

Introduktion til Optimering

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Alle hjælpemidler må benyttes dog *ikke* lommeregner eller computer. Besvarelsen kan udarbejdes med blyant eller kuglepen.

Opgavesættet består af 19 opgaver, navngivet Q1-Q19. Opgaverne Q2-Q8, Q10-Q11 og Q14-Q18 er *multiple-choice opgaver*, som har netop ét korrekt svar. For at besvare en sådan opgave skal man, uden yderligere forklaring, skrive opgavens nummer samt den korrekte svarmulighed. For eksempel kan opgave Q2 besvares med "2A". Q1, Q9, Q12 og Q19 er sædvanlige *tekstopgaver*, som skal besvares tilstrækkeligt detaljeret til at løsningsmetoden kan følges. Hvert korrekt svar til en *multiple-choice opgave* giver 4 point. Hvert korrekt svar til en *tekstopgave* giver 10 point. Man kan samlet opnå 100 point.

The question paper consists of 19 questions named Q1-Q19. The questions Q2-Q8, Q10-Q11 and Q14-Q18 are *multiple-choice questions*, which have exactly one correct answer. To answer such a question simply write the number of the question and the correct answer. For example question Q2 can be answered with "2A". Q1, Q9, Q12 og Q19 are ordinary *text questions*, which should be answered sufficiently detailed to make it possible to follow the solution method. Each correct answer to a *multiple-choice question* gives 4 points. Each correct answer to a *text question* gives 10 points. You can obtain 100 points in total.

Linear Programming

Consider the following linear program (LP)

$$\begin{aligned} & \text{maximize } 3x_1 + 4x_2 + 2x_3 \\ & \text{subject to } x_1 + x_2 + x_3 \leq 20 \\ & \quad \quad \quad x_1 + 2x_2 + x_3 \leq 30 \\ & \quad \quad \quad x_1, x_2, x_3 \geq 0 \end{aligned}$$

Let x_4 denote the slack variable in the first constraint of LP, and let x_5 denote the slack variable in the second constraint of LP.

Q 1: (text question). Write down the dual of LP. Solve LP with the primal simplex method starting from the basis consisting of x_4 and x_5 . In every iteration:

- Give the entering non-basic variable and the leaving basic variable.
- Give the reduced costs on the non-basic variables.
- Give both the primal and the dual basic solution.

■

Q 2: If the objective function coefficient on x_2 is changed from 4 to $4 - \delta$, for which values of δ does the optimal basis for LP remain optimal?

- | | |
|--------------------------|-------------------------|
| 2A) $\delta \geq 1$ | 2D) $\delta \geq 3$ |
| 2B) $\delta \in [-2, 1]$ | 2E) $\delta \in [1, 3]$ |
| 2C) $\delta \geq -2$ | 2F) $\delta \leq -1$ |

■

Q 3: If the right hand side in the second constraint is changed from 30 to $30 - \mu$, for which values of μ does the optimal basis for LP remain optimal?

- | | |
|-------------------------|-------------------------|
| 3A) $\mu \geq 0$ | 3D) $\mu \in [-20, 10]$ |
| 3B) $\mu \leq -10$ | 3E) $\mu \geq 20$ |
| 3C) $\mu \in [-10, 10]$ | 3F) $\mu \geq 10$ |

■

Now consider the following linear program (LP²)

$$\begin{aligned} & \text{maximize } 5x_1 + 8x_2 + 7x_3 + 4x_4 + 6x_5 \\ & \text{subject to } 2x_1 + 3x_2 + 3x_3 + 2x_4 + 2x_5 \leq 20 \\ & \quad \quad \quad 3x_1 + 5x_2 + 4x_3 + 2x_4 + 4x_5 \leq 30 \\ & \quad \quad \quad x_1, x_2, x_3, x_4, x_5 \geq 0 \end{aligned}$$

Let x_6 denote the slack variable in the first constraint of LP², and let x_7 denote the slack variable in the second constraint of LP².

Q 4: How many bases of LP^2 are dual infeasible? (Also count bases that involve the two slack variables in the two constraints).

- | | |
|-------|-------|
| 4A) 7 | 4D) 3 |
| 4B) 5 | 4E) 9 |
| 4C) 6 | 4F) 4 |

Q 5: How many bases of LP^2 that have x_1 as a basic variable are primal feasible? (Also count bases that involve the two slack variables in the two constraints).

- | | |
|-------|----------|
| 5A) 1 | 5D) 4 |
| 5B) 3 | 5E) 5 |
| 5C) 6 | 5F) none |

Finally consider the following linear program (LP^3)

$$\begin{aligned} & \text{maximize } 2x_1 + 4x_2 \\ & \text{subject to } 2x_1 + x_2 \leq 2 \\ & \quad \quad \quad x_1 - x_2 \geq 2 \\ & \quad \quad \quad x_1, x_2 \geq 0 \end{aligned}$$

Q 6: Let DLP^3 denote the dual of LP^3 . Which of the following statements are true.

