Introduktion til Optimering

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Alle hjælpemidler må benyttes dog *ikke* lommeregner eller computer. Besvarelsen kan udarbejdes med blyant eller kuglepen.

Opgavesættet består af 19 opgaver, navngivet Q1-Q19. Opgaverne Q2-Q8, Q10-Q11 og Q14-Q18 er *multiple-choice opgaver*, som har netop ét korrekt svar. For at besvare en sådan opgave skal man, uden yderligere forklaring, skrive opgavens nummer samt den korrekte svarmulighed. For eksempel kan opgave Q2 besvares med “2A”. Q1, Q9, Q12 og Q19 er sædvanlige *tekstopgaver*, som skal besvares tilstrækkeligt detaljeret til at løsningsmetoden kan følges. Hvert korrekt svar til en *multiple-choice opgave* giver 4 point. Hvert korrekt svar til en *tekstopgave* giver 10 point. Man kan samlet opnå 100 point.

The question paper consists of 19 questions named Q1-Q19. The questions Q2-Q8, Q10-Q11 and Q14-Q18 are *multiple-choice questions*, which have exactly one correct answer. To answer such a question simply write the number of the question and the correct answer. For example question Q2 can be answered with “2A”. Q1, Q9, Q12 og Q19 are ordinary *text questions*, which should be answered sufficiently detailed to make it possible to follow the solution method. Each correct answer to a *multiple-choice question* gives 4 points. Each correct answer to a *text question* gives 10 points. You can obtain 100 points in total.

Linear Programming

Consider the following linear program (LP)

$$\begin{array}{lrrrl} \text{maximize} & 3x_1 & +4x_2 & +2x_3 & \\ \text{subject to} & x_1 & +x_2 & +x_3 & \leq 20 \\ & x_1 & +2x_2 & +x_3 & \leq 30 \\ & & x_1, x_2, x_3 \geq 0 & & \end{array}$$

Let x_4 denote the slack variable in the first constraint of LP, and let x_5 denote the slack variable in the second constraint of LP.

Q 1: **(text question)**. Write down the dual of LP. Solve LP with the primal simplex method starting from the basis consisting of x_4 and x_5. In every iteration:

a) Give the entering non-basic variable and the leaving basic variable.

b) Give the reduced costs on the non-basic variables.

c) Give both the primal and the dual basic solution.

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Q 2: If the objective function coefficient on x_2 is changed from 4 to $4-\delta$, for which values of δ does the optimal basis for LP remain optimal?

2A) $\delta \geq 1$

2B) $\delta \in [-2, 1]$

2C) $\delta \geq -2$

2D) $\delta \geq 3$

2E) $\delta \in [1, 3]$

2F) $\delta \leq -1$

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Q 3: If the right hand side in the second constraint is changed from 30 to $30-\mu$, for which values of μ does the optimal basis for LP remain optimal?

3A) $\mu \geq 0$

3B) $\mu \leq -10$

3C) $\mu \in [-10, 10]$

3D) $\mu \in [-20, 10]$

3E) $\mu \geq 20$

3F) $\mu \geq 10$

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Now consider the following linear program (LP^2)

$$\begin{array}{lrrrrrl} \text{maximize} & 5x_1 + 8x_2 & +7x_3 + 4x_4 & +6x_5 & \\ \text{subject to} & 2x_1 + 3x_2 & +3x_3 + 2x_4 & +2x_5 & \leq 20 \\ & 3x_1 + 5x_2 & +4x_3 + 2x_4 & +4x_5 & \leq 30 \\ & x_1, x_2, x_3, x_4, x_5 \geq 0 & & & \end{array}$$

Let x_6 denote the slack variable in the first constraint of LP^2, and let x_7 denote the slack variable in the second constraint of LP^2.

Q 4: How many bases of LP^2 are dual infeasible? (Also count bases that involve the two slack variables in the two constraints).

4A) 7
4B) 5
■ 4C) 6
4D) 3
4E) 9
4F) 4

Q 5: How many bases of LP^2 that have x_1 as a basic variable are primal feasible? (Also count bases that involve the two slack variables in the two constraints).

5A) 1
5B) 3
■ 5C) 6
5D) 4
5E) 5
5F) none

Finally consider the following linear program (LP^3)

$$\begin{array}{rrrl} \text{maximize} & 2x_1 & +4x_2 & \\ \text{subject to} & 2x_1 & +x_2 & \leq 2 \\ & x_1 & -x_2 & \geq 2 \\ & & x_1, x_2 \geq 0 & \end{array}$$

Q 6: Let DLP^3 denote the dual of LP^3. Which of the following statements are true.

