Introduction to Optimization:

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Your assignment

20 different questions Q1-Q20 are posed on the subsequent pages. Q1-Q8 and Q11-Q18 are *multiple choice questions*. For each of these, the only correct answer is one of the answers proposed. To answer a specific question, you are requested without further explanation to write, for example, "7.b" as your answer to question Q7. Q9-Q10 and Q19-Q20 are ordinary *text questions*. Each correct answer to a multiple choice question gives 4 points, to a text question gives 9 points. The maximum score is thus 100 points.

Note: only the last 10 questions are available

Cover inequalities

A problem where all variables $x_1, x_2, x_3, x_4, x_5 \in \{0, 1\}$ contains the following constraint:

$$3x_1 + 7x_2 + 8x_3 + 6x_4 + 10x_5 \leq 15$$

Q 11: Which of the following inequalities is *not* a minimal cover inequality

11.a) $x_2 + x_3 + x_4 \leq 2$
11.b) $x_3 + x_5 \leq 1$
11.c) $x_2 + x_5 \leq 1$
11.d) $x_1 + x_4 + x_5 \leq 2$
11.e) $x_1 + x_2 + x_4 \leq 2$
11.f) $x_1 + x_2 + x_3 \leq 2$

Q 12: What is the largest value of α such that the constraint

$$x_2 + x_3 + x_4 + \alpha x_5 \leq 2$$

is a valid inequality

12.a) $\alpha = 0$
12.b) $\alpha = 1$
12.c) $\alpha = 2$
12.d) $\alpha = 3$
12.e) $\alpha = 4$
12.f) $\alpha = 5$

Dimension

A polyhedron P is defined by the following constraints

$$P = \operatorname{conv}\left\{\begin{array}{rcrcrcr} -2x_1 & + & 2x_2 & & & \leq & 1 \\ & & & & 2x_3 & \leq & 1 \\ & & 2x_2 & & & \leq & 7 \\ 3x_1 & - & 3x_2 & & & \leq & 2 \\ \multicolumn{5}{l}{x_1, x_2, x_3 \geq 0, \text{ integer}} \end{array}\right\}$$

Q 13: What is the dimension $\dim(P)$ of the polyhedron P

13.a) $\dim(P) = 0$
13.b) $\dim(P) = 1$
13.c) $\dim(P) = 2$
13.d) $\dim(P) = 3$
13.e) $\dim(P) = 4$
13.f) $\dim(P) = 5$

Production planning

Professor P_1 (also known as Professor Williams) is planning the production of Rolls-Royce cars. There are four models available, but complicated production constraints limit the output of the factory. In order to maximize the profit, Professor P_1 writes up the following maximization problem

$$\begin{array}{lrcrcrcrcr} \text{maximize} & 8x_1 & + & 2x_2 & - & 3x_3 & + & 5x_4 & & \\ \text{subject to} & 1x_1 & + & 0x_2 & + & 0x_3 & + & 1x_4 & \leq & 7 \\ & 0x_1 & - & 1x_2 & - & 1x_3 & + & 0x_4 & \leq & 3 \\ & -1x_1 & + & 0x_2 & + & 1x_3 & + & 0x_4 & \leq & -1 \\ & 0x_1 & + & 1x_2 & + & 0x_3 & - & 1x_4 & \leq & 2 \\ & 1x_1 & + & 1x_2 & + & 0x_3 & + & 0x_4 & \leq & 12 \\ & 0x_1 & + & 0x_2 & - & 1x_3 & - & 1x_4 & \leq & 7 \\ & \multicolumn{9}{l}{x_1, x_2, x_3, x_4 \geq 0} \end{array}$$

Q 14: What is the optimal solution value of the problem

14.a) $z = \frac{92}{3}$
14.b) $z = \frac{78}{5}$
14.c) $z = \frac{1}{45}$
14.d) $z = 38\frac{3}{7}$
14.e) $z = 60$
14.f) $z = \frac{157}{2}$

Gomory cut

The following optimization problem is given

$$\begin{array}{lrcrcrcl} \text{maximize} & 60x_1 & + & 40x_2 & + & 20x_3 & & \\ \text{subject to} & 8x_1 & + & 6x_2 & + & x_3 & \leq & 48 \\ & 8x_1 & + & 4x_2 & + & 3x_3 & \leq & 40 \\ & 4x_1 & + & 3x_2 & + & x_3 & \leq & 16 \\ & \multicolumn{7}{l}{x_1, x_2, x_3 \geq 0, \text{ integer}} \end{array}$$