- 6A) Both LP^3 and DLP^3 are feasible and bounded.
- 6B) Both LP^3 and DLP^3 are infeasible.
- 6C) Both LP^3 and DLP^3 are unbounded.
- 6D) LP^3 is infeasible and DLP^3 is unbounded.
- 6E) LP^3 is unbounded and DLP^3 is infeasible.

Cuts

Consider the following IP-problem

$$\begin{aligned} & \text{minimize } -x_1 - x_2 \\ & \text{subject to } -3x_1 + 12x_2 \leq 30 \\ & \quad \quad \quad 6x_1 - 3x_2 \leq 8 \\ & \quad \quad \quad x_1, x_2 \in \mathbb{Z}^+ \end{aligned}$$

By adding slack variables $s_1, s_2 \geq 0$ to the two constraints, and solving the LP-relaxed problem by use of the Simplex algorithm the following two constraints appear:

$$\begin{aligned} x_1 & + \frac{1}{21}s_1 + \frac{4}{21}s_2 = \frac{62}{21} \\ x_2 & + \frac{2}{21}s_1 + \frac{1}{21}s_2 = \frac{68}{21} \end{aligned}$$

Derive a Gomory cut (Wolsey calls it Chvatal-Gomory cut) from the first Simplex equation in which the basis variable is fractional

Q 7: Which inequality appears after the slack variables have been eliminated?

7A) $x_1 \leq 2$

7D) $-2x_1 + 2x_2 \leq 7$

7B) $x_2 \geq 3$

7E) $-x_1 + x_2 \leq 1$

7C) $x_1 \leq \frac{1}{3}$

7F) $3x_1 + 4x_2 \leq 4$

■

Q 8: Derive the inequality $x_2 \leq 3$ as a Chvatal cut. Which multipliers (u_1, u_2) should be used:

8A) (2,-1)

8D) (3,0)

8B) (4,3)

8E) (0,1)

8C) (3,2)

8F) (2,1)

■

Model building

The following puzzle is taken from “The Lady or the Tiger” by Raymond Smullyan: A prisoner is faced with a decision where he must open one of two doors. Behind each door is either a lady or a tiger. They may be both tigers, both ladies or one of each.

Each of the doors has a sign bearing a statement that may be either true or false.

- The statement on door one says, ”Both rooms contain ladies.”
- The statement on door two says, ”Both rooms contain ladies.”

If a lady is in room one then the statement on that door is true, otherwise it is false. If a lady is in room two then the statement on that door is false, otherwise it is true.

Q 9: (text question) (*Hint:* read the whole puzzle carefully.)

- a) If we let the binary variable t_i be 1 iff the sign of room i is true, and the binary variable x_i be 1 iff room i contains a lady, formulate the above constraints as an integer-linear model. (The “raw” inequalities expressing the relations should be reported.)
- b) What is the dimension of the solution space X defined by the model.
- c) Is the inequality $x_1 + x_2 \leq 1$ a facet-defining valid inequality?

■

Multiple-choice knapsack problem

We are given n classes N_1, \dots, N_n of items. Each item $j \in N_i$ has an associated profit p_{ij} and a weight w_{ij} . The objective of the problem is to choose exactly one item from each class N_i such that the profit sum of the chosen items is maximized, while the weight sum of the chosen items cannot exceed a given capacity c .

If we introduce the binary variables x_{ij} to indicate if item j is chosen in class N_i , the problem can be formulated as the following integer-linear model:

$$\begin{aligned} \max \quad & \sum_{i=1}^n \sum_{j \in N_i} p_{ij} x_{ij} \\ \text{s.t.} \quad & \sum_{i=1}^n \sum_{j \in N_i} w_{ij} x_{ij} \leq c \\ & \sum_{j \in N_i} x_{ij} = 1, \quad i = 1, \dots, k \\ & x_{ij} \in \{0, 1\}, \quad i = 1, \dots, n, j \in N_i \end{aligned} \quad (1)$$

In the following instance we have $n = 3$ classes, and the capacity is $c = 9$.

$$N_1 = \{1, 2, 3\} \quad N_2 = \{1, 2, 3\} \quad N_3 = \{1, 2\}$$

| | | | | | | | | | | |
|-----------|---|---|---|-----------|---|---|---|-----------|---|---|
| j | 1 | 2 | 3 | j | 1 | 2 | 3 | j | 1 | 2 |
| $p_{1,j}$ | 0 | 4 | 6 | $p_{2,j}$ | 1 | 2 | 3 | $p_{3,j}$ | 0 | 2 |
| $w_{1,j}$ | 0 | 2 | 3 | $w_{2,j}$ | 2 | 3 | 4 | $w_{3,j}$ | 1 | 3 |

Q 10: Solve the above problem to integer optimality. What is the optimal solution value z ?

