Introduction to Optimization, DIKU 2007-08
Monday 12 November
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Lecture 1

What is OR, linear models, standard form, slack form, simplex repetition, graphical interpretation, extreme points, basic solution, CPLEX introduction.

Taha sections

- 2.1, 2.2, 2.3 (briefly), 2.4 (briefly)
- 3.1, 3.2, 3.3, 3.4, 3.6
- all examples can be read briefly

Overview of course

1 What is OR, linear models, standard form, slack form, simplex repetition, graphical interpretation, extreme points, basic solution, CPLEX introduction.

2 Revised simplex algorithm, bounded variables

3 Duality, shadow prices, sensitivity analysis, post-optimal analysis, complementary slackness, KKT optimality constraints, CPLEX sensitivity

4 Network problems, transportation model, total unimodular (perfect matrices, interval matrices, property P), max-flow min-cut duality.

5 Interior point methods, Simplex vs. interior point

6 External speaker: Erling Andersen, MOSEK

Projektopgaver

1 Sensitivity analysis.
2 Multicommodity flow problem.

Assumption

Knowledge of matrix notation (appendix D in Taha)

Overview of course

Linear Programming:

VA	INT-OPT
simplex, equation	simplex, tabular
duality theorem	complementary slack condition
proof of termination	
	revised simplex
	interpretation of tabular
	sensitivity analysis
	post-optimal analysis
	worst-case complexity of simplex
	interior-point methods

What is Operations Research

1. The application of scientific methods
2. By interdisciplinary teams
3. To problems involving the control of organized man-machine systems so as to provide solutions which best serve the purpose of the organization as a whole.

(R.L.Ackoff)

OR is concerned with scientifically deciding how to best design and operate man-machine systems, usually under conditions requiring the allocation of scarce resources (ORSA definition, 1976)

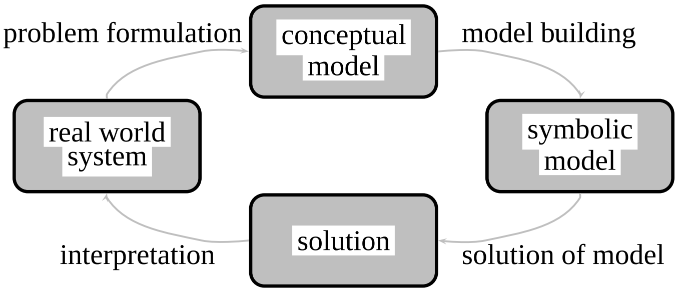
OR is the activity carried on by members of the OR society; its methods are those reported in our journal
(P.M. Morse, 1953)

Operations Research: The science of better
Time-starved executives are making bolder decisions with less risk and better outcomes. Their secret: operations research. (INFORMS, definition 2004)

OR is the discipline of applying advanced analytical methods to help make better decisions. (INFORMS, definition 2004)

OR is an expensive way of being insulted by someone half your age
(Stafford Beer)

Operations Research



problem formulation

conceptual model

model building

real world system

symbolic model

interpretation

solution

solution of model

Model building is an art

The Reddy Mikks Company

	Tons of raw per ton of		Max availability
	Ext. Paint (EP)	Int. Paint (IP)	(tons)
Raw, M1	6	4	24
Raw, M2	1	2	6
Profit/ton	5	4	

demand of IP cannot exceed that of EP by more than 1 ton
maximum daily demand of IP is 2 tons

Any OR model has three basic components

Decision variables	that we seek to determine
Objective (goal)	that we aim to optimize
Constraints	that we need to satisfy

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Raw, M1	6	4	24
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Profit/ton	5	4	

demand of IP cannot exceed that of EP by more than 1 ton
maximum daily demand of IP is 2 tons

- Decision variables
  x_1 tons produced daily of EP
  x_2 tons produced daily of IP
- Objective
  maximize $z = 5x_1 + 4x_2$
- Constraints
  $6x_1 + 4x_2 \leq 24$ (raw M1)
  $x_1 + 2x_2 \leq 6$ (raw M2)
  $-x_1 + x_2 \leq 1$ ($x_2 \leq x_1 + 1$)
  $x_2 \leq 2$
- Nonnegativity constraints (implicitly given)
  $x_1 \geq 0$
  $x_2 \geq 0$

