Introduction to Optimization, DIKU 2007-08
Wednesday 14 November
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Lecture 2

Revised simplex algorithm, bounded variables

Taha sections

- 7.1, 7.2, 7.3
- all examples can be read briefly

Terminology

	Taha	INTOPT
j'th column in A	P_j	A_j
basis	B	A_B
reduced cost	$z_j - c_j$	$\bar{c}_j$

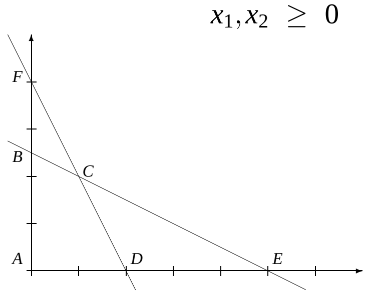
Revised Simplex

Motivation:

- minimal needed information to implement simplex algorithm
- use matrix notation
- simplex iterations are expressed in terms of *original* variables
- better control of machine roundoff error
- revised simplex can be used in column generation

Linear Programming (Taha example 3.2.1)

$$\begin{array}{llcll} \text{maximize} & 2x_1 & + & 3x_2 & \\ \text{subject to} & 2x_1 & + & x_2 & \le 4 \\ & x_1 & + & 2x_2 & \le 5 \\ & & & x_1, x_2 & \ge 0 \end{array}$$



Add slack variables

$$\begin{array}{lllll} \text{maximize} & 2x_1 + 3x_2 & & & \\ \text{subject to} & 2x_1 + x_2 + x_3 & & = & 4 \\ & x_1 + 2x_2 & + x_4 & = & 5 \\ & & x_1, x_2, x_3, x_4 & \ge & 0 \end{array}$$

The set of constraints form a polyhedral.

Non-basic	Basic	Basic Solution	Corner Point	Feasible	Objective
(x_1, x_2)	(x_3, x_4)	$(4, 5)$	A	yes	0
(x_1, x_3)	(x_2, x_4)	$(4, -3)$	F	no	–
(x_1, x_4)	(x_2, x_3)	$(2.5, 1.5)$	B	yes	7.5
(x_2, x_3)	(x_1, x_4)	$(2, 3)$	D	yes	4
(x_2, x_4)	(x_1, x_3)	$(5, -6)$	E	no	–
(x_3, x_4)	(x_1, x_2)	$(1, 2)$	C	yes	8

Basis, basis feasible solution

Since we have added slack variables, the number of variables n is larger than the number of constraints m.

$$\begin{array}{ll} \text{maximize} & cx \\ \text{subject to} & Ax = b \\ & x \ge 0 \end{array}$$

Choose m linearly independent columns from A. The corresponding set $B = \{i_1, i_2, \ldots, i_m\}$ is called a *basis*.

Reformulation

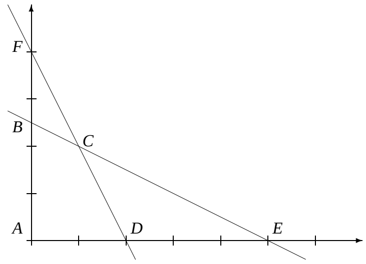
$$\begin{array}{ll} \text{maximize} & c_B x_B + c_N x_N \\ \text{subject to} & A_B x_B + A_N x_N = b \\ & x \ge 0 \end{array}$$

A *Basis feasible solution* is obtained by setting $x_N = 0$.

$$\begin{array}{rcl} A_B x_B + A_N 0 & = & b \\ x_B & = & A_B^{-1} b \end{array}$$

x_B is well defined since A_B is an $m \times m$ matrix and columns are linearly independent.

Corner points and basis feasible solutions



Algorithm Search through all corner points
Basis can be chosen in $C_m^n = \frac{n!}{m!(n-m)!}$ ways

Adjacent basis feasible solutions Two basis feasible solutions x^1 and x^2 are adjacent if B^1 and B^2 have $m-1$ common elements.
(one *entering*, one *leaving* variable)

Simplex algorithm is a greedy algorithm which works as follows: Move from basis feasible solution to adjacent basis feasible solution such that objective function is ”increased most possible” in each step.

