

**Lecture 3, Duality and sensitivity analysis**

Duality, shadow prices, sensitivity analysis, post-optimal analysis, complementary slackness, KKT optimality condition, CPLEX sensitivity

Taha section

- 3.6
- 4.1, 4.2, 4.3, 4.4, 4.5
- all examples can be read briefly

**Duality, motivation**

Looking for an upper bound on LP:

$$\begin{aligned} &\text{maximize } 2x_1 + 3x_2 \\ &\text{subject to } 2x_1 + x_2 \leq 4 \\ &\quad \quad \quad x_1 + 2x_2 \leq 5 \\ &\quad \quad \quad x_1, x_2 \geq 0 \end{aligned}$$

- Multiply second constraint by two

$$2x_1 + 4x_2 \leq 10$$

gives upper bound 10

- Add first and second constraint

$$3x_1 + 3x_2 \leq 9$$

gives upper bound 9

- Linear combination of constraints

$$y_1(2x_1 + x_2) + y_2(x_1 + 2x_2) \leq 4y_1 + 5y_2$$

rewritten to

$$(2y_1 + y_2)x_1 + (y_1 + 2y_2)x_2 \leq 4y_1 + 5y_2$$

- *Dual problem*

$$\begin{aligned} &\text{minimize } 4y_1 + 5y_2 \\ &\text{subject to } 2y_1 + y_2 \geq 2 \\ &\quad \quad \quad y_1 + 2y_2 \geq 3 \\ &\quad \quad \quad y_1, y_2 \geq 0 \end{aligned}$$

**Duality**

Primal problem

$$\begin{aligned} &\max \sum_{j=1}^n c_j x_j \\ &\text{s.t. } \sum_{j=1}^n a_{ij} x_j \leq b_i \quad i = 1, \dots, m \\ &\quad \quad \quad x_j \geq 0 \quad j = 1, \dots, n \end{aligned}$$

Dual problem

$$\begin{aligned} &\min \sum_{i=1}^m b_i y_i \\ &\text{s.t. } \sum_{i=1}^m a_{ij} y_i \geq c_j \quad j = 1, \dots, n \\ &\quad \quad \quad y_i \geq 0 \quad i = 1, \dots, m \end{aligned}$$

**Weak duality**

For every primal feasible  $x$ , dual feasible  $y$

$$\sum_{j=1}^n c_j x_j \leq \sum_{i=1}^m b_i y_i$$

Proof

$$\sum_{j=1}^n c_j x_j \leq \sum_{j=1}^n \left( \sum_{i=1}^m a_{ij} y_i \right) x_j = \sum_{i=1}^m \left( \sum_{j=1}^n a_{ij} x_j \right) y_i \leq \sum_{i=1}^m b_i y_i$$

If  $x$  primal feasible,  $y$  dual feasible,  $\sum_{j=1}^n c_j x_j = \sum_{i=1}^m b_i y_i$  then  $x$  is primal optimal, and  $y$  is dual optimal.

**Strong duality**

For every bounded, feasible LP with primal solution  $x^*$ , there exists dual feasible  $y^*$  such that

$$\sum_{j=1}^n c_j x_j^* = \sum_{i=1}^m b_i y_i^*$$

Proof: Shown at VA course.

## Dualization (Taha table 4.3)

Maximization problem		Minimization Problem	
Constraints		Variables	
$\geq$	$\Leftrightarrow$	$\leq 0$	
$\leq$	$\Leftrightarrow$	$\geq 0$	
$=$	$\Leftrightarrow$	unrestricted	
Variables		Constraints	
$\geq 0$	$\Leftrightarrow$	$\geq$	
$\leq 0$	$\Leftrightarrow$	$\leq$	
unrestricted	$\Leftrightarrow$	$=$	