Proportionality and additivity ⇒ linearity

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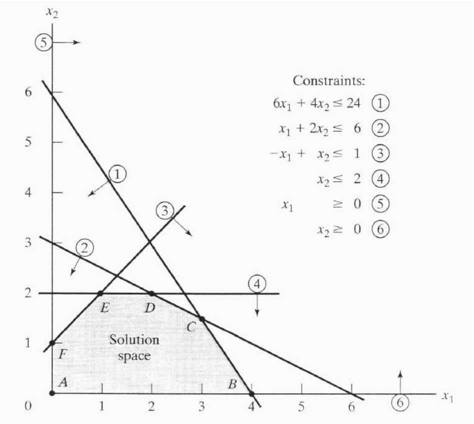
Two decision variables, solution space can be drawn in 2D



Convex solution space

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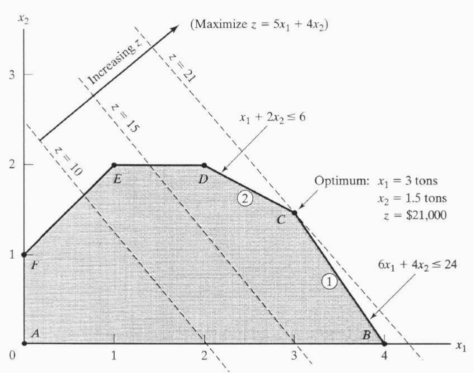
Graphical solution



- Isoprofit line
- Optimal solution in corner point (or side)

$$x^* = (3, \frac{3}{2}) \qquad z^* = 5 \cdot 3 + 4 \cdot \frac{3}{2} = 21$$

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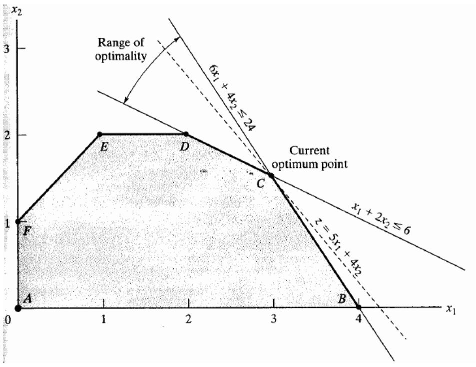
Sensitivity Analysis, objective $z = 5x_1 + 4x_2$



Solution remains optimal between

$$z = x_1 + 2x_2 \qquad z = 6x_1 + 4x_2$$

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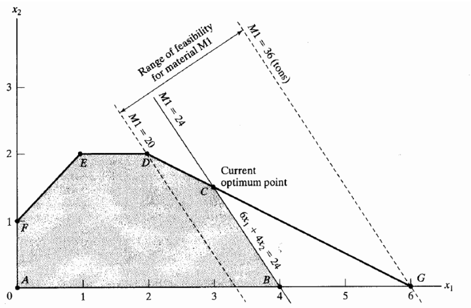
Sensitivity Analysis, constraint M1



range

$$20 \leq M1 \leq 36$$

profit of increasing M1 one unit

$$\frac{z_G - z_C}{36 - 24} = \frac{30 - 21}{12} = \frac{3}{4}$$

dual variable corresponding to M1

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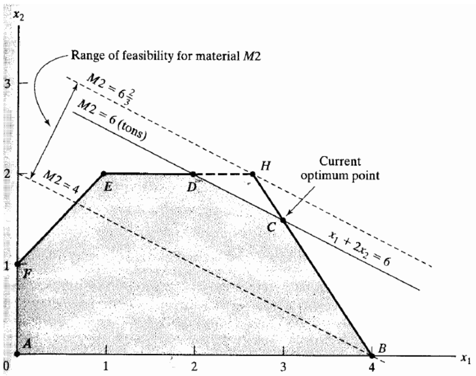
Sensitivity Analysis, constraint M2



range

$$4 \leq M2 \leq 6\frac{2}{3}$$

profit of increasing M2 one unit

$$\frac{z_C - z_B}{6 - 4} = \frac{21 - 20}{2} = \frac{1}{2}$$

dual variable corresponding to M2

Standard form and Slack form

LP problem

$$\begin{array}{ll} \max & c_1x_1 + c_2x_2 + \dots c_nx_n \\ \text{s.t.} & a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \leq b_1 \\ & a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \leq b_2 \\ & \vdots \\ & a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \leq b_m \\ & x_1, x_2, \dots, x_n \geq 0 \end{array}$$