Having added slack variables s_1, s_2, s_3 to the three constraints, and solving the LP-relaxed problem with the Simplex algorithm the following tableau appears

$$\begin{array}{rrrrrrrrrrrrrr} z & + & 4x_1 & & & & & & + & 8s_2 & + & 16s_3 & = & 288 \\ & & 1.6x_1 & & & & & + \; s_1 & + & 1.2s_2 & - & 5.6s_3 & = & 27.2 \\ & & 1.6x_1 & & & + & x_3 & & + & 1.2s_2 & - & 1.6s_3 & = & 11.2 \\ & & 0.8x_1 & + & x_2 & & & & - & 0.4s_2 & + & 1.2s_3 & = & 1.6 \end{array}$$

Derive a Gomory cut from the first inequality having a fractional solution variable.

Q 15: Which inequality appears:

15.a) $\frac{1}{5}x_1 + \frac{2}{5}s_2 + \frac{3}{5}s_3 \geq \frac{1}{5}$
15.b) $\frac{3}{5}x_1 + s_1 + \frac{1}{5}s_2 + \frac{2}{5}s_3 \geq 27$
15.c) $\frac{3}{5}x_1 + \frac{1}{5}s_2 + \frac{2}{5}s_3 \geq \frac{1}{5}$
15.d) $s_1 \geq \frac{1}{5}$
15.e) $\frac{3}{5}x_1 - s_1 + \frac{1}{5}s_2 + \frac{3}{5}s_3 \geq 27.2$
15.f) $\frac{3}{5}x_1 + \frac{1}{5}s_2 - \frac{3}{5}s_3 \geq \frac{1}{5}$

Playing Bridge

Playing bridge is relaxing occupation to most people but professors P_1 to P_8 are very serious players: Before playing each card an LP-model is formulated to maximize the probability of getting the next trick. A model $\mathcal{A}$ is given by

$$\begin{array}{lrcrcrl} \text{maximize} & x_1 & + & 4x_2 & & & \\ \text{subject to} & x_1 & + & x_2 & \leq & 5 & \quad (a) \\ & x_1 & + & 5x_2 & \leq & 20 & \quad (b) \\ & 5x_1 & + & 3x_2 & \leq & 20 & \quad (c) \\ & x_1 & & & \leq & 3 & \quad (d) \\ & x_1 & & & \geq & 0 & \quad (e) \\ & & & x_2 & \geq & 0 & \quad (f) \\ & \multicolumn{5}{l}{x_1, x_2 \text{ integer}} & \end{array}$$

Q 16: Which of the following statements is correct

16.a) inequalities (a), (e), (f) are facet defining and inequality (d) is redundant.
16.b) inequalities (d), (e), (f) are facet defining and inequality (a) is redundant.
16.c) inequalities (a), (d), (e), (f) are facet defining.
16.d) all the inequalities (a), (b), (c), (d), (e), (f) are facet defining.
16.e) inequalities (e), (f) are facet defining and inequalities (b), (c) are redundant.
16.f) inequalities (c), (d), (e), (f) are redundant.

Q 17: Consider the problem $\mathcal{A}$. Derive the facet defining inequality

$$x_1 + 2x_2 \leq 8$$

as a Chvatal-Gomory cut using the multipliers $u = (u_a, u_b, u_c, u_d, u_e, u_f)$ for the six inequalities (a) to (f). What is the correct choice of multipliers:

17.a) $u = (\frac{1}{4}, \frac{3}{4}, 0, 0, 2, 1)$
17.b) $u = (3, 4, 0, 0, -1, -1)$
17.c) $u = (1, 0, \frac{1}{2}, \frac{3}{4}, 0, -1)$
17.d) $u = (2, 3, 1, 0, 0, 0)$
17.e) $u = (0, \frac{1}{5}, \frac{3}{5}, , 0, -1, 0)$
17.f) $u = (\frac{3}{4}, \frac{1}{4}, 0, 0, 0, 0)$

Q 18: Lagrangian relax constraint (b) in problem $\mathcal{A}$ using a mulitiplier $\lambda \geq 0$. What is the resulting problem?