### Example

$$\begin{aligned}
 \max \quad & 3x_1 + 2x_2 + 5x_3 \\
 \text{s.t.} \quad & 5x_1 + 3x_2 + x_3 = -8 \quad (y_1) \\
 & 4x_1 + 2x_2 + 8x_3 \leq 23 \quad (y_2) \\
 & 6x_1 + 7x_2 + 3x_3 \geq 1 \quad (y_3) \\
 & x_1 \leq 4 \quad (y_4) \\
 & x_3 \geq 0 \\
 & x_1, x_2 \in \mathbb{R} \\
 \\ 
 \min \quad & -8y_1 + 23y_2 + y_3 + 4y_4 \\
 \text{s.t.} \quad & 5y_1 + 4y_2 + 6y_3 + y_4 = 3 \quad (x_1) \\
 & 3y_1 + 2y_2 + 7y_3 = 2 \quad (x_2) \\
 & y_1 + 8y_2 + 3y_3 \geq 5 \quad (x_3) \\
 & y_2 \geq 0 \\
 & y_3 \leq 0 \\
 & y_4 \leq 0 \\
 & y_1 \in \mathbb{R}
 \end{aligned}$$

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## Primal-Dual Relation

		Dual		
		Optimal	Infeasible	Unbounded
Primal	Optimal	Possible	Impossible	Impossible
	Infeasible	Impossible	Possible	Possible
	Unbounded	Impossible	Possible	Impossible

## Economic interpretation of Dual variables (Taha 4.3.1)

$y_i$ , dual variables, "shadow prices", "marginal prices"

If constraint  $i$  express a limit on resource  $i$  then dual variable  $y_i$  represents the worth per unit of resource  $i$

If primal problem

$$\begin{aligned}
 \max \quad & \sum_{j=1}^n c_j x_j \\
 \text{s.t.} \quad & \sum_{j=1}^n a_{ij} x_j \leq b_i \quad i = 1, \dots, m \\
 & x_j \geq 0 \quad j = 1, \dots, n
 \end{aligned}$$

has optimal solution value  $z^*$ . Then

$$\begin{aligned}
 \max \quad & \sum_{j=1}^n c_j x_j \\
 \text{s.t.} \quad & \sum_{j=1}^n a_{ij} x_j \leq b_i + \epsilon_i \quad i = 1, \dots, m \\
 & x_j \geq 0 \quad j = 1, \dots, n
 \end{aligned}$$

has optimal solution value  $z^* + \sum_{i=1}^m y_i \epsilon_i$

The changes need to be sufficient small, such that current solution remains optimal

Weak duality theorem:

$$\text{sum of profits} \leq \text{sum of worth of resources}$$

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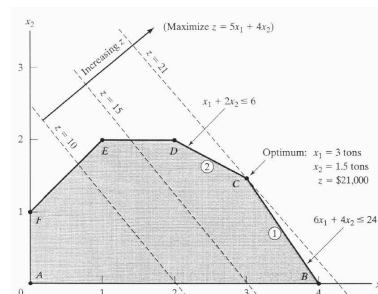
## Reddy Mikks

Primal

$$\begin{aligned}
 \max z = \quad & 5x_1 + 4x_2 \\
 \text{s.t.} \quad & 6x_1 + 4x_2 \leq 24 \quad (\text{resource M1}) \\
 & x_1 + 2x_2 \leq 6 \quad (\text{resource M2}) \\
 & -x_1 + x_2 \leq 1 \quad (\text{market}) \\
 & x_2 \leq 2 \quad (\text{demand}) \\
 & x_1, x_2 \geq 0
 \end{aligned}$$

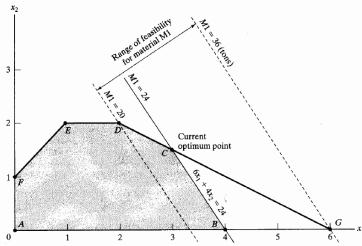
Dual

$$\begin{aligned}
 \min w = \quad & 24y_1 + 6y_2 + y_3 + 2y_4 \\
 \text{s.t.} \quad & 6y_1 + y_2 - y_3 \geq 5 \\
 & 4y_1 + 2y_2 + y_3 + y_4 \geq 4 \\
 & y_1, y_2, y_3, y_4 \geq 0
 \end{aligned}$$

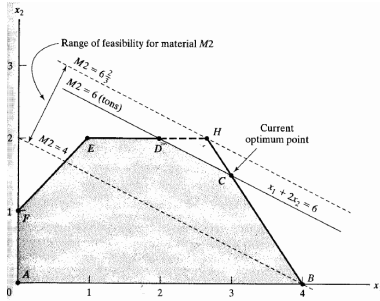


$$x^* = \left(3, \frac{3}{2}\right) \quad y^* = \left(\frac{3}{4}, \frac{1}{2}, 0, 0\right) \quad z^* = 21$$

