Answers, Written Exam, June 2004, David Pisinger

Q11: In class N_3, the first item can be removed since it has profit $p_{3,1} = 0$. The last item in the same class can also be removed since if we choose $x_{3,3} = 1$ then the only other solution is $x_{1,1} = x_{2,1} = 1$, having a too small profit sum.

Knowing that the second item from class 3 should be chosen, we find by inspection that the optimal solution is $x_{1,3} = x_{2,1} = x_{3,2} = 1$ with all other variables set to zero. This gives the objective value $z = 11$. Hence, answer 11D) is correct.

Q12: We should maximize the profit sum

$$\sum_{i=1}^{k} \sum_{j \in N_i} p_{ij} x_{ij}$$

Exactly one item should be chosen from each class

$$\sum_{j \in N_i} x_{ij} = 1 \quad i = 1, \ldots, k$$

The sum of the chosen items may not exceed the capacity c

$$\sum_{i=1}^{k} \sum_{j \in N_i} w_{ij} x_{ij} \le c$$

The decision variables are binary

$$x_{ij} \in \{0,1\} \quad i = 1, \ldots, k, j \in N_i$$

Hence, the correct formulation is 12A).

Q13: Since not all items in C can be in the same knapsack, at most $|C| - 1 = k - 1$ of the items can be chosen. Hence, 13F) is the correct answer.

Q14: This is a straightforward generalization of the separation algorithm for normal cover inequalities.

Q15: Since a convex combination of items a and c gives at least as good a profit as item b, the latter item will never be used in an LP-solution.

More formally, if $x_{i,b} > 0$ then solving

$$x_{ia} + x_{ic} = x_{ib}$$
$$w_{ia} x_{ia} + w_{ic} x_{ic} = w_{ib} x_{ib}$$

gives a convex combination of item a and c with the same weight contribution as item b. The solution to the above set of equations is

$$x_{ic} = x_{ib} \frac{w_{ib} - w_{ia}}{w_{ic} - w_{ia}}$$
$$x_{ia} = x_{ib} \frac{w_{ic} - w_{ib}}{w_{ic} - w_{ia}}$$

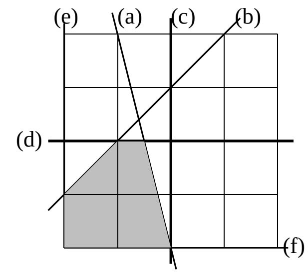
The profit sum satisfies

$$\begin{aligned} & p_{ia} x_{ia} + p_{ic} x_{ic} \ge p_{ib} x_{ib} \Leftrightarrow \\ & p_{ia} x_{ib} \tfrac{w_{ic} - w_{ib}}{w_{ic} - w_{ia}} + p_{ic} x_{ib} \tfrac{w_{ib} - w_{ia}}{w_{ic} - w_{ia}} \ge p_{ib} x_{ib} \Leftrightarrow \\ & p_{ia}(w_{ic} - w_{ib}) + p_{ic}(w_{ib} - w_{ia}) \ge p_{ib}(w_{ic} - w_{ia}) \end{aligned}$$

since from the stated we have

$$\begin{aligned} & \tfrac{p_{ic} - p_{ia}}{w_{ic} - w_{ia}} \ge \tfrac{p_{ib} - p_{ia}}{w_{ib} - w_{ia}} \Leftrightarrow \\ & (p_{ic} - p_{ia})(w_{ib} - w_{ia}) \ge (p_{ib} - p_{ia})(w_{ic} - w_{ia}) \Leftrightarrow \\ & p_{ic}(w_{ib} - w_{ia}) - p_{ia}(w_{ib} - w_{ia}) \ge p_{ib}(w_{ic} - w_{ia}) - p_{ia}(w_{ic} - w_{ia}) \Leftrightarrow \\ & p_{ia}(w_{ic} - w_{ib}) + p_{ic}(w_{ib} - w_{ia}) \ge p_{ia}(w_{ic} - w_{ia}) \end{aligned}$$

Q16: The problem is defined in two variables so we can draw it in the plane as follows



It is easily seen that inequalities (b), (e), (f) are facet-defining. Hence, answer 16F) is correct.