10A) $z = 7$

10D) $z = 10$

10B) $z = 8$

10E) $z = 11$

10C) $z = 9$

10F) $z = 12$

We will solve the multiple-choice knapsack problem through *dynamic programming*. Let $f_k(d)$ be an optimal solution to (1), where the capacity is limited to d and where only the first k classes are considered. In other words

$$f_k(d) = \max \left\{ \sum_{i=1}^k \sum_{j \in N_i} p_{ij} x_{ij} : \sum_{i=1}^k \sum_{j \in N_i} w_{ij} x_{ij} \leq d; \sum_{j \in N_i} x_{ij} = 1, i = 1, \dots, k; x_{ij} \in \{0, 1\} \right\} \quad (2)$$

for $k = 0, \dots, n$ og $d = 0, \dots, c$. For $k = 0$ one can only obtain the profit sum 0 for any value of d so we have

$$f_0(d) = 0 \text{ for } d = 0, \dots, c \quad (3)$$

If we know the optimal solution for f_{k-1} , we can find the optimal solutions for f_k by using a dynamic programming recursion.

Q 11: What is the correct recursion? (We assume that $f_{k-1}(d) = -\infty$ if $d < 0$)

11A) $f_k(d) = f_{k-1}(d - w_{kj}) + p_{kj}$

11B) $f_k(d) = \max_{j \in N_k} \{f_{k-1}(d)\}$

11C) $f_k(d) = \max_{j \in N_k} \{f_{k-1}(d) + p_{kj}\}$

11D) $f_k(d) = \max_{j \in N_k} \{f_{k-1}(d - w_{kj}) + p_{kj}\}$

■

Using the recursion we get the following (incomplete) table:

| $d \backslash k$ | 1 | 2 | 3 |
|------------------|---|-----------|-----------|
| 0 | 0 | $-\infty$ | $-\infty$ |
| 1 | 0 | $-\infty$ | $-\infty$ |
| 2 | 4 | 1 | $-\infty$ |
| 3 | 6 | 2 | 1 |
| 4 | 6 | 5 | 2 |
| 5 | 6 | 7 | |
| 6 | 6 | 8 | |
| 7 | 6 | 9 | |
| 8 | 6 | 9 | |
| 9 | 6 | 9 | |

Q 12: What are the five missing entries in column $k = 3$

12A) (5, 7, 8, 9, 9)

12D) (5, 7, 8, 9, 10)

12B) (5, 6, 7, 8, 10)

12E) (5, 5, 9, 9, 10)

12C) (7, 9, 10, 11, 11)

12F) (7, 9, 10, 11, 12)

■

Q 13: (text question)

- Assume that we for each item j in class N_3 wish to find the smallest and largest value of p_{ij} such that the current IP-solution is unchanged. Derive a formal criteria which can be used to determine the limits. It can be advantageous to distinguish between the cases where $x_{3j} = 0$ or $x_{3j} = 1$ in the optimal solution.
- Use the above formal criteria for all $j \in N_3$ to determine the smallest and largest value of p_{ij} such that the current IP-solution is unchanged.

■

Bin Packing

The *bin packing problem* is to pack n items in the smallest number of bins, such that the capacity c of each bin is respected. Each item $j = 1, \dots, n$ has an associated weight w_j . If we use the binary variables x_{ij} to indicate whether item j is placed in bin i , and v_i to indicate whether bin i is used, we get the formulation:

$$\text{minimize } \sum_{i=1}^n v_i \quad (4)$$

$$\text{subject to } \sum_{j=1}^n w_j x_{ij} \leq cv_i, \quad i = 1, \dots, n \quad (5)$$

$$\sum_{i=1}^n x_{ij} \geq 1, \quad j = 1, \dots, n \quad (6)$$

$$x_{ij} \in \{0, 1\}, \quad i, j = 1, \dots, n \quad (7)$$

$$v_i \in \{0, 1\}, \quad i = 1, \dots, n \quad (8)$$

In the following example we have capacity $c = 9$ and $n = 5$ items with the following weights:

| | | | | | |
|-------|---|---|---|---|---|
| j | 1 | 2 | 3 | 4 | 5 |
| w_j | 2 | 4 | 6 | 7 | 8 |

Q 14: Solve the LP-relaxation of the above problem. What is the optimal solution value?