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profit of increasing M1 one unit  $\frac{z_G - z_C}{36 - 24} = \frac{30 - 21}{12} = \frac{3}{4}$



profit of increasing M2 one unit  $\frac{z_C - z_B}{6 - 4} = \frac{21 - 20}{2} = \frac{1}{2}$

primal (P)	dual (D)
max $cx$	min $yb$
s.t. $Ax \leq b$	s.t. $yA \geq c$
$x \geq 0$	$y \geq 0$

Assume  $x$  feasible to (P) and  $y$  feasible to (D)

Necessary and sufficient conditions for  $x, y$  to be optimal:  
 and  $(\sum_{j=1}^n a_{ij}x_j = b_i \text{ or } y_i = 0) \text{ for } i = 1, \dots, m$   
 and  $(\sum_{i=1}^m a_{ij}y_i = c_j \text{ or } x_j = 0) \text{ for } j = 1, \dots, n$

**Economical interpretation**

$$\sum_{j=1}^n a_{ij}x_j < b_i \Rightarrow y_i = 0$$

If resource  $b_i$  is not fully used, then no additional profit can be obtained by increasing the resource (i.e.  $y_i = 0$ ).

$$\sum_{i=1}^m a_{ij}y_i > c_j \Rightarrow x_j = 0$$

If costs of resources for producing item  $j$  exceed selling cost, then we should not produce item  $j$  (i.e.  $x_j = 0$ ).

**KKT optimality condition**

Karush-Kuhn-Tucker optimality condition for general non-linear constrained minimization problem For LP-problems

max $cx$	min $yb$
s.t. $Ax \leq b$	s.t. $yA \geq c$
$x \geq 0$	$y \geq 0$

the KKT optimality condition becomes

- $Ax \leq b, x \geq 0$  (primal feasible)
- $yA \leq c, y \geq 0$  (dual feasible)
- $y(Ax - b) = 0, x(c - yA) = 0$  (complementary slack)

**Finding the dual variables**

- if the  $i$ th constraint is an “ $\leq$ ” inequality then the  $y_i$  will appear in the column associated with the corresponding slack variable.
- if the  $i$ th constraint is a “ $\geq$ ” inequality then the absolute value of  $y_i$  will appear in the column associated with the corresponding surplus variable.

**Example, Reddy Mikks**

	basic	$z$	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	solution
Iteration 0:	$z$	1	-5	-4	0	0	0	0	0
	$x_3$	0	6	4	1	0	0	0	24
	$x_4$	0	1	2	0	1	0	0	6
	$x_5$	0	-1	1	0	0	1	0	1
	$x_6$	0	0	1	0	0	0	1	2
	Iteration 2:	basic	$z$	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$
$z$		1	0	0	$\frac{3}{4}$	$\frac{1}{2}$	0	0	21
$x_1$		0	1	0	$\frac{1}{4}$	$-\frac{1}{2}$	0	0	3
$x_2$		0	0	0	$-\frac{1}{8}$	$\frac{3}{4}$	0	0	$\frac{3}{2}$
$x_5$		0	0	0	$\frac{3}{8}$	$-\frac{5}{4}$	1	0	$\frac{5}{2}$
$x_6$		0	0	1	$\frac{1}{8}$	$-\frac{3}{4}$	0	1	$\frac{1}{2}$

$y^* = (\frac{3}{4}, \frac{1}{2}, 0, 0)$

## When Simplex terminates

Assume that problem is bounded and feasible. When simplex terminates we have

- solution value 
$$z^* = c_B A_B^{-1} b$$
- objective function 
$$z + (c_B A_B^{-1} A_N - c_N) x_N = (c_B A_B^{-1}) b$$
- reduced costs 
$$\bar{c} = c_B A_B^{-1} A_N - c_N \geq 0$$
- basis equations 
$$x_B + A_B^{-1} A_N x_N = A_B^{-1} b$$
- nonnegativity of all variables 
$$x \geq 0$$