Can be written in standard form

$$\begin{array}{lll} \max & \sum_{j=1}^{n} c_j x_j & \\ \text{s.t.} & \sum_{j=1}^{n} a_{ij} x_j \leq b_i & i = 1, \dots, m \\ & x_j \geq 0 & j = 1, \dots, n \end{array}$$

Slack form

$$\begin{array}{ll} z = \sum_{j=1}^{n} c_j x_j & \\ \sum_{j=1}^{n} a_{ij} x_j + x_{n+i} = b_i & i = 1, \dots, m \end{array}$$

omit max
omit $x_j \geq 0$
introduce slack variables x_{n+i}
set z equal to objective, and treat as ordinary constraint

Matrix representation

LP in standard form

$$\begin{array}{ll} \max & c_1x_1 + c_2x_2 + \dots c_nx_n \\ \text{s.t.} & a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \leq b_1 \\ & a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \leq b_2 \\ & \vdots \\ & a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \leq b_m \\ & x_1, x_2, \dots, x_n \geq 0 \end{array}$$

Written in matrix form

$$\begin{array}{ll} \max & (c_1, c_2, \dots, c_n) \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \\ \text{s.t.} & \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \leq \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} \\ & \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \end{array}$$

Short matrix representation

$$\begin{array}{ll} \max & cx \\ \text{s.t.} & Ax \leq b \\ & x \geq 0 \end{array}$$

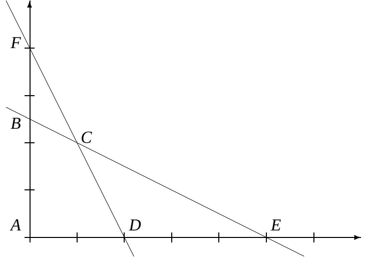
Standard form and Slack form

Any LP can be written as
standard form $\max\{cx : Ax \leq b, x \geq 0\}$
slack form $\max\{z = cx : Ax + s = b, x, s \geq 0\}$

min	$\rightarrow$	max	$\min z = -\max -z$
$\geq$	$\rightarrow$	$\leq$	$p \geq q \Leftrightarrow -p \leq -q$
$=$	$\rightarrow$	$\leq$	$p = q \Leftrightarrow p \leq q, -p \leq -q$
$\leq$	$\rightarrow$	$=$	$p \leq q \Leftrightarrow p + x_s = q, x_s \geq 0$ (slack var)
$\geq$	$\rightarrow$	$=$	$p \geq q \Leftrightarrow p - x_s = q, x_s \geq 0$ (surplus var)
free	$\rightarrow$	nonneg	x_j free $\Leftrightarrow x_j = x'_j - x''_j, x'_j, x''_j \geq 0$

Linear Programming (Taha example 3.2.1)

$$\begin{array}{lrcrcl} \text{maximize} & 2x_1 & + & 3x_2 & & \\ \text{subject to} & 2x_1 & + & x_2 & \leq & 4 \\ & x_1 & + & 2x_2 & \leq & 5 \\ & & & x_1, x_2 & \geq & 0 \end{array}$$



Add slack variables

$$\begin{array}{lrcrcrcrcl} \text{maximize} & 2x_1 & + & 3x_2 & & & & & & \\ \text{subject to} & 2x_1 & + & x_2 & + & x_3 & & & = & 4 \\ & x_1 & + & 2x_2 & & & + & x_4 & = & 5 \\ & & & & & & x_1, x_2, & x_3, x_4 & \geq & 0 \end{array}$$

The set of constraints form a polyhedral
Optimal solution is found at corner point

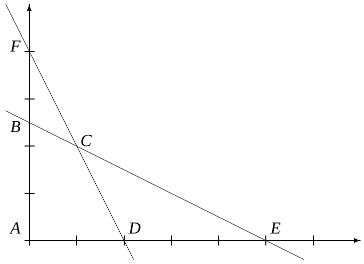
Corner point	Name	Feasible	Objective
$(0, 0, 4, 5)$	A	yes	0
$(0, 4, 0, -3)$	F	no	–
$(0, 2.5, 1.5, 0)$	B	yes	7.5
$(2, 0, 0, 3)$	D	yes	4
$(5, 0, -6, 0)$	E	no	–
$(1, 2, 0, 0)$	C	yes	8

Linear Programming

Corner points are found by setting $n-m$ variables to 0, and solving remaining m equations with m variables.
Number of corner points, when $m < n$

$$C^n_m = \frac{n!}{m!(n-m)!}$$

(exponential in n)



