Introduktion til Optimering

DIKU, 4 timers skriftlig eksamen, 23. januar 2008

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Alle hjælpemidler må benyttes dog *ikke* lommeregner eller computer. Besvarelsen kan udarbejdes med blyant eller kuglepen.

Opgavesættet består af 19 opgaver, navngivet Q1-Q19. Opgaverne Q1-Q10 og Q13-Q18 er *multiple-choice opgaver*, som har netop ét korrekt svar. For at besvare en sådan opgave skal man, uden yderligere forklaring, skrive opgavens nummer samt den korrekte svarmulighed. For eksempel kan opgave Q1 besvares med "1A". Q11, Q12, og Q19 er sædvanlige *tekstopgaver*, som skal besvares tilstrækkeligt detaljeret til at løsningsmetoden kan følges. Hvert korrekt svar til en *multiple-choice opgave* giver 4 point. Hvert korrekt svar til en *tekstopgave* giver 12 point. Man kan samlet opnå 100 point.

The question paper consists of 19 questions named Q1-Q19. The questions Q1-Q10 and Q13-Q18 are *multiple-choice questions*, which have exactly one correct answer. To answer such a question simply write the number of the question and the correct answer. For example question Q1 can be answered with "1A". Q11, Q12 og Q19 are ordinary *text questions*, which should be answered sufficiently detailed to make it possible to follow the solution method. Each correct answer to a *multiple-choice question* gives 4 points. Each correct answer to a *text question* gives 12 points. You can obtain 100 points in total.

Sensitivity Analysis

Consider the following linear program (LP)

$$\begin{array}{llllll} \text{maximize} & 3x_1 & +2x_2 & +2x_3 & & \\ \text{subject to} & 2x_1 & +4x_2 & & \leq & 10 \\ & 2x_1 & -2x_2 & +x_3 & = & 5 \\ & x_1, x_2, x_3 \geq 0 & & & & \end{array} \tag{1}$$

(notice the "$\leq$" and "$=$" constraints). Introduce slack variable x_4 in the first constraint.

Q 1: What is the final Simplex tableau when solving the above problem (using Taha's notation)?

1A)

basic	x_1	x_2	x_3	x_4	solution
z	-3	-2	-2	0	0
x_4	2	4	0	1	10
x_3	2	-2	1	0	5

1D)

basic	x_1	x_2	x_3	x_4	solution
z	0	-5	$-\frac{1}{2}$	0	$\frac{15}{2}$
x_4	0	6	-1	1	5
x_1	1	-1	$\frac{1}{2}$	0	$\frac{5}{2}$

1B)

basic	x_1	x_2	x_3	x_4	solution
z	0	0	$-\frac{4}{3}$	$\frac{5}{6}$	$\frac{35}{3}$
x_2	0	1	$-\frac{1}{6}$	$\frac{1}{6}$	$\frac{5}{6}$
x_1	1	0	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{10}{3}$

1E)

basic	x_1	x_2	x_3	x_4	solution
z	2	0	0	$\frac{2}{3}$	15
x_2	$\frac{1}{2}$	1	0	$\frac{1}{4}$	$\frac{5}{2}$
x_3	1	0	1	$\frac{3}{2}$	5

1C)

basic	x_1	x_2	x_3	x_4	solution
z	-4	0	0	$\frac{3}{2}$	25
x_2	$\frac{1}{2}$	1	0	$\frac{1}{4}$	$\frac{5}{2}$
x_3	6	0	1	1	20

1F)

basic	x_1	x_2	x_3	x_4	solution
z	4	0	0	$\frac{3}{2}$	25
x_2	$\frac{1}{2}$	1	0	$\frac{1}{4}$	$\frac{5}{2}$
x_3	3	0	1	$\frac{1}{2}$	10

■

Q 2: What is the dual problem associated with (1)? (NB: Specify the full domain of y_2)

2A)

$$\begin{array}{ll} \text{minimize} & 10y_1 + 5y_2 \\ \text{subject to} & 2y_1 + 2y_2 \geq 3 \\ & 4y_1 - 2y_2 \geq 2 \\ & y_2 \geq 2 \\ & y_1 \geq 0, y_2 \in \mathbb{R} \end{array}$$