## Finding the dual variables in general

$$\begin{array}{ll} (P) & (D) \\ \max & cx & \min & yb \\ \text{s.t.} & Ax = b & \text{s.t.} & yA \geq c \\ & x \geq 0 & & \end{array}$$

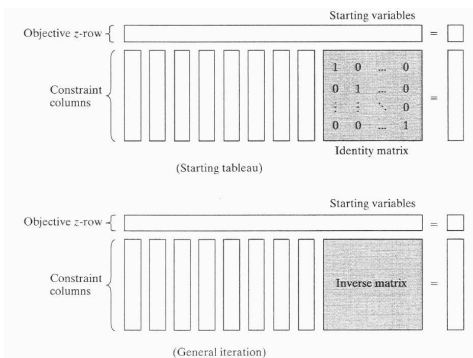
Assume that  $x$  is an optimal solution. Then dual variables can be found as  $y = c_B A_B^{-1}$

Let  $y = c_B A_B^{-1}$ , verify strong duality theorem:

$$yb = c_B A_B^{-1} b = z^* \quad \text{same solution}$$

$$\begin{aligned} yA &= c_B A_B^{-1} A && \text{dual feasible} \\ &= c_B A_B^{-1} (A_B \mid A_N) \\ &= (c_B \mid c_B A_B^{-1} A_N) \\ &\geq (c_B \mid c_N) = c \end{aligned}$$

## Understanding Simplex Tableau (Taha 4.2.2)



$$\begin{array}{ll} \max & cx + ds \\ \text{s.t.} & Ax + Is = b \\ & x, s \geq 0 \end{array}$$

basic solution in Simplex

$$A_B x_B + A_N x_N + Is = b \Leftrightarrow x_B + A_B^{-1} A_N x_N - A_B^{-1} s = A_B^{-1} b$$

objective function

$$z + (c_B A_B^{-1} A_N - c_N) x_N + (c_B A_B^{-1} I - d) s = (c_B A_B^{-1}) b$$

## Optimal Dual Solution (Taha 4.2.3)

### Method 1

$$\left( \begin{array}{l} \text{Optimal value} \\ \text{of dual variable} \\ y_i \end{array} \right) = \left( \begin{array}{l} \text{Optimal primal z-coefficient of} \\ \text{starting variable } x_i + \text{original} \\ \text{objective coefficient } x_i \end{array} \right)$$

$$z + (c_B A_B^{-1} A_N - c_N) x_N + (c_B A_B^{-1} I - d) s = (c_B A_B^{-1}) b$$

### Method 2

$$\left( \begin{array}{l} \text{Optimal values of dual} \\ \text{variables} \end{array} \right) = \left( \begin{array}{l} \text{Row vector of original} \\ \text{objective coefficients of} \\ \text{optimal primal basic} \\ \text{variables} \end{array} \right) \times \left( \begin{array}{l} \text{Optimal} \\ \text{primal} \\ \text{inverse} \end{array} \right)$$

$$y = c_B A_B^{-1}$$

### Example

Primal

$$\begin{aligned} \max z &= 5x_1 + 12x_2 + 4x_3 && -MR \\ \text{s.t.} & x_1 + 2x_2 + x_3 + x_4 && = 10 \\ & 2x_1 - x_2 + 3x_3 && + R = 8 \\ & x_1, x_2, x_3, x_4, R && \geq 0 \end{aligned}$$

Dual

$$\begin{aligned} \min w &= 10y_1 + 8y_2 \\ \text{s.t.} & y_1 + 2y_2 \geq 5 \\ & 2y_1 - y_2 \geq 12 \\ & y_1 + 3y_2 \geq 4 \\ & y_1 \geq 0 \\ & y_2 \geq -M \end{aligned}$$

Final Simplex table

basic	$x_1$	$x_2$	$x_3$	$x_4$	$R$	solution
$z$	0	0	$\frac{3}{5}$	$\frac{29}{5}$	$-\frac{2}{5} + M$	$54\frac{4}{5}$
$x_2$	0	1	$-\frac{1}{5}$	$\frac{2}{5}$	$-\frac{1}{5}$	$\frac{12}{5}$
$x_1$	1	0	$\frac{7}{5}$	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{26}{5}$