Q17: The LP-relaxed problem is

$$\begin{array}{llll} \max & x_1 + 2x_2 & & \\ \text{s.t.} & 4x_1 + x_2 \le 8 & & (a) \\ & -x_1 + x_2 \le 1 & & (b) \\ & x_1 \le 2 & & (c) \\ & x_2 \le 2 & & (d) \\ & -x_1 \le 0 & & (e) \\ & -x_2 \le 0 & & (f) \\ & x_1, x_2 \in \mathbb{R} & & \end{array}$$

the dual problem becomes

$$\begin{array}{llllllll} \min & 8y_a + y_b + 2y_c + 2y_d \\ \text{s.t.} & 4y_a - y_b + y_c - y_e \geq 1 \\ & y_a + y_b + y_d - y_f \geq 2 \\ & y_a, y_b, y_c, y_d, y_e, y_f \in \mathbb{R} \end{array}$$

The primal solution is $x_1 = \frac{3}{2}$, $x_2 = 2$ with objective value $z = \frac{3}{2} + 2 \cdot 2 = 5\frac{1}{2}$. From complementary slackness we get:

$$\begin{aligned} 4y_a - y_b + y_c - y_e &= 1 \\ y_a + y_b + y_d - y_f &= 2 \\ y_b &= 0 \\ y_c &= 0 \\ y_e &= 0 \\ y_f &= 0 \end{aligned}$$

Giving the equations

$$\begin{aligned} 4y_a &= 1 \\ y_a + y_d &= 2 \end{aligned}$$

with optimal solution $y_a = \frac{1}{4}$, $y_d = \frac{7}{4}$, and solution value $z = 8 \cdot \frac{1}{4} + 2 \cdot \frac{7}{4} = 5\frac{1}{2}$. Hence, answer 17D) is correct.

Q18: Lagrangian relaxing constraints (a) and (b) we notice that the remaining constraints define the convex hull of the integer solutions to (c), (d), (e) and (f). Hence, the optimal choice of Lagrangian multipliers λ_a, λ_b correspond to the dual variables y_a, y_b. Hence, $\lambda_a = \frac{1}{4}, \lambda_b = 0$. The correct answer is then 18E).

Q19: We should minimize the number of planes, i.e.

$$\sum_{i=1}^{n} y_i$$

If plane i is used (i.e. $y_i = 1$) then the sum of the weights should not exceed the plane capacity q. If the plane is not used (i.e. $y_i = 0$) then no capacity is available in this plane.

$$\sum_{j=1}^{n} w_j x_{ij} \leq y_i q$$

There should be at least 4 different items in each plane i

$$\sum_{j=1}^{n} x_{ij} \geq 4$$

All items should go into some plane

$$\sum_{i=1}^{n} x_{ij} = 1$$

Finally, all decision variables are binary, hence formulation 19A) is correct.

Q20: We use $\delta = 1$ to indicate that items 1 and 2 go by the same plane, and $\delta = 0$ to indicate that items 1, 2, 3 go by different planes.

The first constraint is mathematically formulated as

$$\delta = 1 \Rightarrow x_{i1} - x_{i2} = 0$$

for each value of $i = 1, \ldots, n$, which in linear form becomes

$$\begin{aligned} x_{i1} - x_{i2} + \delta &\leq 1 \\ x_{i1} - x_{i2} - \delta &\geq -1 \end{aligned}$$

The latter constraint is mathematically formulated as

$$\delta = 0 \Rightarrow (x_{i1} + x_{i2} \leq 1) \wedge (x_{i1} + x_{i3} \leq 1) \wedge (x_{i2} + x_{i3} \leq 1)$$

for each value of $i = 1, \ldots, n$, which in linear form becomes

$$\begin{aligned} x_{i1} + x_{i2} + (1 - \delta) &\leq 2 \\ x_{i1} + x_{i3} + (1 - \delta) &\leq 2 \\ x_{i2} + x_{i3} + (1 - \delta) &\leq 2 \end{aligned}$$

Hence, answer 20D) is correct.