14A) $z = 1$

14D) $z = 4$

14B) $z = 2$

14E) $z = 5$

14C) $z = 3$

14F) $z = 6$

■

Q 15: Find the most violated cover inequality corresponding to constraint (5) for $i = 1$ if we in the primal solution have $v_1 = 1$ and $x_{11} = 1, x_{12} = \frac{1}{2}, x_{13} = \frac{1}{3}, x_{14} = \frac{1}{7}, x_{15} = \frac{1}{4}$ (the remaining variables have some other values).

15A) $x_{11} + x_{12} + x_{13} + x_{14} \leq 3$

15D) $x_{11} + x_{13} + x_{14} + x_{15} \leq 3$

15B) $x_{11} + x_{21} + x_{31} \leq 2$

15E) $x_{11} + x_{15} \leq 1$

15C) $x_{11} + x_{12} + x_{13} + x_{14} + x_{15} \leq 4$

15F) $x_{11} + x_{12} + x_{13} \leq 2$

■

To find a tighter lower bound we consider the Dantzig-Wolfe decomposed problem. Let R be the set of packings of a single bin. Moreover, let a_{ij} be a binary value which indicates whether item j is used in packing i . If we use the binary variable x'_i to determine whether packing $i \in R$ is used, we get the following model:

$$\begin{aligned} \min \quad & \sum_{i \in R} x'_i \\ \text{s.t.} \quad & \sum_{i \in R} a_{ij} x'_i \geq 1 \quad j = 1, \dots, n \\ & x'_i \in \{0, 1\} \quad i \in R \end{aligned} \quad (9)$$

As the above model may be exponentially large, we solve the LP-relaxed problem through column generation. We start with the trivial formulation:

$$\begin{array}{rcl}
 \min & x'_1 + x'_2 + x'_3 + x'_4 + x'_5 & \\
 \text{s.t.} & x'_1 & \geq 1 \\
 & x'_2 & \geq 1 \\
 & x'_3 & \geq 1 \\
 & x'_4 & \geq 1 \\
 & x'_5 & \geq 1
 \end{array} \tag{10}$$

where all variables $x'_j \geq 0$. Let y_j be the dual variable corresponding to constraint (10) for item j .

Q 16: Find the dual variables corresponding to the above problem

- | | |
|--|--|
| 16A) $y_1 = 0, y_2 = 0, y_3 = 0, y_4 = 0, y_5 = 0$ | 16D) $y_1 = 2, y_2 = 1, y_3 = 0, y_4 = 2, y_5 = 0$ |
| 16B) $y_1 = 2, y_2 = 0, y_3 = 1, y_4 = 0, y_5 = 2$ | 16E) $y_1 = 1, y_2 = 1, y_3 = 1, y_4 = 1, y_5 = 0$ |
| 16C) $y_1 = \frac{1}{5}, y_2 = \frac{1}{5}, y_3 = \frac{1}{5}, y_4 = \frac{1}{5}, y_5 = \frac{1}{5}$ | 16F) $y_1 = 1, y_2 = 1, y_3 = 1, y_4 = 1, y_5 = 1$ |

■

Q 17: Which of the following columns is the next to be added to the model (10)? The columns are specified by the involved set of items j

- | | |
|----------|-------------|
| 17A) {1} | 17E) {5} |
| 17B) {2} | 17F) {1, 2} |
| 17C) {3} | 17G) {1, 5} |
| 17D) {4} | 17H) {2, 3} |

■

Add the column to the formulation (10). Solving the LP-model the dual variables become $y_1 = 0, y_2 = y_3 = y_4 = y_5 = 1$.

Q 18: What is the next column to be added to (10)

- | | |
|--|-------------|
| 18A) {1} | 18F) {5} |
| 18B) {2} | 18G) {1, 2} |
| 18C) {3} | 18H) {1, 5} |
| 18D) {4} | 18I) {2, 3} |
| 18E) none (column generation terminates) | |

■

If we instead Lagrange relax constraints (6) in the simple formulation of the bin packing problem using multipliers $\lambda = (\lambda_1, \dots, \lambda_n)$ we get a relaxed problem.

Q 19: (text question)

- a) What is the Lagrangian relaxed problem, and the domain of the multipliers λ ? *Hint*: check that we get a valid lower bound when stating the domain of λ .
- b) What is the best choice of Lagrangian multipliers λ , i.e. the solution to the Lagrangian dual problem for the considered instance. *Hint*: use your knowledge of the strength of the Lagrangian dual problem.
- c) What is the solution value of the Lagrangian relaxed problem for this choice of Lagrangian multipliers?