Method 1

$$y_1 = \frac{29}{5} + 0 = \frac{29}{5}, \quad y_2 = -\frac{2}{5} + M + (-M) = -\frac{2}{5}$$

Method 2

$$(y_1, y_2) = (12, 5) \begin{pmatrix} \frac{2}{5} & -\frac{1}{5} \\ \frac{7}{5} & \frac{3}{5} \end{pmatrix} = \left(\frac{29}{5}, -\frac{2}{5}\right)$$

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### Sensitivity Analysis (Taha 3.6)

- Sensitivity Analysis: find limits of parameters such that current solution remains optimal
- Post-optimal Analysis: determine new optimum resulting from targeted changes in input data

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### Changes in the right-hand side (Taha 3.6.2)

- Let  $D_i$  be change to constraint  $i$
- Add  $D_i$  to right-hand side
- Write up feasibility conditions for current solution to remain optimal
- Determine limits of  $D_i$  for which current solution remains optimal

TOYCO model

$$\begin{aligned} \max z &= 3x_1 + 2x_2 + 5x_3 \\ \text{s.t.} & x_1 + 2x_2 + x_3 \leq 430 \\ & 3x_1 + 2x_3 \leq 460 \\ & x_1 + 4x_2 \leq 420 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

Final Simplex table

basic	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	solution
$z$	4	0	0	1	2	0	1350
$x_2$	$-\frac{1}{4}$	1	0	$\frac{1}{2}$	$-\frac{1}{4}$	0	100
$x_3$	$\frac{3}{2}$	0	1	0	$\frac{1}{2}$	0	230
$x_6$	2	0	0	-2	1	1	20

Slack variables  $x_4, x_5, x_6$ , their coefficients are dual variables  $y_1 = 1, y_2 = 2, y_3 = 0$

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### Feasibility Range

$$\begin{aligned} \max z &= 3x_1 + 2x_2 + 5x_3 \\ \text{s.t.} & x_1 + 2x_2 + x_3 \leq 430 + D_1 \\ & 3x_1 + 2x_3 \leq 460 + D_2 \\ & x_1 + 4x_2 \leq 420 + D_3 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

in matrix form after adding slack variables

$$Ax = b + d$$

basic solution in Simplex

$$\begin{aligned} A_B x_B + A_N x_N &= b + d \quad \Leftrightarrow \\ x_B + A_B^{-1} A_N x_N &= A_B^{-1} (b + d) \end{aligned}$$

optimal table

$$\begin{aligned} x_2 &= 100 + \frac{1}{2}D_1 - \frac{1}{4}D_2 \\ x_3 &= 230 + \frac{1}{2}D_2 \\ x_6 &= 20 - 2D_1 + D_2 + D_3 \end{aligned}$$

Current solution remains feasible as long as  $x_2, x_3, x_6 \geq 0$

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## Feasibility Range

- Changing operation 1 time from 430 to  $430 + D_1$

$$\begin{aligned}x_2 &= 100 + \frac{1}{2}D_1 \geq 0 \\x_3 &= 230 \geq 0 \\x_6 &= 20 - 2D_1 \geq 0\end{aligned}$$

$$\text{Implying } -200 \leq D_1 \leq 10$$

- Changing operation 2 time from 460 to  $460 + D_2$   
Implying  $-20 \leq D_2 \leq 400$

- Changing operation 3 time from 420 to  $420 + D_3$   
Implying  $-20 \leq D_3 \leq \infty$

Checking feasibility of change

- Checking whether  $D_1 = -30, D_2 = -12, D_3 = 10$

$$\begin{aligned}x_2 &= 100 + \frac{1}{2}D_1 - \frac{1}{4}D_2 = 88 \geq 0 \\x_3 &= 230 + \frac{1}{2}D_2 = 224 \geq 0 \\x_6 &= 20 - 2D_1 + D_2 + D_3 = 78 \geq 0\end{aligned}$$

feasible

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## Changes in the objective function (Taha 3.6.3)

- Let  $d_i$  be change to coefficient  $c_i$
- Add  $d_i$  to respective coefficient
- Write up feasibility conditions for current solution to remain optimal
- Determine limits of  $d_i$  for which current solution remains optimal