Simplex, example 3.2.1

Iteration 0:

basic	z	x_1	x_2	x_3	x_4	solution
z	1	-2	-3	0	0	0
x_3	0	2	1	1	0	4
x_4	0	1	2	0	1	5

Entering variable x_2
maximum value of entering variable $\min\{\frac{4}{1}, \frac{5}{2}\} = \frac{5}{2}$
leaving variable is x_4
Pivot row 3 with $(\frac{3}{2}, -\frac{1}{2}, \frac{1}{2})$

Iteration 1:

basic	z	x_1	x_2	x_3	x_4	solution
z	1	$-\frac{1}{2}$	0	0	$\frac{3}{2}$	$\frac{15}{2}$
x_3	0	$\frac{3}{2}$	0	1	$-\frac{1}{2}$	$\frac{3}{2}$
x_2	0	$\frac{1}{2}$	1	0	$\frac{1}{2}$	$\frac{5}{2}$

Entering variable x_1
maximum value of entering variable $\min\{\frac{\frac{3}{2}}{\frac{3}{2}}, \frac{\frac{5}{2}}{\frac{1}{2}}\} = 1$
leaving variable is x_3
Pivot row 2 with $(\frac{1}{3}, \frac{2}{3}, -\frac{1}{3})$

Iteration 2:

basic	z	x_1	x_2	x_3	x_4	solution
z	1	0	0	$\frac{1}{3}$	$\frac{4}{3}$	8
x_1	0	1	0	$\frac{2}{3}$	$-\frac{1}{3}$	1
x_2	0	0	1	$-\frac{1}{3}$	$\frac{2}{3}$	2

All reduced costs in objective are positive, hence stop

Canonical form (Taha notation)

Objective function $\bar{c}x$ expressed in nonbasis variables only (*canonical form*)

$$\bar{c}_B = 0$$

This can be obtained by considering the objective function as an ordinary constraint

$$\begin{aligned} & z - cx = 0 \\ & Ax = b \\ & x \geq 0 \end{aligned}$$

Multiplying any constraint i by a real number π_i and adding it to some other constraint j does not change the problem. In particular, we can add any constraint to the objective function.

$$\begin{aligned} & z - cx + \pi(Ax - b) = 0 \\ & z + (\pi A - c)x = \pi b \\ & z + (\pi A_B - c_B)x_B + (\pi A_N - c_N)x_N = \pi b \end{aligned}$$

To have a canonical form $\bar{c}_B = 0$ so we must have $\pi A_B - c_B = 0$. From this we can determine π as

$$\pi = c_B A_B^{-1}$$

Thus the objective function becomes

$$\begin{aligned} & z - (c_B A_B^{-1} A_N - c_N)x_N = (c_B A_B^{-1})b \\ & z + \bar{c}_N x_N = z_0 \end{aligned}$$

where $\bar{c}_N = c_B A_B^{-1} A_N - c_N$ are the *reduced costs*.

Canonical form (Cormen notation)

The objective function $\tilde{c}x$ is in *canonical form* if

$$\tilde{c}_B = 0$$

This can be obtained by considering the objective function as an ordinary constraint

$$\begin{aligned} & z = cx \\ & Ax = b \\ & x \geq 0 \end{aligned}$$

Multiplying any constraint i by a real number π_i and adding it to some other constraint j does not change the problem. In particular, we can add any constraint to the objective function.

$$\begin{aligned} z &= cx - \pi(Ax - b) \\ &= (c - \pi A)x + \pi b \\ z &= (c_B - \pi A_B)x_B + (c_N - \pi A_N)x_N + \pi b \end{aligned}$$

To have a canonical form $\tilde{c}_B = 0$ so we must have $(c_B - \pi A_B) = 0$. From this we can determine π as

$$\pi = c_B A_B^{-1}$$

Thus the objective function becomes

$$\begin{aligned} z &= (c_N - c_B A_B^{-1} A_N)x_N + (c_B A_B^{-1})b \\ &= c^{\pi} x_N + z_0 \end{aligned}$$

where $c^{\pi} = c_N - c_B A_B^{-1} A_N$ are the *reduced costs*.