6A) Both LP^3 and DLP^3 are feasible and bounded.

6B) Both LP^3 and DLP^3 are infeasible.

6C) Both LP^3 and DLP^3 are unbounded.

6D) LP^3 is infeasible and DLP^3 is unbounded.

■ 6E) LP^3 is unbounded and DLP^3 is infeasible.

Cuts

Consider the following IP-problem

$$\begin{array}{rrcrl} \text{minimize} & -x_1 & - & x_2 & \\ \text{subject to} & -3x_1 & + & 12x_2 & \leq 30 \\ & 6x_1 & - & 3x_2 & \leq 8 \\ & & x_1, x_2 \in \mathbb{Z}^+ & & \end{array}$$

By adding slack variables $s_1, s_2 \geq 0$ to the two constraints, and solving the LP-relaxed problem by use of the Simplex algorithm the following two constraints appear:

$$\begin{array}{rrrrl} x_1 & & + \frac{1}{21}s_1 & + \frac{4}{21}s_2 & = \frac{62}{21} \\ & x_2 & + \frac{2}{21}s_1 & + \frac{1}{21}s_2 & = \frac{68}{21} \end{array}$$

Derive a Gomory cut (Wolsey calls it Chvatal-Gomory cut) from the first Simplex equation in which the basis variable is fractional

Q 7: Which inequality appears after the slack variables have been eliminated?

7A) $x_1 \leq 2$
7B) $x_2 \geq 3$
7C) $x_1 \leq \frac{1}{3}$
7D) $-2x_1 + 2x_2 \leq 7$
7E) $-x_1 + x_2 \leq 1$
7F) $3x_1 + 4x_2 \leq 4$

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Q 8: Derive the inequality $x_2 \leq 3$ as a Chvatal cut. Which multipliers (u_1, u_2) should be used:

8A) (2,-1)
8B) (4,3)
8C) (3,2)
8D) (3,0)
8E) (0,1)
8F) (2,1)

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Model building

The following puzzle is taken from “The Lady or the Tiger” by Raymond Smullyan: A prisoner is faced with a decision where he must open one of two doors. Behind each door is either a lady or a tiger. They may be both tigers, both ladies or one of each.

Each of the doors has a sign bearing a statement that may be either true or false.

- The statement on door one says, ”Both rooms contain ladies.”
- The statement on door two says, ”Both rooms contain ladies.”

If a lady is in room one then the statement on that door is true, otherwise it is false. If a lady is in room two then the statement on that door is false, otherwise it is true.

Q 9: **(text question)** (*Hint:* read the whole puzzle carefully.)

a) If we let the binary variable t_i be 1 iff the sign of room i is true, and the binary variable x_i be 1 iff room i contains a lady, formulate the above constraints as an integer-linear model. (The “raw” inequalities expressing the relations should be reported.)

b) What is the dimension of the solution space X defined by the model.

c) Is the inequality $x_1 + x_2 \leq 1$ a facet-defining valid inequality?

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Multiple-choice knapsack problem

We are given n classes $N_1, \ldots, N_n$ of items. Each item $j \in N_i$ has an associated profit p_{ij} and a weight w_{ij}. The objective of the problem is to choose exactly one item from each class N_i such that the profit sum of the chosen items is maximized, while the weight sum of the chosen items cannot exceed a given capacity c.

If we introduce the binary variables x_{ij} to indicate if item j is chosen in class N_i, the problem can be formulated as the following integer-linear model:

$$\begin{aligned} \max \ & \sum_{i=1}^{n} \sum_{j \in N_i} p_{ij} x_{ij} \\ \text{s.t.} \ & \sum_{i=1}^{n} \sum_{j \in N_i} w_{ij} x_{ij} \le c \\ & \sum_{j \in N_i} x_{ij} = 1, \quad i = 1, \ldots, k \\ & x_{ij} \in \{0,1\}, \quad i = 1, \ldots, n, j \in N_i \end{aligned} \tag{1}$$

In the following instance we have $n = 3$ classes, and the capacity is $c = 9$.