### TOYCO model

$$\begin{aligned}\max z &= 3x_1 + 2x_2 + 5x_3 \\ \text{s.t.} \quad &x_1 + 2x_2 + x_3 \leq 430 \\ &3x_1 + x_3 \leq 460 \\ &x_1 + 4x_2 \leq 420 \\ &x_1, x_2, x_3 \geq 0\end{aligned}$$

Final Simplex table

basic	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	solution
$z$	4	0	0	1	2	0	1350
$x_2$	$-\frac{1}{4}$	1	0	$\frac{1}{2}$	$-\frac{1}{4}$	0	100
$x_3$	$\frac{3}{2}$	0	1	0	$\frac{1}{2}$	0	230
$x_6$	2	0	0	-2	1	1	20

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## Objective Function

$$\begin{aligned}\max z &= (3 + d_1)x_1 + (2 + d_2)x_2 + (5 + d_3)x_3 \\ \text{s.t.} \quad &x_1 + 2x_2 + x_3 \leq 430 \\ &3x_1 + x_3 \leq 460 \\ &x_1 + 4x_2 \leq 420 \\ &x_1, x_2, x_3 \geq 0\end{aligned}$$

Simplex algorithm, finds objective function

$$z + ((c_B + d_B)A_B^{-1}A_N - (c_N + d_N)x_N = ((c_B + d_B)A_B^{-1})b$$

Current solution remains optimal as long as reduced costs are nonnegative for all nonbasis variables

$$\bar{c}'_N = (c_B + d_B)A_B^{-1}A_N - (c_N + d_N) \geq 0$$

reduced costs can be rewritten

$$\begin{aligned}\bar{c}'_N &= (c_B + d_B)A_B^{-1}A_N - (c_N + d_N) \\ &= \bar{c}_N + d_B A_B^{-1}A_N - d_N \\ &= \bar{c}_N + d_B \bar{A}_N - d_N\end{aligned}$$

where  $\bar{A}_N$  is the contents of the final simplex table

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### TOYCO model

Final simplex table with  $d$ -values at appropriate places

		$d_1$	$d_2$	$d_3$	0	0	0	
	basic	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	solution
1	$z$	4	0	0	1	2	0	1350
$d_2$	$x_2$	$-\frac{1}{4}$	1	0	$\frac{1}{2}$	$-\frac{1}{4}$	0	100
$d_3$	$x_3$	$\frac{3}{2}$	0	1	0	$\frac{1}{2}$	0	230
0	$x_6$	2	0	0	-2	1	1	20

current solution remains optimal as long as reduced costs

$$\bar{c}_1 = 4 - \frac{1}{4}d_2 + \frac{3}{2}d_3 - d_1 \geq 0$$

$$\bar{c}_4 = 1 + \frac{1}{2}d_2 \geq 0$$

$$\bar{c}_5 = 2 - \frac{1}{4}d_2 + \frac{1}{2}d_3 \geq 0$$

- maximize  $z = (3 + d_1)x_1 + 2x_2 + 5x_3$   
condition

$$4 - d_1 \geq 0$$

$$1 \geq 0$$

$$2 \geq 0$$

giving  $-\infty < d_1 \leq 4$

- maximize  $z = 3x_1 + (2 + d_2)x_2 + 5x_3$   
giving  $-2 < d_2 \leq 8$

- maximize  $z = 3x_1 + 2x_2 + (5 + d_3)x_3$   
giving  $-\frac{8}{3} < d_3 \leq \infty$

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## Post-optimality analysis (Taha 4.5)

Periodic recalculation of the optimum solution. Post-optimality analysis determines the new solution in an efficient way.

Condition after parameters change	Recommended action
Current solution remains optimal and feasible	No further action necessary
Current solution becomes infeasible	Use dual simplex to recover feasibility
Current solution becomes nonoptimal	Use primal simplex to recover feasibility
Current solution becomes nonoptimal and infeasible	Use generalized simplex to obtain new solution

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## Dual Simplex

Dual simplex starts from (better than) optimal infeasible basis solution. Searches for feasible solution.