Solutions

Answer 1 The dual of LP is given by

$$\begin{aligned} & \text{minimize } 20u_1 + 30u_2 \\ & \text{subject to } \quad u_1 + u_2 \geq 3 \quad (x_1) \\ & \quad \quad \quad u_1 + 2u_2 \geq 4 \quad (x_2) \\ & \quad \quad \quad u_1 + u_2 \geq 2 \quad (x_3) \\ & \quad \quad \quad u_1, u_2 \geq 0 \end{aligned}$$

The initial primal basic feasible solution is given by $x_4 = 20$ and $x_5 = 30$. The corresponding dual basic feasible solution is given by $u_1 = 0$ and $u_2 = 0$ (the slacks in the non-negativity constraints on u_1 and u_2 are non-basic). The simplex tableau associated with this basis is

$$\begin{array}{rcccc} -Z & & +3x_1 & +4x_2 & +2x_3 & = & 0 \\ x_4 & & +x_1 & +x_2 & +x_3 & = & 20 \\ x_5 & & +x_1 & +2x_2 & +x_3 & = & 30 \end{array}$$

The reduced costs associated with the nonbasic variables x_1 , x_2 and x_3 are $r_1 = 3$, $r_2 = 4$ and $r_3 = 2$ respectively (the slacks in the dual constraints). The nonbasic variable x_2 has the largest reduced cost, and is selected to enter the basis. The variable x_5 reaches zero first when x_2 is increased from zero (since $\frac{30}{2} < \frac{20}{1}$), and therefore x_5 leaves the basis. The new simplex tableau becomes the following after Gaussian elimination

$$\begin{array}{rcccc} -Z & & +x_1 & & -2x_5 & = & -60 \\ x_4 & & +\frac{1}{2}x_1 & +\frac{1}{2}x_3 & -\frac{1}{2}x_5 & = & 5 \\ x_2 & & +\frac{1}{2}x_1 & +\frac{1}{2}x_3 & +\frac{1}{2}x_5 & = & 15 \end{array}$$

The new primal basic solution is $x_2 = 15$ and $x_4 = 5$. The corresponding dual basic solution is $u_1 = 0$ and $u_2 = 2$ (obtained by setting the dual constraints corresponding to x_2 and x_4 to equalities). The reduced costs corresponding to the nonbasic variables x_1 , x_3 and x_5 are $r_1 = 1$, $r_3 = 0$ and $r_5 = -2$. Since $r_1 > 0$ (the dual constraint corresponding to x_1 is violated), the solution is not optimal. The variable x_1 enters the basis (since $r_1 > 0$). The variable that reaches zero first when x_1 is increased from zero is x_4 (since $\frac{5}{1/2} < \frac{15}{1/2}$), so x_4 leaves the basis. The new simplex tableau obtained after Gaussian elimination is

$$\begin{array}{rcccc} -Z & & -x_3 & -2x_4 & -x_5 & = & -70 \\ x_1 & & +x_3 & +2x_4 & -x_5 & = & 10 \\ x_2 & & & -x_4 & +x_5 & = & 10 \end{array}$$

The new primal basic solution is $x_1 = 10$ and $x_2 = 10$. The corresponding dual basic solution is $u_1 = 2$ and $u_2 = 1$. The reduced costs on the nonbasic variables x_3 , x_4 and x_5 are $r_3 = -1$, $r_4 = -2$ and $r_5 = -1$. Since all reduced costs are negative, the solution is optimal. ■

Answer 2 Now suppose the objective function coefficient on x_2 is changed from 4 to $4 - \delta$. Since we only change an objective function coefficient, the basis consisting of x_1 and x_2 remains primal feasible, so we only need to check dual feasibility. The new corresponding dual basic solution solves the system $u_1 + u_2 = 3$ and $u_1 + 2u_2 = 4 - \delta$, which means $(u_1, u_2) = (2 + \delta, 1 - \delta)$. For this solution to be dual feasible, we must have $u_1 + u_2 \geq 2$, $u_1 \geq 0$ and $u_2 \geq 0$, which means that we must have $\delta \in [-2, 1]$ for the basis (x_1, x_2) to remain an optimal basis (answering question 2). ■

Answer 3 Next suppose we change the right hand side of the second constraint from 30 to $30 - \mu$. Since we only modify the objective function in the problem DLP, the basis consisting of x_1 and x_2 remains dual feasible. We therefore only need to check primal feasibility. The new basic solution obtained from the basis (x_1, x_2) satisfies $x_1 + x_2 = 20$ and $x_1 + 2x_2 = 30 - \mu$, which implies $(x_1, x_2) = (10 + \mu, 10 - \mu)$. For the basis (x_1, x_2) to remain primal feasible, we must have $x_1 = 10 + \mu \geq 0$ and $x_2 = 10 - \mu \geq 0$, or equivalently $\mu \in [-10, 10]$ (answering question 3). ■

Answer 4 The dual DLP² of LP² is given by

$$\begin{aligned} & \text{minimize} && 20u_1 + 30u_2 \\ & \text{subject to} && 2u_1 + 3u_2 \geq 5 && (x_1) \\ & && 3u_1 + 5u_2 \geq 8 && (x_2) \\ & && 3u_1 + 4u_2 \geq 7 && (x_3) \\ & && 2u_1 + 2u_2 \geq 4 && (x_4) \\ & && 2u_1 + 4u_2 \geq 6 && (x_5) \\ & && u_1, u_2 \geq 0 \end{aligned}$$