Current solution value

The basis feasible solution is found by setting $x_N = 0$. In this case the objective value becomes:

$$z + \bar{c}_N x_N = z_0$$

Reduced costs

The reduced costs $\bar{c}_j$ represent the gain by increasing the value of a non-basis variable.

Iterative step

Choose the variable $s \in N$ with largest negative value of $\bar{c}_s < 0$ to enter basis.

Optimality criteria

If all the reduced costs $\bar{c}_j \geq 0$ for a given basis feasible solution x, then x is an optimal solution.

Iteration 1:

basic	z	x_1	x_2	x_3	x_4	solution
z	1	$-\frac{1}{2}$	0	0	$\frac{3}{2}$	$\frac{15}{2}$
x_3	0	$\frac{3}{2}$	0	1	$-\frac{1}{2}$	$\frac{3}{2}$
x_2	0	$\frac{1}{2}$	1	0	$\frac{1}{2}$	$\frac{5}{2}$

Leaving variable

Iteration 1:

basic	z	x_1	x_2	x_3	x_4	solution
z	1	$-\frac{1}{2}$	0	0	$\frac{3}{2}$	$\frac{15}{2}$
x_3	0	$\frac{3}{2}$	0	1	$-\frac{1}{2}$	$\frac{3}{2}$
x_2	0	$\frac{1}{2}$	1	0	$\frac{1}{2}$	$\frac{5}{2}$

Most promising variable x_1

Keeping all other nonbasic variables at 0, constraints

$$\begin{array}{lll} x_3 \geq 0, & x_3 = \frac{3}{2} - \frac{3}{2}x_1 & \Rightarrow \quad \frac{3}{2} - \frac{3}{2}x_1 \geq 0 \\ x_2 \geq 0, & x_2 = \frac{5}{2} - \frac{1}{2}x_1 & \Rightarrow \quad \frac{5}{2} - \frac{1}{2}x_1 \geq 0 \end{array}$$

implying $x_1 \leq 1$.

When $x_1 = 1$ we have $x_3 = 0$ (x_3 leaves basis)

In general Assume $s \in N$ is entering variable, calculate

$$\bar{A}_s = A_B^{-1} A_s, \qquad \bar{b} = A_B^{-1} b$$

Keeping $x_j = 0$ for $j \in N \setminus \{s\}$ the current equations say

$$x_B + x_s \bar{A}_s = \bar{b}$$

To maintain $x_B \geq 0$ we must have

$$\bar{b} - x_s \bar{A}_s \geq 0$$

hence $\bar{b}_i - x_s \bar{a}_{is} \geq 0$ for all i. Most binding constraint

$$k = \arg\min_{i=1,\ldots,m} \left\{ \frac{\bar{b}_i}{\bar{a}_{is}} \middle| \bar{a}_{is} > 0 \right\}$$

leaving variable r is k'th variable in basis B.

Revised simplex algorithm, maximization (Taha 7.2.2)

0) Construct a starting basis feasible solution and let A_B and c_B be its associated basis and objective.

1) Compute the inverse A_B^{-1}.

2) For each nonbasis variable $j \in N$ compute

$$\bar{c}_j = z_j - c_j = c_B A_B^{-1} A_j - c_j$$

if $\bar{c}_j \geq 0$ for all nonbasis $j \in N$ stop; optimal solution

$$x_B = A_B^{-1} b, \qquad z = c_B x_B$$

Else, apply optimality condition to find entering variable x_s

$$s = \arg\min_{j \in N} \{\bar{c}_j\}$$

3) Compute $\bar{A}_s = A_B^{-1} A_s$.
If $\bar{A}_s \leq 0$ the problem is unbounded, stop.
Else, compute $\bar{b} = A_B^{-1} b$.
Feasibility check

$$k = \arg\min_{i=1,\ldots,m} \left\{ \frac{\bar{b}_i}{\bar{a}_{is}} \middle| \bar{a}_{is} > 0 \right\}$$

Leaving variable: basis variable corresponding to row k, $r = B_k$.