- leaving variable:  $x_r$  is the basic variable having most negative value.
- optimality criteria: if all basic variables are nonnegative
- entering variable: nonbasic variable with  $\bar{a}_{rj} < 0$

$$\min \left\{ \left| \frac{\bar{c}}{\bar{a}_{rj}} \right|, \bar{a}_{rj} < 0 \right\}$$

## Example

$$\begin{aligned} \min z = & 3x_1 + 2x_2 + x_3 \\ \text{s.t.} & -3x_1 - x_2 - x_3 \leq -3 \\ & 3x_1 - 3x_2 - x_3 \leq -6 \\ & x_1 + x_2 + x_3 \leq 3 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

Simplex table

basic	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	solution
$z$	-3	-2	-1	0	0	0	0
$x_4$	-3	-1	-1	1	0	0	-3
$x_5$	3	-3	-1	0	1	0	-6
$x_6$	1	1	1	0	0	1	3

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## Post-optimality analysis

TOYCO Example 4.5-3

$$\begin{aligned} \max z = & 3x_1 + 2x_2 + 5x_3 \\ \text{s.t.} & x_1 + 2x_2 + x_3 \leq 430 \\ & 3x_1 + 2x_3 \leq 460 \\ & x_1 + 4x_2 \leq 420 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

Solution  $x_1 = 0, x_2 = 100, x_3 = 230$ .

- Add constraint  $3x_1 + x_2 + x_3 \leq 500$   
Redundant
- Add constraint  $3x_1 + 3x_2 + x_3 \leq 500$   
Run dual simplex

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All reduced costs  $\bar{c}_j \leq 0$ .

Infeasible

Leaving variable  $x_5$  as  $-6$  smallest.

$$\min \left\{ \left| \frac{\bar{c}}{\bar{a}_{rj}} \right|, \bar{a}_{rj} < 0 \right\} = \min \left\{ -\frac{2}{3}, 1 \right\} = \frac{2}{3}$$

Entering variable  $x_2$

Iteration 1:

basic	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	solution
$z$	-5	0	$-\frac{1}{3}$	0	$-\frac{2}{3}$	0	4
$x_4$	-4	0	$-\frac{2}{3}$	1	$-\frac{1}{3}$	0	-1
$x_2$	-1	1	$\frac{1}{3}$	0	$-\frac{1}{3}$	0	2
$x_6$	2	0	$\frac{2}{3}$	0	$\frac{1}{3}$	1	1

Leaving variable  $x_4$

Entering variable

$$\min \left\{ \frac{5}{4}, \frac{1}{2}, 2 \right\} = \frac{1}{2}$$

Iteration 2:

basic	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	solution
$z$	-3	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{9}{2}$
$x_3$	1	0	1	$-\frac{3}{2}$	$\frac{1}{2}$	0	$\frac{3}{2}$
$x_2$	0	0	0	$\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{3}{2}$
$x_6$	0	1	0	1	0	1	0

Optimal and feasible

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## Sensitivity with CPLEX

### TOYCO example

```
maximize
  3 x1 + 2 x2 + 5 x3
subject to
  x1 + 2 x2 + x3 <= 430
  3 x1 + 2 x3 <= 460
  x1 + 4 x2 <= 420
bounds
  x1 >= 0
  x2 >= 0
  x3 >= 0
end
```

### Solution

Dual simplex - Optimal: Objective = 1.3500000000e+03  
Solution time = 0.00 sec. Iterations = 3 (2)

```
CPLEX> dis sol var -
Variable Name      Solution Value
x2                  100.000000
x3                  230.000000
All other variables in the range 1-3 are zero.
```

```
CPLEX> dis sol dua -
Constraint Name      Dual Price
c1                   1.000000
c2                   2.000000
All other dual prices in the range 1-3 are zero.
```

## Sensitivity with CPLEX

```
CPLEX> dis sen obj -
```

Variable Name	Reduced Cost	OBJ Sensitivity Ranges		
		Down	Current	Up
x1	-4.0000	-infinity	3.0000	7.0000
x2	zero	zero	2.0000	10.0000
x3	zero	2.3333	5.0000	+infinity

```
CPLEX> dis sen rhs -
```

Constraint Name	Dual Price	RHS Sensitivity Ranges		
		Down	Current	Up
c1	1.0000	230.0000	430.0000	440.0000
c2	2.0000	440.0000	460.0000	860.0000
c3	zero	400.0000	420.0000	+infinity