Observe that $(u_1, u_2) = (1, 1)$ is the dual basic solution for *all* bases that do not include x_6 and x_7 , and that $(u_1, u_2) = (1, 1)$ is dual feasible. Hence every dual infeasible basis must have either $u_1 = 0$ (x_6 basic) or $u_2 = 0$ (x_7 basic). Clearly $(u_1, u_2) = (0, 0)$ is dual infeasible (both x_6 and x_7 basic). The values $(u_1, 0)$ are dual feasible if and only if $u_1 \geq \max\{\frac{5}{2}, \frac{8}{3}, \frac{7}{3}, \frac{4}{2}, \frac{6}{2}\} = \frac{6}{2} = 3$. Since the maximum is achieved uniquely for the dual constraint $2u_1 + 4u_2 \geq 6$ (corresponding to x_5), the only basis that includes x_7 that is dual feasible is the basis consisting of x_5 and x_7 . With a similar argument, the only basis that includes x_6 and is dual feasible is the basis consisting of x_4 and x_6 . It follows that there are 9 bases that are dual infeasible (answering question 4). ■

Answer 5 It is easily checked that the basis consisting of x_1 and x_2 has the primal basic solution $(x_1, x_2) = (10, 0)$. Since the value of x_2 is zero in this primal basic solution, it follows that any other basis that involves x_1 will also have this basic solution. There are a total of 6 bases that involve x_1 , and they are all primal feasible with the same basic solution (answering question 5). ■

Answer 6 We first argue that LP³ is infeasible. If (x_1, x_2) is a feasible solution, then $2x_1 + x_2 \leq 2$ and $x_1 - x_2 \geq 2$, which implies $2x_1 + x_2 \leq 2$ and $-x_1 + x_2 \leq -2$, and therefore (by adding these two inequalities) $x_1 + 2x_2 \leq 0$. Since $x_1, x_2 \geq 0$, this implies $x_1 = x_2 = 0$. However, these values do not satisfy the second constraint, and therefore LP³ is infeasible.

We now argue that DLP^3 is unbounded. The problem DLP^3 is given by

$$\begin{aligned} & \text{minimize } 2u_1 - 2u_2 \\ & \text{subject to } 2u_1 - u_2 \geq 2 \\ & \quad \quad u_1 + u_2 \geq 4 \\ & \quad \quad u_1, u_2 \geq 0 \end{aligned}$$

Since LP^3 is infeasible, DLP^3 is either infeasible or unbounded. Hence, it is sufficient to argue that DLP^3 is feasible. The values $(u_1, u_2) = (2, 2)$ is a feasible solution to DLP^3 (the solution to the system $2u_1 - u_2 = 2$ and $u_1 + u_2 = 4$). Hence DLP^3 is unbounded (answering question 6). ■

Answer 7 From the equations

$$\begin{aligned} x_1 + \frac{1}{21}s_1 + \frac{4}{21}s_2 &= \frac{62}{21} \\ x_2 + \frac{2}{21}s_1 + \frac{1}{21}s_2 &= \frac{68}{21} \end{aligned}$$

we see that the current solution is $x_1 = \frac{62}{21}$ and $x_2 = \frac{68}{21}$. We derive a Gomory cut from the first equation getting

$$\frac{1}{21}s_1 - \left\lfloor -\frac{1}{21}s_1 \right\rfloor + \frac{4}{21}s_2 - \left\lfloor -\frac{4}{21}s_2 \right\rfloor \geq \frac{62}{21} - \left\lfloor -\frac{62}{21} \right\rfloor$$

which after multiplication with 21 gives

$$1s_1 + 4s_2 \geq 20$$

Substituting $s_1 = 30 + 3x_1 - 12x_2$ and $s_2 = 8 - 6x_1 + 3x_2$ we get $30 + 3x_1 - 12x_2 + 4(8 - 6x_1 + 3x_2) \geq 20$ which can be reduced to $62 - 21x_1 \geq 20$ or even shorter, $x_1 \leq 2$. ■

Answer 8 If we use the multipliers $(u_1, u_2) = (2, 1)$ we find:

$$\begin{array}{r} -6x_1 + 24x_2 \leq 60 \\ 6x_1 - 3x_2 \leq 8 \\ \hline + 21x_2 \leq 68 \end{array}$$

dividing by 21 gives $x_2 \leq \lfloor 68/21 \rfloor = 3$. ■

Answer 9 We let the binary variable t_i be 1 iff the sign of room i is true, and the binary variable x_i be 1 iff room i contains a lady.

