Written exam, 27 January 2006

SOLUTION (Q1-Q10)

The opening questions can be approached in different ways. Due to the simplicity of the problems considered, however, Simplex iterations or the like are nowhere necessary.

Although first explicitly requested in **Q3**, a nice gateway is to look at D(α,β,γ,Δ), the dual of P(α,β,γ,Δ), from the very beginning. Let w and z be the objective function values of P(α,β,γ,Δ) and D(α,β,γ,Δ), respectively. Only a single variable called *y* appears in D(α,β,γ,Δ). The dual objective function is

$$\min z = \Delta y,$$

and the three dual constraints d1-d3 are:

d1:	$3y \geq 6\alpha$	or	$y \geq 2\alpha$		
d2:			$y \geq \beta$		
d3:	$(1/\gamma)y \geq -8$	or	$y \geq -8\gamma$	if $\gamma > 0$	(don't miss
		or	$y \leq -8\gamma$	if $\gamma < 0$	this point!)

y itself is a *free* variable, that is, y can take any real value since the only constraint in P(α,β,γ,Δ) is an *equation*.

Q1: For (α,β,γ,Δ) = (4, 8, 1, 3) we have

d1:	$y \geq 8$	
d2:	$y \geq 8$	
d3:	$y \geq -8$	redundant

Since z = 3y is to be minimized, *the* optimal solution to D(4, 8, 1, 3) is to take y=8 yielding $z_{min} = 24$. Only d1 and d2 will then hold as equations thus allowing for x_1 and x_2 to take positive values. The optimal *basic* solutions to P(4, 8, 1, 3) are accordingly $(x_1,x_2,x_3) = (1,0,0)$ and (0,3,0) for both of which w_{max} equals 24.

The same result follows from the observation that P(α,β,γ,Δ) has only one constraint implying that at most one variable can be positive in any basic feasible solution. Obviously, x_3 cannot be positive in any optimal solution. It is furthermore seen that the same value $w_{max} = 24$ obtains for $x_1 > 0$ or $x_2 > 0$.

⇒ **1C)**

Q2: For (α,β,γ,Δ) = (0, 2, -1/8, 0) it suffices to look at d2,d3:

d2: $y \geq 2$,   d3: $y \leq -8\gamma = 1$ (note that $\gamma < 0$)

Since d2,d3 are conflicting constraints, D(0, 2, -1/8, 0) is infeasible which implies that P(0, 2, -1/8, 0) is either infeasible or unbounded.

Note that P(0, 2, -1/8, 0) actually *is* unbounded. For $x_1 = 0$ the objective function reduces to

$$2x_2 - 8x_3 = x_2 + (x_2 - 8x_3) = x_2$$

since $(x_2 - 8x_3) = \Delta = 0$.

$\Rightarrow$ **2A)**.

Q3: Infeasibility of D(0, 2, -1/8, 0) has already been established in **Q2** and extends to all values of α,Δ

$\Rightarrow$ **3A)**.

Q4: For $(\alpha,\beta,\gamma,\Delta) = (-6, -10, 1, 3)$ the dual constraints read

d1:	$y \geq -12$	redundant
d2:	$y \geq -10$	redundant
d3:	$y \geq -8$	

$\Rightarrow \quad y = -8 \Rightarrow \quad \boldsymbol{\Sigma = -8}, \qquad z_{min} = -24$

For the sake of comparison, we find for P(-6, -10, 1, 3):

$$\left.\begin{array}{ll} \max & -36x_1 - 10x_2 - 8x_3 \\ & 3x_1 + x_2 + x_3 = 3 \end{array}\right\} \Rightarrow x_3 = 3, \qquad w_{max} = -24$$

$\Rightarrow$ **4D)**.

Q5: Since $y = -8$ solves D(-6, -10, 1, 3) and, hence, is the *shadow price* associated with the only constraint of P(-6, -10, 1, 3+δ), the answer follows immediately from **Q4**: $w(\delta)-w(0) = -8\delta$.

Alternatively we may evaluate the primal and dual objective functions:

Primal:	$x_3 = 3+\delta$,	$w_{max} = -8(3+\delta) = -24 - 8\delta$
Dual:	d1,d2,d3: no changes.	$z_{min} = -8(3+\delta) = -24 - 8\delta$

$\Rightarrow$ **5C)**.

