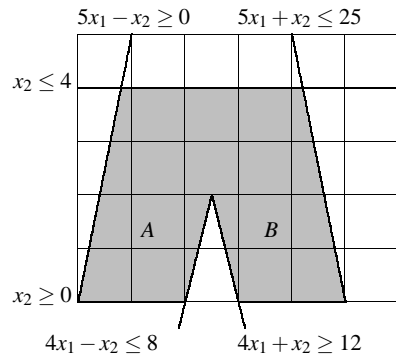


**Answers, Written Exam, December 2003, David Pisinger**

**Q11:** Since the set  $T$  is not convex, we need to introduce some indicator variables.



Let  $\delta_A = 1$  if  $(x_1, x_2) \in A$  and  $\delta_B = 1$  if  $(x_1, x_2) \in B$ . Then we have

$$\delta_A = 1 \Rightarrow 4x_1 - x_2 \leq 8$$

which can be modeled by the inequality

$$4x_1 - x_2 + M_A \delta_A \leq M_A + 8$$

where  $M_A$  is an upper bound on  $4x_1 - x_2 - 8$ . Since  $0 \leq x_1 \leq 5$  and  $0 \leq x_2 \leq 4$  we may choose  $M_A = 12$ , getting the inequality

$$4x_1 - x_2 + 12\delta_A \leq 20$$

In a similar way we have

$$\delta_B = 1 \Rightarrow 4x_1 + x_2 \geq 12$$

which can be modeled by the inequality

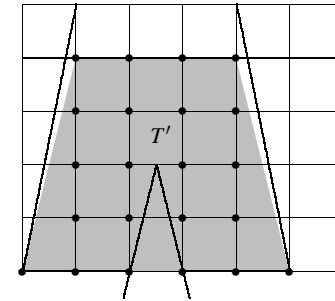
$$4x_1 + x_2 + m_B \delta_B \geq m_B + 12$$

As a lower bound on  $4x_1 + x_2 - 12$  we may choose  $m_B = -12$ , getting the inequality

$$4x_1 + x_2 - 12\delta_B \geq 0$$

Finally we must demand that  $\delta_A + \delta_B \geq 1$  to ensure that  $(x_1, x_2)$  is in at least one of the sets  $A$  or  $B$ . Hence the correct answer is 11B).

**Q12:** The set  $T'$  is graphically depicted as

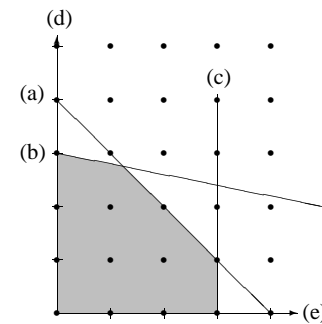


Fortunately, the set  $T'$  is convex, hence it is straightforward to describe as an LP-model:

$$\begin{aligned} x_2 &\geq 0 \\ x_2 &\leq 4 \\ 4x_1 - x_2 &\geq 0 \\ 4x_1 + x_2 &\leq 20 \end{aligned}$$

Hence, the correct answer is 12F).

**Q13:** Since the problem is defined in two variables we may solve it graphically



It is easily seen that  $x_1 = \frac{5}{4}$ ,  $x_2 = \frac{11}{4}$  is the optimal solution, hence 13D) is the correct answer.

**Q14:** The dual problem is

$$\begin{array}{ll} \text{minimize} & 4y_1 + 15y_2 + 3y_3 \\ \text{subject to} & y_1 + y_2 + y_3 \geq 1 \\ & y_1 + 5y_2 \geq 4 \\ & y_1, y_2, y_3 \geq 0 \end{array}$$

From complementary slackness we have

$$\begin{array}{ll} x_1 + x_2 = 4 & \text{hence } y_1 \geq 0 \\ x_1 + 5x_2 = 15 & \text{hence } y_2 \geq 0 \\ x_1 < 3 & \text{hence } y_3 = 0 \end{array}$$

Inserting  $y_3 = 0$  in the dual problem we get

$$\begin{array}{ll} x_1 > 0 & \text{hence } y_1 + y_2 = 1 \\ x_2 > 0 & \text{hence } y_1 + 5y_2 = 4 \end{array}$$

with optimal solution  $y_1 = \frac{1}{4}$  and  $y_2 = \frac{3}{4}$ . Hence the correct table must be 14A).