4) New basis is $B := B \cup \{s\} \setminus \{r\}$. Go to step 1.

When Simplex terminates

Assume that problem is bounded and feasible. When simplex terminates we have

- solution value
$$z^* = c_B A_B^{-1} b$$
- objective function
$$z - (c_B A_B^{-1} A_N - c_N) x_N = (c_B A_B^{-1}) b$$
- reduced costs
$$\bar{c} = c_B A_B^{-1} A_N - c_N \geq 0$$
- basis equations
$$x_B + A_B^{-1} A_N x_N = A_B^{-1} b$$
- nonnegativity of all variables
$$x \geq 0$$

Advantages of revised simplex

- Simplex table expressed in original variables
- Valuable interpretation of all terms in simplex table
- Can avoid to write A explicitly (e.g. delayed column generation)

All commercial simplex algorithms use revised simplex

- Fewer calculations needed, since only maintain A_B^{-1} and right-hand side
- Less storage, due to same arguments
- Rest of A can be stored in compact form (only storing non-zero elements, low precision)
- calculations made on A_B^{-1} with high precision

Bigger example, The Reddy Mikks Company

Problem formulation in standard form

$$\begin{array}{llcrcl} \text{maximize} & 5x_1 & + & 4x_2 & & \\ \text{subject to} & 6x_1 & + & 4x_2 & \leq & 24 \\ & x_1 & + & 2x_2 & \leq & 6 \\ & -x_1 & + & x_2 & \leq & 1 \\ & & & x_2 & \leq & 2 \\ & x_1, x_2 \geq 0 & & & & \end{array}$$

Add slack variables x_3, x_4, x_5, x_6

$$\begin{array}{llcrcrcl} \text{maximize} & 5x_1 & + & 4x_2 & & & & \\ \text{subject to} & 6x_1 & + & 4x_2 & + & x_3 & = & 24 \\ & x_1 & + & 2x_2 & + & x_4 & = & 6 \\ & -x_1 & + & x_2 & + & x_5 & = & 1 \\ & & & x_2 & + & x_6 & = & 2 \\ & x_1, x_2, x_3, x_4, x_5, x_6 \geq 0 & & & & & & \end{array}$$

Bigger example, The Reddy Mikks Company

Iteration 0:

			s						
	basic	z	x_1	x_2	x_3	x_4	x_5	x_6	solution
	z	1	-5	-4	0	0	0	0	0
r	x_3	0	6	4	1	0	0	0	24
	x_4	0	1	2	0	1	0	0	6
	x_5	0	-1	1	0	0	1	0	1
	x_6	0	0	1	0	0	0	1	2

Iteration 1:

				s					
	basic	z	x_1	x_2	x_3	x_4	x_5	x_6	solution
	z	1	0	$-\frac{2}{3}$	$\frac{5}{6}$	0	0	0	20
	x_1	0	1	$\frac{2}{3}$	$\frac{1}{6}$	0	0	0	4
r	x_4	0	0	$\frac{4}{3}$	$-\frac{1}{6}$	1	0	0	2
	x_5	0	0	$\frac{5}{3}$	$\frac{1}{6}$	0	1	0	5
	x_6	0	0	1	0	0	0	1	2

Iteration 2:

basic	z	x_1	x_2	x_3	x_4	x_5	x_6	solution
z	1	0	0	$\frac{3}{4}$	$\frac{1}{2}$	0	0	21
x_1	0	1	0	$\frac{1}{4}$	$-\frac{1}{2}$	0	0	3
x_2	0	0	0	$-\frac{1}{8}$	$\frac{3}{4}$	0	0	$\frac{3}{2}$
x_5	0	0	0	$\frac{3}{8}$	$-\frac{5}{4}$	1	0	$\frac{5}{2}$
x_6	0	0	1	$\frac{1}{8}$	$-\frac{3}{4}$	0	1	$\frac{1}{2}$

Revised simplex algorithm

Example, Reddy Mikks.

$$\max z = (5, 4, 0, 0, 0, 0)(x_1, x_2, x_3, x_4, x_5, x_6)^T$$

subject to

$$\begin{pmatrix} 6 & 4 & 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} 24 \\ 6 \\ 1 \\ 2 \end{pmatrix}$$