Q6: If some x_j is free, the *jth* dual constraint must be an *equation* determining the value of y. Thus, x_2,x_3 free $\Rightarrow$ d2,d3 are equations:

(d2: $y = \beta$, d3: $y = -8\gamma$) $\Rightarrow \beta = -8\gamma$

$\Rightarrow$ all three statements are true

$\Rightarrow$ **6D)**.

Q7: We find:

d1:	$y \geq 2k$	
d2:	$y \geq k$	redundant
d3:	$y \geq -8k$	redundant

$\Rightarrow$ **7C)**.

Q8: Recall that any basic feasible solution to P(k,k,k,k) has at most one positive variable. We need only to look at the equation,

$$3x_1 + x_2 + (1/k)x_3 = k$$

i) $x_1 > 0$: $x_1 = k/3$ $\Rightarrow$ k is divisible by 3
ii) $x_2 > 0$: $x_2 = k$ $\Rightarrow$ k is integral
iii) $x_3 > 0$: $x_3 = k^2$ $\Rightarrow$ k is integral or the square root of an integer

k is divisible by 3 implies that both k and k^2 are integers whereas neither ii) nor iii) implies i)

$\Rightarrow$ **8C).**

Q9 ***(text question):***

The most important key to the answer is that the following must hold for the *reduced cost* t_{ij}:

$$t_{ij} = u_i + v_j - c_{ij} = \begin{cases} 0 & \text{if } x_{ij} \text{ is basic} \\ \leq 0 & \text{if } x_{ij} \text{ is nonbasic} \end{cases}$$

An entry (i,j) with $t_{ij} = 0$ will be called *admissible.*
Disregarding **Q10**, ii) the full solution to **Q9, Q10** is:

u_i, v_j	$v_1 = 3$	$v_2 = ? = 6$	$v_3 = -2$	$v_4 = 5$	Supply
$u_1 = 0$	5 < 0	6 $= 0$ (1)	0 < 0	5 $= 0$ (4)	$a_1 = 5$
$u_2 = 2$	5 $= 0$ (3)	M < 0	M < 0	$c_{24} = ? = 7$ (4)	$a_2 = 7$
$u_3 = 7$	13 < 0	15 < 0	5 $= 0$ (2)	16 < 0	$a_3 = 2$
$u_4 = 4$	8 < 0	10 $= 0$ (5)	2 $= 0$ (3)	11 < 0	$a_4 = 8$
$u_5 = ? = 16$	M < 0	$c_{52} = ?$ (0)	14 $= 0$ (2)	M < 0	$a_5 = ? = 2$
Demand	$b_1 = ? = 3$	$b_2 = 6$	$b_3 = ? = 7$	$b_4 = ? = 8$	

The "Sudoku-flavoured" solution approach embarks from entry (2,1) which appears to be the only admissible entry in column 1. Hence, $x_{21} = a_2 - c_{24} = 7\text{-}4 = 3$ and $b_1 = 3$. $x_{2,4} > 0 \Rightarrow t_{2,4} = 0 \Rightarrow c_{24} = u_2 + v_4 = 2+5 = 7$. Row 1 has two admissible entries $\Rightarrow x_{14} = a_1 - x_{12} = 5\text{-}1 = 4$. Column 4 has two admissible entries $\Rightarrow b_4 = 8$, et cetera. We may also take into account that TP-Booze is *balanced.*

Q10 ***(text question):***

To determine u_5, a_5, b_3, and x_{53}, however, we need the additional information about the objective function value = 163 which enables us first to identify x_{53}:

Primal: $135 + 14x_{53} = \mathbf{163} \quad \Rightarrow \quad x_{53} = 2$

from which the other values follow as shown.

Though not requested, we provide also the dual objective function:

Dual: $145 - 14 + 2\times16 = \mathbf{163}$

Q10, ii): The solution will remain optimal for $t_{52} \leq 0$ or

$16 + 6 - c_{52} \leq 0 \Rightarrow c_{52} \geq 22.$