**Q15:** We have the inequality

$$y_1 + \frac{5}{4}y_3 - \frac{5}{4}y_4 + \frac{1}{4}y_5 = \frac{1}{4}$$

The Gomory cut becomes

$$\frac{1}{4}y_3 + \frac{3}{4}y_4 + \frac{1}{4}y_5 \geq \frac{1}{4}$$

Introducing the slack variables  $y_4 = y_1 + y_2 + y_3 - 1$  and  $y_5 = y_1 + 5y_2 - 4$  we get

$$\frac{1}{4}y_3 + \frac{3}{4}(y_1 + y_2 + y_3 - 1) + \frac{1}{4}(y_1 + 5y_2 - 4) \geq \frac{1}{4}$$

which is reduced to

$$y_1 + 2y_2 + y_3 \geq 2$$

The correct answer is 15E).

**Q16:** The optimal choice of Lagrangian multiplier corresponds to the associated dual variable  $y_2$ . Hence  $\lambda = y_2 = \frac{3}{4}$ , and the correct answer is 16B).

**Q17:** We have the inequality

$$8x_1 + 8x_2 + 7x_3 + 5x_4 + 3x_5 + 2x_6 + 2x_7 \leq 11$$

The cover inequality

$$x_3 + x_4 + x_7 \leq 2$$

is not minimal since if we removed  $x_7$  from the cover we would get the sum  $7 + 5 > 11$ . Hence the cover is not minimal and 17E) is correct.

**Q18:** We have the minimal cover inequality

$$x_4 + x_5 + x_6 + x_7 \leq 3$$

To find the largest value of  $\alpha$  such that the inequality

$$\alpha x_1 + x_4 + x_5 + x_6 + x_7 \leq 3$$

is valid we solve the problem:

$$\begin{array}{ll} \gamma = \text{maximize} & x_4 + x_5 + x_6 + x_7 \\ \text{subject to} & 8 + 5x_4 + 3x_5 + 2x_6 + 2x_7 \leq 11 \\ & x_1, x_2, x_3, x_4, x_5 \in \{0, 1\} \end{array}$$

with optimal solution  $\gamma = 1$ . Hence  $\alpha = 3 - \gamma = 2$ , and thus 18D) is correct.

**Q19:** We first show that  $d \geq n - |B|$ . Construct the  $n - |B|$  points

$$\mathbf{x}^j = (x_1, \dots, x_n) = (0, \dots, \underset{\downarrow}{1}, \dots, 0)$$

for  $j \in N \setminus B$ . Moreover, construct the additional point  $(0, \dots, 0)$ . It is easily checked that the  $n - |B| + 1$  points are affinely independent and that all the points are in  $P$ . Hence  $d \geq n - |B|$ .

We next show that  $d \leq n - |B|$ . This is obvious since any feasible solution must have  $x_j = 0$  for all  $j \in B$ .

**Q20:** Due to the assumption  $a_j \leq b$  for  $j \in N$  we know from Q19 that the dimension of  $P$  is  $\dim(P) = n$ . To show that the inequality

$$x_j \leq 1$$

is facet defining we consider the face  $F = \{x \in P : x_j = 1\}$ . Construct  $n$  affine independent points as follows:

$$\begin{array}{l} \mathbf{x}^j = (x_1, \dots, x_n) = (0, \dots, 0, \underset{\downarrow}{1}, 0, \dots, 0) \\ \mathbf{x}^{ij} = (x_1, \dots, x_n) = (0, \dots, 0, \underset{\downarrow}{1}, 0, \dots, 0, \underset{\downarrow}{1}, 0, \dots, 0) \quad i \in N \setminus \{j\} \end{array}$$

All the points are in  $F$  since  $x_j = 1$  and  $a_i + a_j \leq b$ . Hence  $\dim(F) = n - 1$  and thus the inequality  $x_j \leq 1$  is facet defining.