Answers, Written Exam, 14 December 2001, David Pisinger

Answer 11 Constraint 11.d) saying $x_1+x_4+x_5 \le 2$ is not a minimal cover inequality since x_4+x_5 is also a cover. All the other constraints are minimal cover inequalities. ■

Answer 12 The constraint

$$x_2 + x_3 + x_4 + \alpha x_5 \le 2 \tag{2}$$

is valid if it is satisfied for all binary values of x_1, x_2, x_3, x_4, x_5 satisfying

$$3x_1 + 7x_2 + 8x_3 + 6x_4 + 10x_5 \le 15 \tag{3}$$

If we set $x_5 = 1$ and $x_1 = 0$ then we demand that $x_2 + x_3 + x_4 + \alpha x_5 \le 2$ is valid when $7x_2 + 8x_3 + 6x_4 + 10 \le 15$. The largest value of $x_2 + x_3 + x_4$ subject to the given constraint is found by solving the following maximization problem

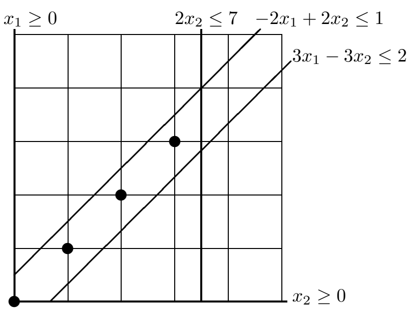
$$\begin{array}{lrcrcrcl} \text{maximize} & \gamma = x_2 & + & x_3 & + & x_4 & & \\ \text{subject to} & 7x_2 & + & 8x_3 & + & 6x_4 & \le & 5 \\ & \multicolumn{5}{l}{x_2, x_3, x_4 \in \{0,1\}} & & \end{array}$$

which has the solution $\gamma = 0$. Hence knowing that $x_2 + x_3 + x_4$ never will exceed 0 we may set $\alpha = 2 - \gamma = 2$.

Now, setting $x_5 = 0$ we notice that the resulting constraint (2) is trivially satisfied, as it was given in advance.

Setting $x_1 = 1$ we notice that (3) gets tighter, and hence (2) is valid whenever (3) is satisfied. Hence answer 12.c) is correct. ■

Answer 13 As $x_3 \ge 0$ and $x_3 \le \frac{1}{2}$ and x_3 is integer, then we may conclude that $x_3 = 0$. The remaining problem has two variables and hence may be depicted as follows



The convex hull enclosing the integer solutions (the black points) is a line, and hence it is only possible to find two affine independent points in P. The dimension is $\dim(P) = 1$ and thus the answer is 13.b). ■

Answer 14 We notice that the constraint matrix is totally unimodular, as every row contains at most two nonzero coefficients, and all coefficients are $0, 1, -1$. Placing columns 1 and 3 into set P_1 and columns 2 and 4 into set P_2 we see that property P is satisfied.

Now, knowing that the constraint matrix is totally unimodular, and noting that the right-hand sides are all integers, an optimal solution must be integral (if it exists). As the coefficients in the objective function are all integers, the solution value will also be integral. The only proposed integer solution is 14.e).

One could also solve the problem using the simplex algorithm, getting the solution $x_1 = 7$, $x_2 = 2$, $x_3 = x_4 = 0$ with objective value $z = 60$. ■

Answer 15 The first inequality with a fractional solution value is

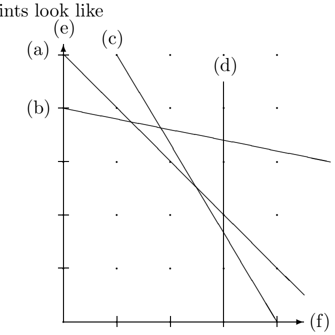
$$\frac{8}{5}x_1 + s_1 + \frac{6}{5}s_2 - \frac{28}{5}s_3 = \frac{136}{5}$$

The Gomory cut becomes

$$\frac{3}{5}x_1 + 0s_1 + \frac{1}{5}s_2 + \frac{2}{5}s_3 \ge \frac{1}{5}$$

hence 15.c) is correct. ■

Answer 16 The constraints look like



It is easily seen that inequalities (d), (e), (f) are facet defining and inequality (a) is redundant. Hence the correct answer is 16.b). ■