$N_1 = \{1,2,3\}$ $\quad N_2 = \{1,2,3\}$ $\quad N_3 = \{1,2\}$

j	1	2	3
$p_{1,j}$	0	4	6
$w_{1,j}$	0	2	3

j	1	2	3
$p_{2,j}$	1	2	3
$w_{2,j}$	2	3	4

j	1	2
$p_{3,j}$	0	2
$w_{3,j}$	1	3

Q 10: Solve the above problem to integer optimality. What is the optimal solution value z?

10A) $z = 7$
10B) $z = 8$
10C) $z = 9$
10D) $z = 10$
10E) $z = 11$
10F) $z = 12$

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We will solve the multiple-choice knapsack problem through *dynamic programming*. Let $f_k(d)$ be an optimal solution to (1), where the capacity is limited to d and where only the first k classes are considered. In other words

$$f_k(d) = \max \left\{ \sum_{i=1}^{k} \sum_{j \in N_i} p_{ij} x_{ij} : \sum_{i=1}^{k} \sum_{j \in N_i} w_{ij} x_{ij} \le d; \ \sum_{j \in N_i} x_{ij} = 1, i = 1, \ldots, k; \ x_{ij} \in \{0,1\} \right\} \tag{2}$$

for $k = 0, \ldots, n$ og $d = 0, \ldots, c$. For $k = 0$ one can only obtain the profit sum 0 for any value of d so we have

$$f_0(d) = 0 \text{ for } d = 0, \ldots, c \tag{3}$$

If we know the optimal solution for f_{k-1}, we can find the optimal solutions for f_k by using a dynamic programming recursion.

Q 11: What is the correct recursion? (We assume that $f_{k-1}(d) = -\infty$ if $d < 0$)

11A) $f_k(d) = f_{k-1}(d - w_{kj}) + p_{kj}$

11B) $f_k(d) = \max_{j \in N_k} \{ f_{k-1}(d) \}$

11C) $f_k(d) = \max_{j \in N_k} \{ f_{k-1}(d) + p_{kj} \}$

11D) $f_k(d) = \max_{j \in N_k} \{ f_{k-1}(d - w_{kj}) + p_{kj} \}$

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Using the recursion we get the following (incomplete) table:

$d \backslash k$	1	2	3
0	0	$-\infty$	$-\infty$
1	0	$-\infty$	$-\infty$
2	4	1	$-\infty$
3	6	2	1
4	6	5	2
5	6	7	
6	6	8	
7	6	9	
8	6	9	
9	6	9	

Q 12: What are the five missing entries in column $k = 3$

12A) $(5,7,8,9,9)$

12B) $(5,6,7,8,10)$

12C) $(7,9,10,11,11)$

12D) $(5,7,8,9,10)$

12E) $(5,5,9,9,10)$

12F) $(7,9,10,11,12)$

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Q 13: **(text question)**

a) Assume that we for each item j in class N_3 wish to find the smallest and largest value of p_{ij} such that the current IP-solution is unchanged. Derive a formal criteria which can be used to determine the limits. It can be advantageous to distinguish between the cases where $x_{3j} = 0$ or $x_{3j} = 1$ in the optimal solution.

b) Use the above formal criteria for all $j \in N_3$ to determine the smallest and largest value of p_{ij} such that the current IP-solution is unchanged.

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Bin Packing

The *bin packing problem* is to pack n items in the smallest number of bins, such that the capacity c of each bin is respected. Each item $j = 1, \ldots, n$ has an associated weight w_j. If we use the binary variables x_{ij} to indicate whether item j is placed in bin i, and v_i to indicate whether bin i is used, we get the formulation:

$$\text{minimize} \quad \sum_{i=1}^{n} v_i \tag{4}$$

$$\text{subject to} \quad \sum_{j=1}^{n} w_j x_{ij} \le c v_i, \quad i = 1, \ldots, n \tag{5}$$

$$\sum_{i=1}^{n} x_{ij} \ge 1, \quad j = 1, \ldots, n \tag{6}$$

$$x_{ij} \in \{0,1\}, \quad i, j = 1, \ldots, n \tag{7}$$

$$v_i \in \{0,1\}, \quad i = 1, \ldots, n \tag{8}$$

In the following example we have capacity $c = 9$ and $n = 5$ items with the following weights:

j	1	2	3	4	5
w_j	2	4	6	7	8

Q 14: Solve the LP-relaxation of the above problem. What is the optimal solution value?