Corner points are basic solutions

Non-basic	Basic	Basic Solution	Corner Point	Feasible	Objective
(x_1,x_2)	(x_3,x_4)	$(4,5)$	A	yes	0
(x_1,x_3)	(x_2,x_4)	$(4,-3)$	F	no	–
(x_1,x_4)	(x_2,x_3)	$(2.5,1.5)$	B	yes	7.5
(x_2,x_3)	(x_1,x_4)	$(2,3)$	D	yes	4
(x_2,x_4)	(x_1,x_3)	$(5,-6)$	E	no	–
(x_3,x_4)	(x_1,x_2)	$(1,2)$	C	yes	8

Basis, basis feasible solution

Since we have added slack variables, the number of variables n is larger than the number of constraints m.

$$\begin{array}{ll} \text{maximize} & cx \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array}$$

Choose m linearly independent columns from A. The corresponding set $B = \{i_1, i_2, \ldots, i_m\}$ is called a *basis*.

- A_B columns in A corresponding to basis variables B.
- A_N columns in A corresponding to non-basis variables N

$$\begin{array}{ll} \text{maximize} & c_B x_B + c_N x_N \\ \text{subject to} & A_B x_B + A_N x_N = b \\ & x \geq 0 \end{array}$$

A *Basis feasible solution* is obtained by setting $x_N = 0$.

$$\begin{array}{rcl} A_B x_B + A_N 0 & = & b \\ x_B & = & A_B^{-1} b \end{array}$$

x_B is well defined:
A_B is an $m \times m$ matrix
columns must be linearly independent

Basis, basis feasible solution

Example:

$$\begin{array}{ll} \text{maximize} & c_B x_B + c_N x_N \\ \text{subject to} & A_B x_B + A_N x_N = b \\ & x \geq 0 \end{array}$$

$n = 4$ variables, m constraints
Assume $B = \{2,4\}$, $N = \{1,3\}$.

$$\max \quad (c_1, c_2, c_3, c_4) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

$$\text{s.t.} \quad \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ \vdots & & & \\ a_{m1} & a_{m2} & a_{m3} & a_{m4} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \leq \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

is equivalent to

$$\max \quad (c_2, c_4) \begin{pmatrix} x_2 \\ x_4 \end{pmatrix} + (c_1, c_3) \begin{pmatrix} x_1 \\ x_3 \end{pmatrix}$$

$$\text{s.t.} \quad \begin{pmatrix} a_{12} & a_{14} \\ a_{22} & a_{24} \\ \vdots & \\ a_{m2} & a_{m4} \end{pmatrix} \begin{pmatrix} x_2 \\ x_4 \end{pmatrix} + \begin{pmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \\ \vdots & \\ a_{m1} & a_{m3} \end{pmatrix} \begin{pmatrix} x_1 \\ x_3 \end{pmatrix} \leq \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

Corner points and basis feasible solutions

Theorem A feasible solution x to

$$\begin{array}{ll} \text{maximize} & cx \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array}$$

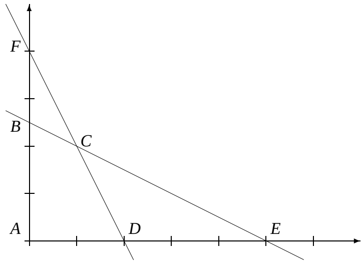
is a corner point if and only if x is a basis feasible solution

Trivial algorithm Search through all corner points

- Basis can be chosen in $C^n_m = \frac{n!}{m!(n-m)!}$ different ways
- Each time invert A_B in time $O(m^3)$

Adjacent basis feasible solutions

Two basis feasible solutions x^1 and x^2 are adjacent if B^1 and B^2 have $m-1$ common elements.



$$\begin{array}{lrcrcrcrcl} \text{maximize} & 2x_1 & + & 3x_2 & & & & & & \\ \text{subject to} & 2x_1 & + & x_2 & + & x_3 & & & = & 4 \\ & x_1 & + & 2x_2 & & & + & x_4 & = & 5 \\ & & & & & & & x_1,x_2,x_3,x_4 & \geq & 0 \end{array}$$

Corner point	Name	Feasible	Objective
$(0,0,4,5)$	A	yes	0
$(0,4,0,-3)$	F	no	–
$(0,2.5,1.5,0)$	B	yes	7.5
$(2,0,0,3)$	D	yes	4
$(5,0,-6,0)$	E	no	–
$(1,2,0,0)$	C	yes	8

Simplex algorithm

Trivial algorithm: Search through all corner points

- Basis can be chosen in $C_m^n = \frac{n!}{m!(n-m)!}$ different ways
- Each time invert A_B in time $O(m^3)$

Better algorithm: Search through corner points

- Greedy approach (hopefully not all basic solutions)
- Consider adjacent solutions (faster to invert A_B)
- A table can be used to maintain information

objective is increased “most possible” in each step.