Iteration 0

$$
\begin{aligned}
&B = \{3,4,5,6\}\\
&N = \{1,2\}\\
&c_B = (0,0,0,0)\\
&A_B = (A_3, A_4, A_5, A_6) = I\\
&A_B^{-1} = I
\end{aligned}
$$

Thus

$$
\begin{aligned}
&x_B = A_B^{-1} b = (24,6,1,2)^T\\
&z = c_B x_B = 0
\end{aligned}
$$

Optimality computation

$$
\begin{aligned}
&c_B A_B^{-1} = (0,0,0,0)\\
&\{\bar{c}_j\}_{j \in N} = c_B A_B^{-1}(A_1, A_2) - (c_1, c_2) = (-5,-4)
\end{aligned}
$$

entering variable

$$s = \arg\min_{j \in N}\{\bar{c}_j\} = \arg\min_{j=1,2}\{-5,-4\} = 1$$

Feasibility computation

$$
\begin{aligned}
&\bar{b} = A_B^{-1} b = (24,6,1,2)^T\\
&\bar{A}_s = A_B^{-1} A_s = (6,1,-1,0)^T
\end{aligned}
$$

leaving variable

$$
\begin{aligned}
k &= \arg\min_{i=1,\dots,m}\left\{\frac{\bar{b}_i}{\bar{a}_{is}} \,\middle|\, \bar{a}_{is} > 0\right\}\\
&= \arg\min\left\{\tfrac{24}{6}, \tfrac{6}{1}, -, -\right\}\\
&= \arg\min\{4,6,-,-\} = 1
\end{aligned}
$$

hence, leaving variable is first element in $B = \{3,4,5,6\}$ so leaving variable is $r = 3$.

basic	x_1	x_2	x_3	x_4	x_5	x_6	solution
z	-5	-4	0	0	0	0	0
x_3	6						24
x_4	1						6
x_5	-1						1
x_6	0						2

Iteration 1

$$
\begin{aligned}
&B = \{1,4,5,6\}\\
&N = \{2,3\}\\
&c_B = (5,0,0,0)\\
&A_B = (A_1, A_4, A_5, A_6) = \begin{pmatrix} 6 & 0 & 0 & 0\\ 1 & 1 & 0 & 0\\ -1 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 \end{pmatrix}
\end{aligned}
$$

we find

$$A_B^{-1} = \frac{1}{6}\begin{pmatrix} 1 & 0 & 0 & 0\\ -1 & 6 & 0 & 0\\ 1 & 0 & 6 & 0\\ 0 & 0 & 0 & 6 \end{pmatrix}$$

thus

$$
\begin{aligned}
&x_B = A_B^{-1} b = (4,2,5,2)^T\\
&z = c_B x_B = 20
\end{aligned}
$$

Optimality computation

$$
\begin{aligned}
&c_B A_B^{-1} = (\tfrac{5}{6}, 0, 0, 0)\\
&\{\bar{c}_j\}_{j \in N} = c_B A_B^{-1}(A_2, A_3) - (c_2, c_3) = (-\tfrac{2}{3}, \tfrac{5}{6})
\end{aligned}
$$

entering variable

$$s = \arg\min_{j \in N}\{\bar{c}_j\} = \arg\min_{j=2,3}\{-\frac{2}{3}, \frac{5}{6}\} = 2$$

Feasibility computation

$$
\begin{aligned}
&\bar{b} = (x_1, x_4, x_5, x_6)^T = (4,2,5,2)^T\\
&\bar{A}_s = A_B^{-1} A_s = (\tfrac{2}{3}, \tfrac{4}{3}, \tfrac{5}{3}, 1)^T
\end{aligned}
$$

leaving variable

$$
\begin{aligned}
k &= \arg\min_{i=1,\dots,m}\left\{\frac{\bar{b}_i}{\bar{a}_{is}} \,\middle|\, \bar{a}_{is} > 0\right\}\\
&= \arg\min\left\{\tfrac{4}{\frac{2}{3}}, \tfrac{2}{\frac{4}{3}}, \tfrac{5}{\frac{5}{3}}, \tfrac{2}{1}\right\}\\
&= \arg\min\{6, \tfrac{3}{2}, 3, 2\} = 2
\end{aligned}
$$

hence, leaving variable is second element in $B = \{1,4,5,6\}$, so leaving variable is $r = 4$.