a) We have the constraints:

$$\begin{aligned} t_1 = 1 &\Leftrightarrow x_1 + x_2 \geq 2 \\ t_2 = 1 &\Leftrightarrow x_1 + x_2 \geq 2 \\ x_1 = 1 &\Leftrightarrow t_1 = 1 \\ x_2 = 1 &\Leftrightarrow t_2 = 0 \end{aligned}$$

Which can be linearized to

$$\begin{aligned}x_1 + x_2 &\geq 2t_1 \\x_1 + x_2 &\leq 1 + t_1 \\x_1 + x_2 &\geq 2t_2 \\x_1 + x_2 &\leq 1 + t_2 \\x_1 &= t_1 \\x_2 &= 1 - t_2 \\x_1, x_2, t_1, t_2 &\in \{0, 1\}\end{aligned}$$

b) By substituting $x_1 = t_1$ and $x_2 = (1 - t_2)$ into the model we get

$$\begin{aligned}t_1 + 1 - t_2 &\geq 2t_1 \\t_1 + 1 - t_2 &\leq 1 + t_1 \\t_1 + 1 - t_2 &\geq 2t_2 \\t_1 + 1 - t_2 &\leq 1 + t_2 \\t_1, t_2 &\in \{0, 1\}\end{aligned}$$

which can be reduced to

$$\begin{aligned}t_1 + t_2 &\leq 1 \\t_2 &\geq 0 \\t_1 - 3t_2 &\geq -1 \\t_1 - 2t_2 &\leq 0 \\t_1, t_2 &\in \{0, 1\}\end{aligned}$$

which has the unique solutions $(x_1, x_2, t_1, t_2) = (0, 1, 0, 0)$. Since we have one affine independent point, the dimension is 0.

c) As the solution space is $X = \{(0, 1, 0, 0)\}$ the inequality $x_1 + x_2 \leq 1$ is valid. However, since the problem is not fully-dimensional it does not make sense to talk about facets. Also, $X \cap \{(x_1, x_2, t_1, t_2) \mid x_1 + x_2 = 1\}$ contains one points. So the inequality is not facet defining.

Notice, that $x_1 = 0$ and $x_2 = 1$, hence the prisoner should choose room 2. ■

Answer 10 We have the instance:

$$N_1 = \{1, 2, 3\} \quad N_2 = \{1, 2, 3\} \quad N_3 = \{1, 2\}$$

$$\begin{array}{c|ccc} j & 1 & 2 & 3 \\ \hline p_{1,j} & 0 & 4 & 6 \\ w_{1,j} & 0 & 2 & 3 \end{array} \quad \begin{array}{c|ccc} j & 1 & 2 & 3 \\ \hline p_{2,j} & 1 & 2 & 3 \\ w_{2,j} & 2 & 3 & 4 \end{array} \quad \begin{array}{c|cc} j & 1 & 2 \\ \hline p_{3,j} & 0 & 2 \\ w_{3,j} & 1 & 3 \end{array}$$

Dynamic programming of the multiple-choice knapsack problem

| | 1 | 2 | 3 |
|---|---|-----------|-----------|
| 0 | 0 | $-\infty$ | $-\infty$ |
| 1 | 0 | $-\infty$ | $-\infty$ |
| 2 | 4 | 1 | $-\infty$ |
| 3 | 6 | 2 | 1 |
| 4 | 6 | 5 | 2 |
| 5 | 6 | 7 | 5 |
| 6 | 6 | 8 | 7 |
| 7 | 6 | 9 | 8 |
| 8 | 6 | 9 | 9 |
| 9 | 6 | 9 | 10 |

Hence, the optimal solution value is $z = 10$. ■

Answer 11 We have the recursion

$$f_k(d) = \max_{j \in N_k} \{f_{k-1}(d - w_{kj}) + p_{kj}\} \quad (11)$$

where it is assumed that $f_{k-1}(d) = -\infty$ if $d < 0$. ■

Answer 12 Using the dynamic programming table from answer 10 we see that the five entries in the last column are (5, 7, 8, 9, 10). ■

Answer 13

a) Assume that we need to find the sensitivity interval for item j' .