Simplex, equation form (Cormen)

$$\text{Iteration 0:} \quad \begin{array}{rclcrcr} \hline z & = & & & 2x_1 & + & 3x_2 \\ x_3 & = & 4 & - & 2x_1 & - & x_2 \\ x_4 & = & 5 & - & x_1 & - & 2x_2 \\ \hline \end{array}$$

Most promising variable x_2
Keeping $x_1 = 0$ (nonbasic), x_2 cannot be increased infinitely

$$\begin{array}{l} x_3 \geq 0, \ \ x_3 = 4 - x_2 \ \ \Rightarrow \ \ 4 - x_2 \geq 0 \\ x_4 \geq 0, \ \ x_4 = 5 - 2x_2 \ \Rightarrow \ \ 5 - 2x_2 \geq 0 \end{array}$$

implying $x_2 \leq \frac{5}{2}$.
When $x_2 = \frac{5}{2}$ we have $x_4 = 0$ (x_4 leaves basis)
Gauss elimination $x_2 = \frac{5}{2} - \frac{1}{2}x_1 - \frac{1}{2}x_4$

$$\text{Iteration 1:} \quad \begin{array}{rclcrcr} \hline z & = & \frac{15}{2} & + & \frac{1}{2}x_1 & - & \frac{3}{2}x_4 \\ x_3 & = & \frac{3}{2} & - & \frac{3}{2}x_1 & + & \frac{1}{2}x_4 \\ x_2 & = & \frac{5}{2} & - & \frac{1}{2}x_1 & - & \frac{1}{2}x_4 \\ \hline \end{array}$$

Most promising variable x_1
Keeping $x_4 = 0$ (nonbasic), x_1 cannot be increased infinitely

$$\begin{array}{l} x_3 \geq 0, \ \ x_3 = \frac{3}{2} - \frac{3}{2}x_1 \ \Rightarrow \ \frac{3}{2} - \frac{3}{2}x_1 \geq 0 \\ x_2 \geq 0, \ \ x_2 = \frac{5}{2} - \frac{1}{2}x_1 \ \Rightarrow \ \frac{5}{2} - \frac{1}{2}x_1 \geq 0 \end{array}$$

implying $x_1 \leq 1$.
When $x_1 = \frac{3}{2}$ we have $x_3 = 0$ (x_3 leaves basis)

Gauss elimination $x_1 = 1 - \frac{2}{3}x_3 + \frac{1}{3}x_4$

$$\text{Iteration 2:} \quad \begin{array}{rclcrcr} \hline z & = & 8 & - & \frac{1}{3}x_3 & - & \frac{4}{3}x_4 \\ x_1 & = & 1 & - & \frac{2}{3}x_3 & + & \frac{1}{3}x_4 \\ x_2 & = & 2 & + & \frac{1}{3}x_3 & + & \frac{2}{3}x_4 \\ \hline \end{array}$$

All costs in objective are negative, hence stop

Simplex, in tabular form

Taha is using the equations

$$\begin{array}{rcrcrcl} z & - & 2x_1 & - & 3x_2 & = & 0 \\ x_3 & + & 2x_1 & + & x_2 & = & 4 \\ x_4 & + & x_1 & + & 2x_2 & = & 5 \end{array}$$

this leads to tabular form

Iteration 0:

basic	z	x_1	x_2	x_3	x_4	solution
z	1	−2	−3	0	0	0
x_3	0	2	1	1	0	4
x_4	0	1	2	0	1	5

Entering variable x_2
maximum value of entering variable $\min\{\frac{4}{1}, \frac{5}{2}\} = \frac{5}{2}$
leaving variable is x_4
Multiply third row by $\frac{3}{2}$ and add to first row
Multiply third row by $-\frac{1}{2}$ and add to second row
Multiply third row by $\frac{1}{2}$