Answer 17 Using the proposed multipliers from answer 17.f) we get

$$\begin{array}{rcrcr} \frac{3}{4}x_1 & + & \frac{3}{4}x_2 & \le & \frac{15}{4} \\ \frac{1}{4}x_1 & + & \frac{5}{4}x_2 & \le & \frac{20}{4} \\ \hline x_1 & + & 2x_2 & \le & \frac{35}{4} \end{array}$$

rounding down the coefficients on both sides we get

$$x_1 + 2x_2 \le 8$$

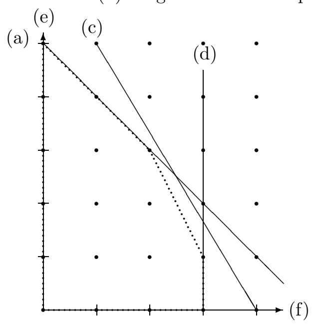
■

Answer 18 Lagrangian relaxing constraint (b) using multiplier $\lambda \ge 0$ we get

$$\begin{array}{lrcrcr} \text{maximize} & x_1 & + & 4x_2 & - & \lambda(x_1 + 5x_2 - 20) \\ \text{subject to} & x_1 & + & x_2 & \le & 5 \\ & 5x_1 & + & 3x_2 & \le & 20 \\ & x_1 & & & \le & 3 \\ & \multicolumn{5}{l}{x_1, x_2 \ge 0 \text{ integer}} \end{array}$$

reducing the objective function, one gets the problem in 18.a). ■

Answer 19 Having relaxed constraint (b) we get the solution space



where the dotted lines mark the convex hull. The lagrangian dual function has discontinuities when the optimal solution (x_1, x_2) of the lagrangian relaxed problem change. This happens when the isoprofit line is parallel to one of the sides of the convex hull, i.e. when $\lambda = \frac{3}{4}$, $\lambda = \frac{7}{9}$, $\lambda = \frac{4}{5}$ and when $\lambda = 1$.

The objective function at these points is

$$\begin{array}{lcl} \lambda = 0 & : & x_1 + 4x_2 \\ \lambda = \frac{3}{4} & : & \frac{1}{4}x_1 + \frac{1}{4}x_2 + 15 \\ \lambda = \frac{7}{9} & : & \frac{2}{9}x_1 + \frac{1}{9}x_2 + \frac{140}{9} \\ \lambda = \frac{4}{5} & : & \frac{1}{5}x_1 + 16 \\ \lambda = 1 & : & -x_2 + 20 \end{array}$$

For $\lambda = 0$, the optimal solution is $(0, 5)$ having objective 20.00.
For $\lambda = \frac{3}{4}$, one of the optimal solutions is $(2, 3)$ having objective $16 + \frac{1}{4} = 16.25$.
For $\lambda = \frac{7}{9}$, one of the optimal solutions is $(3, 1)$ having objective $16 + \frac{1}{3} = 16.33$.
For $\lambda = \frac{4}{5}$, one of the optimal solutions is $(3, 0)$ having objective $16 + \frac{3}{5} = 16.60$.
For $\lambda = 1$, one of the optimal solutions is $(0, 0)$ having objective 20.00.

Having these breakpoints it is easy to draw the lagrangian dual function in the interval 0 to 1. ■

Answer 20 To formulate the problem as a MIP-model we introduce the variables

- $\ell_{ij} = 1$ iff rectangle i is located left to j
- $r_{ij} = 1$ iff rectangle i is located right to j
- $b_{ij} = 1$ iff rectangle i is located below j
- $a_{ij} = 1$ iff rectangle i is located above j
- (x_i, y_i) are the lower left coordinates of rectangle i.

To ensure that no two rectangles overlap, we demand that

$$\ell_{ij} + r_{ij} + a_{ij} + b_{ij} \geq 1 \qquad i, j \in I, i < j$$

Depending on the relative position, the coordinates must satisfy

$$\begin{array}{lcl} \ell_{ij} = 1 & \Rightarrow & x_i + w_i \leq x_j \\ r_{ij} = 1 & \Rightarrow & x_j + w_j \leq x_i \\ b_{ij} = 1 & \Rightarrow & y_i + h_i \leq y_j \\ a_{ij} = 1 & \Rightarrow & y_j + h_j \leq y_i \end{array}$$

An upper bound M on $(x_i + w_i) - x_j$ is $M = W \leq V$. Hence we may model the first constraint as

$$x_i - x_j + V\ell_{ij} \leq V - w_i$$

The following constraints are modelled in a similar way.

No part of the rectangles may exceed the sheet, hence we demand that $0 \leq x_i \leq W - w_i$ and $0 \leq y_i \leq H - h_i$.

The problem can now be formulated as the following MIP problem

$$\begin{array}{lll} \min & W & \\ \text{s.t.} & \ell_{ij} + r_{ij} + b_{ij} + a_{ij} \geq 1 & i, j \in I, i < j \\ & x_i - x_j + V\ell_{ij} \leq V - w_i & i, j \in I, i < j \\ & x_j - x_i + Vr_{ij} \leq V - w_j & i, j \in I, i < j \\ & y_i - y_j + Hb_{ij} \leq H - h_i & i, j \in I, i < j \\ & y_j - y_i + Ha_{ij} \leq H - h_j & i, j \in I, i < j \\ & x_i \geq 0 & i \in I \\ & W - x_i \geq w_i & i \in I \\ & y_i \geq 0 & i \in I \\ & y_i \leq H - h_i & i \in I \\ & \ell_{ij}, r_{ij}, b_{ij}, a_{ij} \in \{0, 1\} & i, j \in I, i < j \end{array} \qquad (4)$$

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