basic	x_1	x_2	x_3	x_4	x_5	x_6	solution
z	0	$-\frac{2}{3}$	$\frac{5}{6}$	0	0	0	20
x_1		$\frac{2}{3}$					4
x_4		$\frac{4}{3}$					2
x_5		$\frac{5}{3}$					5
x_6		1					2

Iteration 2

$$B = \{1,2,5,6\}$$
$$N = \{3,4\}$$
$$c_B = (5,4,0,0)$$
$$A_B = (A_1, A_2, A_5, A_6) = \begin{pmatrix} 6 & 4 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

we find

$$A_B^{-1} = \frac{1}{8}\begin{pmatrix} 2 & -4 & 0 & 0 \\ -1 & 6 & 0 & 0 \\ 3 & -10 & 8 & 0 \\ 1 & -6 & 0 & 8 \end{pmatrix}$$

thus

$$x_B = A_B^{-1} b = (3, \tfrac{3}{2}, \tfrac{5}{2}, \tfrac{1}{2})^T$$
$$z = c_B x_B = 21$$

Optimality computation

$$c_B A_B^{-1} = (\tfrac{3}{4}, \tfrac{1}{2}, 0, 0)$$
$$\{\bar{c}_j\}_{j \in N} = c_B A_B^{-1}(A_3, A_4) - (c_3, c_4) = (\tfrac{3}{4}, \tfrac{1}{2})$$

Thus B is optimal, stop.
Optimal solution

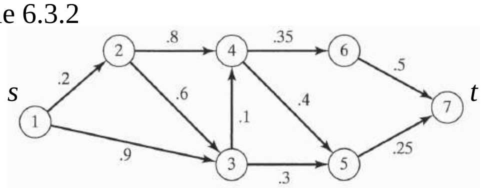
$$x_1 = 3$$
$$x_2 = \tfrac{3}{2}$$
$$z = 21$$

simplex tableau

basic	x_1	x_2	x_3	x_4	x_5	x_6	solution
z	$\frac{3}{4}$	$-\frac{1}{2}$	0	0	0	0	21
x_1							3
x_2							$\frac{3}{2}$
x_5							$\frac{5}{2}$
x_6							$\frac{1}{2}$

Applications of OR

Example 6.3.2



- weighted graph $G = (V, E)$
- each edge (i, j) has probability of success p_{ij}
- find most reliable route $s \to t$

Introduce $x_{ij} = 1$ iff edge (i, j) is used

- flow conservation: $\sum_{j \in V} x_{ij} - \sum_{j \in V} x_{ji} = 0$
- one edge leaving s, one edge entering t
- objective

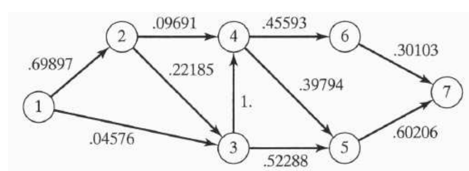
$$\max \prod_{x_{ij}=1} p_{ij}$$

	x_{12}	x_{13}	x_{23}	x_{24}	x_{34}	x_{35}	x_{45}	x_{46}	x_{57}	x_{67}	
s	1	1									= 1
t									-1	-1	= -1
2	-1		1	1							= 0
3		-1	-1		1	1					= 0
4				-1	-1		1	1			= 0
5						-1	-1		1		= 0
6								-1		1	= 0

Applications of OR

Objective

$$\max \log \prod_{x_{ij}=1} p_{ij} = \max \sum_{(i,j)\in E} \log p_{ij} x_{ij} = \min \sum_{(i,j)\in E} -\log p_{ij} x_{ij}$$