Iteration 1:

basic	z	x_1	x_2	x_3	x_4	solution
z	1	$-\frac{1}{2}$	0	0	$\frac{3}{2}$	$\frac{15}{2}$
x_3	0	$\frac{3}{2}$	0	1	$-\frac{1}{2}$	$\frac{3}{2}$
x_2	0	$\frac{1}{2}$	1	0	$\frac{1}{2}$	$\frac{5}{2}$

Entering variable x_1
maximum value of entering variable $\min\{\frac{\frac{3}{2}}{\frac{3}{2}}, \frac{\frac{5}{2}}{\frac{1}{2}}\} = 1$

leaving variable is x_3
Multiply second row by $\frac{1}{3}$ and add to first row
Multiply second row by $-\frac{1}{3}$ and add to third row
Multiply second row by $\frac{2}{3}$

Iteration 2:

basic	z	x_1	x_2	x_3	x_4	solution
z	1	0	0	$\frac{1}{3}$	$\frac{4}{3}$	8
x_1	0	1	0	$\frac{2}{3}$	$-\frac{1}{3}$	1
x_2	0	0	1	$-\frac{1}{3}$	$\frac{2}{3}$	2

All reduced costs in objective are positive, hence stop

Initial solution

Simplex moves from basis solution to neighboring basis solution. Initial basis solution:

- If m slack variables, set these to zero
- or add m variables $\tilde{x}$ having a large negative cost in objective function

$$\begin{array}{ll} \text{maximize} & cx - M\tilde{x} \\ \text{subject to} & Ax + \tilde{x} = b \\ & x, \tilde{x} \geq 0 \end{array}$$

then $\tilde{x}$ is a basis feasible solution.

Iterative step

Reduced cost used for choosing the next variable to enter basis

Termination criteria

All reduced costs are positive (TAHA)
All reduced costs are negative (Cormen)

Proof of optimality

Shown at DAT2A.

LP basis solution

1 The solution to $A_B x_B = b$ is uniquely determined

The solution is $x_B = A_B^{-1} b$ where A_B^{-1} is the inverse matrix of A_B.

- $x_B = A_B^{-1}$ is called a basis solution
- m variables in x_B are called basic variables
- $n - m$ remaining variables nonbasic variables

If all basic variables are nonnegative, then x_B is called basic feasible solution
If some of the basic variables are zero, we talk of a degenerate solution
If some of the basic variables are negative, the basic solution is infeasible

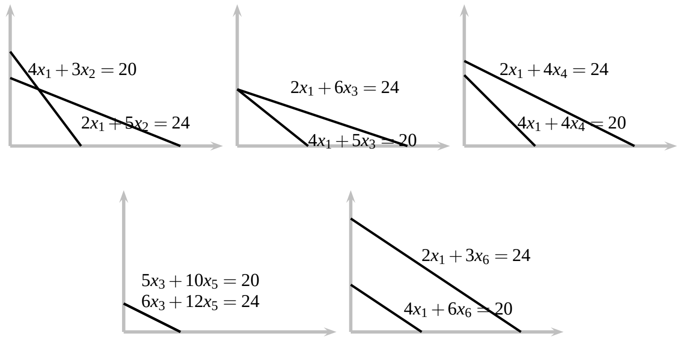
2 $A_B x_B = b$ has infinitely many solutions

3 $A_B x_B = b$ has no solutions

LP basis solution, example

$$\begin{array}{rcl} 4x_1+3x_2+5x_3+4x_4+10x_5+6x_6 & = & 20 \\ 2x_1+5x_2+6x_3+4x_4+12x_5+3x_6 & = & 24 \end{array}$$

- Case 1.1 $x_B=(x_1,x_2)$ Basic feasible solution $x_1=2, x_2=4$ (nondegenerate)
- Case 1.2 $x_B=(x_1,x_3)$ Basic feasible solution $x_1=0, x_2=4$ (degenerate)
- Case 1.3 $x_B=(x_1,x_4)$ Basic infeasible solution $x_1=-2, x_4=7$
- Case 2 $x_B=(x_3,x_5)$ Infinitely many solutions
- Case 3 $x_B=(x_1,x_6)$ No solution



LP basis solution

An LP-problem may have

a) No feasible solutions (in case of conflicting constraints)

b) Feasible solutions but no optimal solution (in case of unboundedness such as $\max x, x \geq 1$)

c) Feasible and one or more optimal solutions. In this case the problem also has an optimal basic solution.