18.a)
$$\begin{array}{llrl} \max & (1-\lambda)x_1 + & (4-5\lambda)x_2 + 20\lambda & \\ \text{s.t.} & x_1 + & x_2 \leq & 5 \\ & 5x_1 + & 3x_2 \leq & 20 \\ & x_1 & \leq & 3 \\ & \multicolumn{3}{l}{x_1, x_2 \geq 0 \text{ integer}} \end{array}$$

18.b)
$$\begin{array}{llrl} \max & x_1 + & x_2 & \\ \text{s.t.} & (1+\lambda)x_1 + & (1+5\lambda)x_2 \leq & 5 + 20\lambda \\ & 5x_1 + & 3x_2 \leq & 20 \\ & x_1 & \leq & 3 \\ & \multicolumn{3}{l}{x_1, x_2 \geq 0 \text{ integer}} \end{array}$$

18.c)
$$\begin{array}{ll} \max & (1-\lambda)x_1 + (4-5\lambda)x_2 - 20\lambda \\ \text{s.t.} & x_1, x_2 \geq 0 \text{ integer} \end{array}$$

18.d)
$$\begin{array}{llrl} \max & (1-\lambda)x_1 + & (4-5\lambda)x_2 & \\ \text{s.t.} & x_1 + & x_2 \leq & 5 \\ & 5x_1 + & 3x_2 \leq & 20 \\ & x_1 & \leq & 3 \\ & \multicolumn{3}{l}{x_1, x_2 \geq 0 \text{ integer}} \end{array}$$

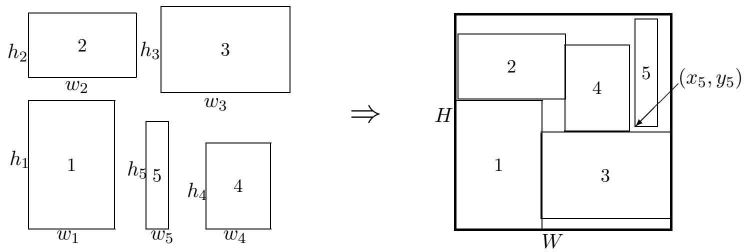
18.e)
$$\begin{array}{llrl} \max & -\lambda x_1 - & 5\lambda x_2 + & 20\lambda \\ \text{s.t.} & x_1 + & x_2 \leq & 5 \\ & 5x_1 + & 3x_2 \leq & 20 \\ & x_1 & \leq & 3 \\ & \multicolumn{3}{l}{x_1, x_2 \geq 0 \text{ integer}} \end{array}$$

18.f)
$$\begin{array}{llrl} \max & 5x_1 + & 7\lambda x_2 + & 20\lambda \\ \text{s.t.} & x_1 + & x_2 \leq & 5 \\ & 5x_1 + & 3x_2 \leq & 20 \\ & x_1 & \leq & 3 \\ & \multicolumn{3}{l}{x_1, x_2 \geq 0 \text{ integer}} \end{array}$$

Lagrangian dual problem (text question)

Q 19: Considering the lagrangian relaxation from the previous question, draw the lagrangian dual problem as function of λ in the interval $0 \leq \lambda \leq 1$. Indicate for which values of λ the slope changes.

Strip packing problem (text question)



The strip packing problem asks for an arrangement of n rectangels into a sheet of minimum width. Rectangle i has width w_i and height h_i. The sheet has width W and height H, where W is to be minimised and H is a constant. There is a number of constraints to be observed: no two rectangles may overlap, and no rectangle may be placed outside the sheet.

We introduce the variables (x_i, y_i) to denote the position of the lower left corner of rectangle i. Moreover, for $i < j$, we introduce some relations $\ell_{ij}, r_{ij}, a_{ij}, b_{ij}$ to denote whether rectangle i is *left*, *right*, *above* or *below* rectangle j. To avoid overlaps, at least one of these relations must hold true for each pair of rectangles. Depending on the relative position of the rectangles, we must demand that

$$\begin{aligned} \ell_{ij} = 1 &\Rightarrow x_i + w_i \le x_j \\ r_{ij} = 1 &\Rightarrow x_j + w_j \le x_i \\ b_{ij} = 1 &\Rightarrow y_i + h_i \le y_j \\ a_{ij} = 1 &\Rightarrow y_j + h_j \le y_i \end{aligned} \tag{1}$$

Q 20: Formulate the problem as a mixed-integer linear programming model. You may assume that $W \le V$, where V is a given upper bound (constant).