent zero

$$\begin{pmatrix} 0 & 6 & 1 & 0 & 2 \\ 0 & 4 & 2 & 1 & 1 \\ 1 & 0 & 3 & 3 & 0 \\ 0 & 4 & 0 & 2 & 2 \\ 0 & 1 & 0 & 0 & \boxed{0} \end{pmatrix}$$

Hungarian Algorithm

Mark independent set S of zero’s

$$\begin{pmatrix} 0 & 6 & 1 & \boxed{0} & 2 \\ \boxed{0} & 4 & 2 & 1 & 1 \\ 1 & \boxed{0} & 3 & 3 & 0 \\ 0 & 4 & \boxed{0} & 2 & 2 \\ 0 & 1 & 0 & 0 & \boxed{0} \end{pmatrix}$$

Find dual costs

$$\begin{array}{cc} \begin{array}{r} 1 \\ 2 \\ -\Delta + 3 \\ 2 \\ 3 \end{array} & \begin{pmatrix} 0 & 6 & 1 & 0 & 2 \\ 0 & 4 & 2 & 1 & 1 \\ 1 & 0 & 3 & 3 & 0 \\ 0 & 4 & 0 & 2 & 2 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \\ & \begin{array}{ccccc} 0 & 1 & 2 & 1 & 2 \\ & + & & & + \\ & \Delta & & & \Delta \end{array} \end{array}$$

Proof of algorithm

- Hungarian algorithm will terminate (one new independent zero in each iteration)
- In each iteration, entries are nonnegative i.e.
  $$c_{ij} - u_i - v_j \geq 0 \quad \Leftrightarrow \quad u_i + v_j \leq c_{ij}$$
  (u, v) is dual feasible
- Proof of optimality follows from complementary slack
  $$x_{ij} = 0 \quad \text{or} \quad u_i + v_j = c_{ij} \qquad \text{for all } i, j$$
  $$x_{ij} \neq 1 \quad \text{or} \quad c_{ij} - u_i - v_j = 0 \qquad \text{for all } i, j$$
  "we have 0 in the matrix or the assignment is not chosen"

$$\begin{array}{c} c_{ij} \\ \begin{pmatrix} 1 & 9 & 4 & 2 & 6 \\ 2 & 8 & 6 & 4 & 6 \\ 3 & 4 & 7 & 6 & 5 \\ 2 & 8 & 4 & 5 & 7 \\ 3 & 6 & 5 & 4 & 6 \end{pmatrix} \end{array} \qquad \begin{array}{cc} & c_{ij} - u_i - v_j \\ \begin{array}{c} 1 \\ 2 \\ 2 \\ 2 \\ 3 \end{array} & \begin{pmatrix} 0 & 6 & 1 & \underline{0} & 2 \\ \underline{0} & 4 & 2 & 1 & 1 \\ 1 & \underline{0} & 3 & 3 & 0 \\ 0 & 4 & \underline{0} & 2 & 2 \\ 0 & 1 & 0 & 0 & \underline{0} \end{pmatrix} \\ & \begin{array}{ccccc} 0 & 2 & 2 & 1 & 3 \end{array} \end{array}$$

$$\boxed{\begin{array}{ll} \min & \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij} \\ \text{s.t.} & \sum_{j=1}^{n} x_{ij} = 1 \quad \forall i \\ & \sum_{i=1}^{n} x_{ij} = 1 \quad \forall j \\ & x_{ij} \in \{0, 1\} \quad \forall i, j \end{array}} \qquad \boxed{\begin{array}{ll} \max & \sum_{i=1}^{n} u_i + \sum_{j=1}^{n} v_j \\ \text{s.t.} & u_i + v_j \leq c_{ij} \qquad \forall i, j \\ & u_i, v_j \in \mathbb{R} \end{array}}$$

Hungarian algorithm

- Dual feasibility is maintained
- Complementary slack is respected
- Searching for primal feasible

Dual algorithm