Bigger example, The Reddy Mikks Company

$$\begin{array}{rrcrcr} \text{maximize} & 5x_1 & + & 4x_2 & & \\ \text{subject to} & 6x_1 & + & 4x_2 & \leq & 24 \\ & x_1 & + & 2x_2 & \leq & 6 \\ & -x_1 & + & x_2 & \leq & 1 \\ & & & x_2 & \leq & 2 \\ & & x_1,x_2 & \geq & 0 & \end{array}$$

Add slack variables x_3, x_4, x_5, x_6

$$\begin{array}{rrcrcrcr} \text{maximize} & 5x_1 & + & 4x_2 & & & & \\ \text{subject to} & 6x_1 & + & 4x_2 & + & x_3 & = & 24 \\ & x_1 & + & 2x_2 & + & x_4 & = & 6 \\ & -x_1 & + & x_2 & + & x_5 & = & 1 \\ & & & x_2 & + & x_6 & = & 2 \\ & & & & & \multicolumn{3}{r}{x_1,x_2,x_3,x_4,x_5,x_6 \geq 0} \end{array}$$

Bigger example, The Reddy Mikks Company

Write up equation form (Cormen style)
Iteration 0:

$$\begin{array}{rcrcrcr} \hline z & = & & & 5x_1 & + & 4x_2 \\ x_3 & = & 24 & - & 6x_1 & - & 4x_2 \\ x_4 & = & 6 & - & x_1 & - & 2x_2 \\ x_5 & = & 1 & + & x_1 & - & x_2 \\ x_6 & = & 2 & & & - & x_2 \\ \hline \end{array}$$

Iteration 1:

$$\begin{array}{rcrcrcr} \hline z & = & 20 & + & \frac{2}{3}x_2 & - & \frac{5}{6}x_3 \\ x_1 & = & 4 & - & \frac{2}{3}x_2 & - & \frac{1}{6}x_3 \\ x_4 & = & 2 & - & \frac{4}{3}x_2 & + & \frac{1}{6}x_3 \\ x_5 & = & 5 & + & \frac{5}{3}x_2 & - & \frac{1}{6}x_3 \\ x_6 & = & 2 & & 1x_2 & & \\ \hline \end{array}$$

Iteration 2:

$$\begin{array}{rcrcrcr} \hline z & = & 21 & - & \frac{3}{4}x_3 & - & \frac{1}{2}x_4 \\ x_1 & = & 3 & - & \frac{1}{4}x_3 & + & \frac{1}{2}x_4 \\ x_2 & = & \frac{3}{2} & + & \frac{1}{8}x_3 & - & \frac{3}{4}x_4 \\ x_5 & = & \frac{5}{2} & - & \frac{3}{8}x_3 & + & \frac{5}{4}x_4 \\ x_6 & = & \frac{1}{2} & - & \frac{1}{8}x_3 & + & \frac{3}{4}x_4 \\ \hline \end{array}$$

Bigger example, The Reddy Mikks Company

Iteration 0:

basic	z	x_1	x_2	x_3	x_4	x_5	x_6	solution
z	1	-5	-4	0	0	0	0	0
x_3	0	6	4	1	0	0	0	24
x_4	0	1	2	0	1	0	0	6
x_5	0	-1	1	0	0	1	0	1
x_6	0	0	1	0	0	0	1	2

Iteration 1:

basic	z	x_1	x_2	x_3	x_4	x_5	x_6	solution
z	1	0	$-\frac{2}{3}$	$\frac{5}{6}$	0	0	0	20
x_1	0	1	$\frac{2}{3}$	$\frac{1}{6}$	0	0	0	4
x_4	0	0	$\frac{4}{3}$	$-\frac{1}{6}$	1	0	0	2
x_5	0	0	$\frac{5}{3}$	$\frac{1}{6}$	0	1	0	5
x_6	0	0	1	0	0	0	1	2

Iteration 2:

basic	z	x_1	x_2	x_3	x_4	x_5	x_6	solution
z	1	0	0	$\frac{3}{4}$	$\frac{1}{2}$	0	0	21
x_1	0	1	0	$\frac{1}{4}$	$-\frac{1}{2}$	0	0	3
x_2	0	0	0	$-\frac{1}{8}$	$\frac{3}{4}$	0	0	$\frac{3}{2}$
x_5	0	0	0	$\frac{3}{8}$	$-\frac{5}{4}$	1	0	$\frac{5}{2}$
x_6	0	0	1	$\frac{1}{8}$	$-\frac{3}{4}$	0	1	$\frac{1}{2}$