14A) $z = 1$
14B) $z = 2$
14C) $z = 3$
14D) $z = 4$
14E) $z = 5$
14F) $z = 6$

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Q 15: Find the most violated cover inequality corresponding to constraint (5) for $i = 1$ if we in the primal solution have $v_1 = 1$ and $x_{11} = 1$, $x_{12} = \frac{1}{2}$, $x_{13} = \frac{1}{3}$, $x_{14} = \frac{1}{7}$, $x_{15} = \frac{1}{4}$ (the remaining variables have some other values).

15A) $x_{11} + x_{12} + x_{13} + x_{14} \le 3$
15B) $x_{11} + x_{21} + x_{31} \le 2$
15C) $x_{11} + x_{12} + x_{13} + x_{14} + x_{15} \le 4$
15D) $x_{11} + x_{13} + x_{14} + x_{15} \le 3$
15E) $x_{11} + x_{15} \le 1$
15F) $x_{11} + x_{12} + x_{13} \le 2$

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To find a tighter lower bound we consider the Dantzig-Wolfe decomposed problem. Let R be the set of packings of a single bin. Moreover, let a_{ij} be a binary value which indicates whether item j is used in packing i. If we use the binary variable x'_i to determine whether packing $i \in R$ is used, we get the following model:

$$\begin{aligned} \min \quad & \sum_{i \in R} x'_i \\ \text{s.t} \quad & \sum_{i \in R} a_{ij} x'_i \ge 1 \quad j = 1, \ldots, n \\ & x'_i \in \{0,1\} \quad i \in R \end{aligned} \tag{9}$$

As the above model may be exponentially large, we solve the LP-relaxed problem through column generation. We start with the trivial formulation:

$$\begin{array}{llllll} \min & x'_1 + x'_2 + x'_3 + x'_4 + x'_5 \\ \text{s.t.} & x'_1 & & & & \geq 1 \\ & & x'_2 & & & \geq 1 \\ & & & x'_3 & & \geq 1 \\ & & & & x'_4 & \geq 1 \\ & & & & & x'_5 \geq 1 \end{array} \tag{10}$$

where all variables $x'_j \geq 0$. Let y_j be the dual variable corresponding to constraint (10) for item j.

Q 16: Find the dual variables corresponding to the above problem

16A) $y_1 = 0, y_2 = 0, y_3 = 0, y_4 = 0, y_5 = 0$
16B) $y_1 = 2, y_2 = 0, y_3 = 1, y_4 = 0, y_5 = 2$
16C) $y_1 = \frac{1}{5}, y_2 = \frac{1}{5}, y_3 = \frac{1}{5}, y_4 = \frac{1}{5}, y_5 = \frac{1}{5}$
16D) $y_1 = 2, y_2 = 1, y_3 = 0, y_4 = 2, y_5 = 0$
16E) $y_1 = 1, y_2 = 1, y_3 = 1, y_4 = 1, y_5 = 0$
16F) $y_1 = 1, y_2 = 1, y_3 = 1, y_4 = 1, y_5 = 1$

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Q 17: Which of the following columns is the next to be added to the model (10)? The columns are specified by the involved set of items j

17A) $\{1\}$
17B) $\{2\}$
17C) $\{3\}$
17D) $\{4\}$
17E) $\{5\}$
17F) $\{1,2\}$
17G) $\{1,5\}$
17H) $\{2,3\}$

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Add the column to the formulation (10). Solving the LP-model the dual variables become $y_1 = 0, y_2 = y_3 = y_4 = y_5 = 1$.

Q 18: What is the next column to be added to (10)

18A) $\{1\}$
18B) $\{2\}$
18C) $\{3\}$
18D) $\{4\}$
18E) none (column generation terminates)
18F) $\{5\}$
18G) $\{1,2\}$
18H) $\{1,5\}$
18I) $\{2,3\}$

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If we instead Lagrange relax constraints (6) in the simple formulation of the bin packing problem using multipliers $\lambda = (\lambda_1, \dots, \lambda_n)$ we get a relaxed problem.

Q 19: (**text question**)

a) What is the Lagrangian relaxed problem, and the domain of the multipliers λ? *Hint:* check that we get a valid lower bound when stating the domain of λ.

b) What is the best choice of Lagrangian multipliers λ, i.e. the solution to the Lagrangian dual problem for the considered instance. *Hint:* use your knowledge of the strength of the Lagrangian dual problem.

c) What is the solution value of the Lagrangian relaxed problem for this choice of Lagrangian multipliers?

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