Size of input

- V nodes, E edges with cost

Size of model

- $E = O(V^2)$ variables
- $O(V)$ constraints
- Size of A-matrix $O(V^3)$

Revised simplex

- Only store A_B of size $O(V^2)$
- Columns in A can be generated “on the fly” from graph

Note

Cormen is using a better LP-formulation of shortest-path

Advanced comments

Pivot operation corresponds to multiplying current A_B^{-1} with

$$E = \begin{pmatrix} 1 & & v_1 & & \\ & \ddots & \vdots & & \\ & & 1 & & \\ & & v_r & & \\ & & & 1 & \\ & & \vdots & \ddots & \\ & & v_m & & 1 \end{pmatrix}$$

If current basis inverse is A_B^{-1} and right-hand side is $\overline{b}$ then new basis and right-hand side is

$$EA_B^{-1} \qquad E\overline{b}$$

If initial basis is I and operations $E_1, E_2, \dots, E_k$ then

$$A_B^{-1} = E_k \dots, E_2 E_1 I \qquad \overline{b} = E_k \dots E_2 E_1 b$$

Advanced algorithms

- Maintain A_B^{-1} in product form
- Only column $v_1, \dots, v_m$ is stored from E
- When $A_B^{-1} = E_k \dots, E_2 E_1$ becomes too complex to calculate, store result, start over again.

Simplex, example 3.2.1

Iteration 0:

basic	z	x_1	x_2	x_3	x_4	solution
z	1	-2	-3	0	0	0
x_3	0	2	1	1	0	4
x_4	0	1	2	0	1	5

Entering variable x_2
maximum value of entering variable $\min\{\frac{4}{1}, \frac{5}{2}\} = \frac{5}{2}$
leaving variable is x_4
Pivot row 3 with $(\frac{3}{2}, -\frac{1}{2}, \frac{1}{2})$

Iteration 1:

basic	z	x_1	x_2	x_3	x_4	solution
z	1	$-\frac{1}{2}$	0	0	$\frac{3}{2}$	$\frac{15}{2}$
x_3	0	$\frac{3}{2}$	0	1	$-\frac{1}{2}$	$\frac{3}{2}$
x_2	0	$\frac{1}{2}$	1	0	$\frac{1}{2}$	$\frac{5}{2}$

Entering variable x_1
maximum value of entering variable $\min\{\frac{\frac{3}{2}}{\frac{3}{2}}, \frac{\frac{5}{2}}{\frac{1}{2}}\} = 1$
leaving variable is x_3
Pivot row 2 with $(\frac{1}{3}, \frac{2}{3}, -\frac{1}{3})$

Iteration 2:

basic	z	x_1	x_2	x_3	x_4	solution
z	1	0	0	$\frac{1}{3}$	$\frac{4}{3}$	8
x_1	0	1	0	$\frac{2}{3}$	$-\frac{1}{3}$	1
x_2	0	0	1	$-\frac{1}{3}$	$\frac{2}{3}$	2

All reduced costs in objective are positive, hence stop

Example

iteration 1

$$E_1 = \begin{pmatrix} 1 & -\frac{1}{2} \\ 0 & \frac{1}{2} \end{pmatrix}$$

iteration 2

$$E_2 = \begin{pmatrix} \frac{2}{3} & 0 \\ -\frac{1}{3} & 1 \end{pmatrix}$$

we have

$$A_B^{-1} = E_2 E_1 = \begin{pmatrix} \frac{2}{3} & 0 \\ -\frac{1}{3} & 1 \end{pmatrix} \begin{pmatrix} 1 & -\frac{1}{2} \\ 0 & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{pmatrix}$$

Bounded variables

In production planning, branch-and-bound we frequently have

$$\ell \le x \le u$$

Lower bound

$$x \ge \ell$$

substitute

$$x = \ell + x'$$
$$x' \ge 0$$

solve problem in terms of x'. Back-substitute original variables $x = x' + \ell$

Bounded variables

Upper bound

Handle when computing max value of entering variable.

$$x_j \le \arg\min_{i=1,\dots,m} \left\{ \frac{\overline{b}_i}{\overline{a}_{is}} \middle| \overline{a}_{is} > 0 \right\}$$