If $x_{nj'} = 1$ we need to ensure that

$$f_{n-1}(c - w_{nj'}) + p_{nj'} \geq f_{n-1}(c - w_{nj}) + p_{nj} \text{ for all } j \in N_n, j \neq j'$$

If $x_{nj'} = 0$ we need to ensure that

$$f_{n-1}(c - w_{nj'}) + p_{nj'} \leq f_{n-1}(c - w_{nj}) + p_{nj} \text{ for at least one } j \in N_n, j \neq j'$$

b) Using the above equations we consider class $n = 3$. We have that $x_{31} = 0$ hence

$$p_{\max} = 1; p_{\min} = -\infty$$

Moreover, we have that $x_{32} = 1$, hence

$$p_{\max} = \infty; p_{\min} = 1$$

■

Answer 14 If we use the LP-solution $x_{11} = 1, x_{12} = 1, x_{13} = \frac{1}{2}, x_{23} = \frac{1}{2}, x_{24} = \frac{6}{7}, x_{34} = \frac{1}{7}, x_{35} = 1$, and $v_1 = v_2 = v_3 = 1$ we find $z = 3$. ■

Answer 15 The most violated cover inequality is found by solving the problem

$$\text{minimize } \gamma = (1-1)\delta_1 + (1-\frac{1}{2})\delta_2 + (1-\frac{1}{3})\delta_3 + (1-\frac{1}{7})\delta_4 + (1-\frac{1}{4})\delta_5 \quad (12)$$

$$\text{subject to } 2\delta_1 + 4\delta_2 + 6\delta_3 + 7\delta_4 + 8\delta_5 \geq 10 \quad (13)$$

$$\delta_j \in \{0, 1\}, \quad j = 1, \dots, 5 \quad (14)$$

which has the solution $\delta_1 = \delta_5 = 1$ and objective value $\gamma = \frac{3}{4}$. This leads to the cover inequality

$$x_{11} + x_{15} \leq 1$$

Since $\gamma < 1$ the inequality is violated. ■

Answer 16 The dual variables are $y_1 = 1, y_2 = 1, y_3 = 1, y_4 = 1, y_5 = 1$ ■

Answer 17 To find the next column to be added to the model, we solve the pricing problem

$$\text{minimize } 1 - (1x_1 + 1x_2 + 1x_3 + 1x_4 + 1x_5) \quad (15)$$

$$\text{subject to } 2x_1 + 4x_2 + 6x_3 + 7x_4 + 8x_5 \leq 9 \quad (16)$$

$$x_1, x_2, x_3, x_4, x_5 \in \{0, 1\} \quad (17)$$

which has the solution $x_1 = x_2 = 1$, with solution value $z = -1$. (other solutions exist, but these were not proposed). ■

Answer 18 Solving the above pricing problem we find

$$\text{minimize } 1 - (0x_1 + 1x_2 + 1x_3 + 1x_4 + 1x_5) \quad (18)$$

$$\text{subject to } 2x_1 + 4x_2 + 6x_3 + 7x_4 + 8x_5 \leq 9 \quad (19)$$

$$x_1, x_2, x_3, x_4, x_5 \in \{0, 1\} \quad (20)$$

which has the solution value $z = 0$. Hence, the column generation terminates. ■

Answer 19

a) If we Lagrange relax constraint (6) using multipliers $\lambda \geq 0$ we get the objective function

$$\text{minimize } \sum_{i=1}^n v_i - \sum_{j=1}^n \lambda_j \left(\sum_{i=1}^n x_{ij} - 1 \right)$$

which can be reduced to

$$\text{minimize } \sum_{i=1}^n \left(v_i - \sum_{j=1}^n \lambda_j x_{ij} \right) + \sum_{j=1}^n \lambda_j$$

- b) Since the strength of the Lagrangian dual is the same as the strength of the LP-relaxed Dantzig-Wolfe decomposed problem, the optimal choice of Lagrangian multipliers corresponds to the dual variables of the LP-relaxed Dantzig-Wolfe decomposed problem.

This means that $\lambda_1 = 0$ while $\lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = 1$.

- c) The relaxed problem is:

$$\text{minimize } \sum_{i=1}^n \left(v_i - \sum_{j=1}^n \lambda_j x_{ij} \right) + \sum_{j=1}^n \lambda_j \quad (21)$$

$$\text{subject to } \sum_{j=1}^n w_j x_{ij} \leq c v_i, \quad i = 1, \dots, n \quad (22)$$

$$x_{ij} \in \{0, 1\}, \quad i, j = 1, \dots, n \quad (23)$$

$$v_i \in \{0, 1\}, \quad i = 1, \dots, n \quad (24)$$

Notice that the problem is separable, i.e. all variables with index i are independent of variables with index i' . Hence, we may solve the problem for each i independently.

If $v_i = 0$ then $x_{ij} = 0$ for all j due to (22). If $v_i = 1$ then we have the optimization problem

$$\text{minimize } v_i - \sum_{j=1}^n \lambda_j x_{ij} \quad (25)$$

$$\text{subject to } \sum_{j=1}^n w_j x_{ij} \leq c \quad (26)$$

$$x_{ij} \in \{0, 1\}, \quad j = 1, \dots, n \quad (27)$$

which can be recognized as the last pricing problem, with solution value 0. Hence the objective is

$$\sum_{j=1}^n \lambda_j = 4$$

■