CPLEX

LP-solver

- simplex
- dual simplex
- network simplex
- interior point methods

IP/MIP-solver

- branch-and-bound
- cuts

Modes of use

- interactive mode
- file mode
- callable library
- called from AMPL

License

`bach` and `kand` computers

```
rlogin bach-1
cplex
ILOG CPLEX 9.100, licensed to "university-copenhagen", options: e m b
```

Interactive use

- add variables
- add constraints
- change bounds

File mode

- several file formats
- ".lp" is natural form
- type in data to file
- read file in CPLEX
- optimize
- display solution

CPLEX Commands

- `help`
  displays a list of all commands
- `read filename.lp`
  reads a file in "natural format"
- `write`
  write problem or solution info to a file
- `optimize`
  run optimizer (LP or IP depending on formulation)
- `display problem all`
  show the problem
- `display solution variables -`
  show primal solution
- `display solution dual -`
  show dual solution
- `display sensitivity objective -`
  show objective sensitivity range
- `display sensitivity rhs -`
  show right-hand-side sensitivity range
- `display sensitivity lb -`
  show lower bound sensitivity range
- `display sensitivity ub -`
  show upper bound sensitivity range
- `quit`
  leave CPLEX

Commands may be abbreviated e.g. `dis sol var -`

LP file format

- everything following a backslash is a comment
- variables have form `x12, y, antalskibe`
- `minimize` or `maximize` objective function
- `subject to`
- constraints should be in linear form with constant on right side
  $$2\,x1 + 3\,x2 <= 25$$
  relational operators $<$, $<=$, $>=$, $>$ are interpreted as $\leq, \leq, \geq, \geq$
- `bounds`
  $$0 <= x3 <= \texttt{+infty}$$
  (warning: bounds have no associated dual variables!)
- `integer` (warning: means binary, so do not use!)
- `general` variables which must be integer variables
- `binary` variables which must be 0-1 variables
- `end`

Using CPLEX

Reddy Mikks, assuming decision variables are integers:

```
maximize
  5 x1 + 4 x2
subject to
  6 x1 + 4 x2 <= 24
    x1 + 2 x2 <=  6
  - x1 +   x2 <=  1
           x2 <=  2
bounds
  x1 >= 0
  x2 >= 0
general
  x1 x2
end

kand-1 > cplex
ILOG CPLEX 9.100, licensed to "university-copenhagen", options: e m b

Welcome to CPLEX Interactive Optimizer 9.1.0
  with Simplex, Mixed Integer \& Barrier Optimizers
Copyright (c) ILOG 1997-2005
CPLEX is a registered trademark of ILOG

Type 'help' for a list of available commands.
Type 'help' followed by a command name for more
information on commands.

CPLEX> read reddymix.lp
Problem 'reddymix.lp' read.
Read time =    0.00 sec.

CPLEX> opt
Tried aggregator 1 time.
MIP Presolve eliminated 1 rows and 0 columns.
MIP Presolve modified 1 coefficients.
Reduced MIP has 3 rows, 2 columns, and 6 nonzeros.
Presolve time =    0.00 sec.
MIP emphasis: balance optimality and feasibility
Root relaxation solution time =    0.00 sec.
```

```
        Nodes                                          Cuts/
   Node  Left     Objective  IInf  Best Integer     Best Node    ItCnt

      0     0       21.0000     1                     21.0000        2
*     0+    0                   0       20.0000       21.0000        2
               infeasible              20.0000      Cuts:  2        2

Mixed integer rounding cuts applied:  1

Integer optimal solution:  Objective =    2.0000000000e+01
Solution time =    0.00 sec.  Iterations = 2  Nodes = 0

CPLEX> dis sol var -
Variable Name           Solution Value
x1                            4.000000
All other variables in the range 1-2 are zero.
```

Using CPLEX from AMPL

- AMPL is a modeling language for mathematical programming
- AMPL is described in Appendix A of Taha
- AMPL can use several solvers. To use CPLEX, see section A.8 in Taha

```
option solver cplex;
option cplex_options 'sensitivity';
```

Solvers available on CD-ROM:

- CPLEX
- KNITRO
- LPSOLVE
- LOQO
- MINOS