New constraints
We need to ensure $x_i \ge 0$ for $i \in B$

$$x_j \le \theta_j^1 = \min_{i=1,\dots,m} \left\{ \frac{\overline{b}_i}{\overline{a}_{is}} \middle| \overline{a}_{is} > 0 \right\}$$

and to ensure that $x_i \le u_i$ for $i \in B$

$$x_j \le \theta_j^2 = \min_{i=1,\dots,m} \left\{ \frac{\overline{b}_i - u_i}{\overline{a}_{is}} \middle| \overline{a}_{is} > 0 \right\}$$

combining the three restrictions

$$x_j = \min(\theta_j^1, \theta_j^2, u_j)$$

Bounded variables

Pitfalls:

- Since upper bounds $x_j \le u_j$ are handled implicit, no dual variable are calculated corresponding to the constraint
- Duality theorem, complementary slackness seem to not work

Only use bounds if you know what you are doing, otherwise use explicit constraint

Complexity of Simplex

Klee and Minty (1975) proved that the Simplex algorithm may use exponential time

maximize
$$2^{n-1}x_1 + 2^{n-2}x_2 + \dots + 2x_{n-1} + 1x_n$$
subject to
$$\begin{array}{rcrcrcrcrcl}
1x_1 & + & & + & & + & & + & & \le & 5 \\
4x_1 & + & 1x_2 & + & & + & & + & & \le & 5^2 \\
8x_1 & + & 4x_2 & + & 1x_3 & + & & + & & \le & 5^3 \\
\vdots & + & & + & & + & & + & & \le & \vdots \\
2^n x_1 & + & 2^{n-1}x_2 & + & \dots & + & 4x_{n-1} & + & 1x_n & \le & 5^n
\end{array}$$
$x_i \ge 0, i = 1, \dots, n$

The problem has

- n variables
- n constraints
- 2^n extreme points
- Simplex, starting at $x = (0, \dots, 0)$, visits all extreme points
- optimal solution $(0, 0, \dots, 0, 5^n)$

Complexity of Simplex

For $n = 3$ simplex visits $2^3 = 8$ extreme points
Assume (s_1, s_2, s_3) slack variables:

basis	nonbasis x_1	x_2	x_3	RHS
z	-4	-2	-1	0
s_1	1^*			5
s_2	4	1		25
s_3	8	4	1	125

basis	nonbasis s_1	x_2	x_3	RHS
z	4	-2	-1	20
x_1	1			5
s_2	-4	1^*		5
s_3	-8	4	1	85

basis	nonbasis s_1	s_2	x_3	RHS
z	-4	2	-1	30
x_1	1^*			5
x_2	-4	1		5
s_3	8	-4	1	65

basis	nonbasis x_1	s_2	x_3	RHS
z	4	2	-1	50
s_1	1			5
x_2	4	1		25
s_3	-8	-4	1^*	25

basis	nonbasis x_1	s_2	s_3	RHS
z	-4	-2	1	75
s_1	1^*			5
x_2	4	1		25
x_3	-8	-4	1	25

basis	nonbasis s_1	s_2	s_3	RHS
z	4	-2	1	95
x_1	1			5
x_2	-4	1^*		5
x_3	8	-4	1	65

basis	nonbasis s_1	x_2	s_3	RHS
z	-4	2	1	105
x_1	1^*			5
s_2	-4	1		5
x_3	-8	4	1	85

basis	nonbasis x_1	x_2	s_3	RHS
z	4	2	1	125
s_1	1^*			5
s_2	4	1		25
x_3	8	4	1	125

Complexity of Simplex

- Worst-case complexity is exponential (instances have been constructed which ”fools” the greedy strategy to visit nearly all corner points).
- Several heuristics are used in commercial simplex implementations

Avg. number of iterations, using largest-coefficient rule

$m \backslash n$	10	20	30	40	50
10	9.4	14.2	17.4	19.4	20.2
20		25.2	30.7	38.0	41.5
30			44.4	52.7	62.9
40				67.6	78.7
50					95.2

Source: Avis and Chvatal (1978).