2D)

$$\begin{array}{ll} \text{minimize} & 10y_1 + 5y_2 \\ \text{subject to} & 2y_1 + 2y_2 \geq 3 \\ & 4y_1 - 2y_2 = 2 \\ & y_2 \geq 2 \\ & y_1 \geq 0, y_2 \in \mathbb{R} \end{array}$$

2B)

$$\begin{array}{ll} \text{minimize} & 10y_1 + 5y_2 \\ \text{subject to} & 2y_1 + 2y_2 \geq 3 \\ & 4y_1 - 2y_2 \geq 2 \\ & y_2 \geq 2 \\ & y_1, y_2 \in \mathbb{R} \end{array}$$

2E)

$$\begin{array}{ll} \text{minimize} & 10y_1 + 5y_2 \\ \text{subject to} & 2y_1 + 2y_2 \geq 3 \\ & 4y_1 - 2y_2 = 2 \\ & y_2 \geq 2 \\ & y_1, y_2 \in \mathbb{R} \end{array}$$

2C)

$$\begin{array}{ll} \text{minimize} & 10y_1 + 5y_2 \\ \text{subject to} & 2y_1 + 2y_2 \geq 3 \\ & 4y_1 - 2y_2 \geq 2 \\ & y_2 \geq 2 \\ & y_1 \geq 0, y_2 \geq 0 \end{array}$$

2F)

$$\begin{array}{ll} \text{minimize} & 10y_1 + 5y_2 \\ \text{subject to} & 2y_1 + 2y_2 \geq 3 \\ & 4y_1 - 2y_2 = 2 \\ & y_2 \geq 2 \\ & y_1 \geq 0, y_2 \geq 0 \end{array}$$

■

Q 3: What is the optimal solution to the above dual problem?

3A)	$y_1 = 0, y_2 = 5$	3D)	$y_1 = 2, y_2 = 2$
3B)	$y_1 = -\frac{3}{2}, y_2 = 3$	3E)	$y_1 = \frac{3}{2}, y_2 = 2$
3C)	$y_1 = 2, y_2 = 1$	3F)	$y_1 = \frac{1}{2}, y_2 = 4$

■

Q 4: Assume that x_1 is changed to $x_1 = 2$. What is the loss/increase in profit?

4A)	objective is unchanged	4D)	increase of 2
4B)	increase of 4	4E)	increase of 6
4C)	loss of 4	4F)	loss of 8

■

Q 5: Assume that we added a real number d_1 to the first constraint in (1) getting the constraint $2x_1 + 4x_2 \leq 10 + d_1$. In which interval can d_1 vary without changing the optimal basis?

5A)	$d_1 \geq -10$	5D)	$d_1 \geq 10$
5B)	$d_1 \leq -5$	5E)	$d_1 \geq 5$
5C)	$0 \leq d_1 \leq 5$	5F)	$-10 \leq d_1 \leq 10$

■

Q 6: What is the marginal value of decreasing the second constraint by one unit (i.e. changing 5 to 4).

6A)	0	6D)	1
6B)	2	6E)	3
6C)	-2	6F)	-4

■

Q 7: If we introduce a new product x_4 which demands 3 units of constraint 1, and 1 unit of constraint 2 we get the following model

$$\begin{array}{lllllll} \text{maximize} & 3x_1 & +2x_2 & +2x_3 & + & c_4x_4 & \\ \text{subject to} & 2x_1 & +4x_2 & & + & 3x_4 & \leq 10 \\ & 2x_1 & -2x_2 & +x_3 & + & 1x_4 & = \ \ 5 \\ & x_1, x_2, x_3 \geq 0 & & & & & \end{array} \tag{2}$$

What is the minimum value of c_4 needed to make the new product worth producing?

7A)	$c_4 > 0$	7D)	$c_4 > \frac{13}{2}$
7B)	$c_4 > \frac{1}{2}$	7E)	$c_4 > \frac{1}{4}$
7C)	$c_4 < 1$	7F)	$c_4 < 2$

■

Q 8: Due to increased production costs, the objective function in (1) is changed to

$$(2-d)x_1 + (3-d)x_2 + (2-d)x_3$$

For which values of d will the current LP solution remain optimal?

8A)	$d \leq 1$	8D)	$d \leq \frac{1}{2}$
8B)	$d \geq 2$	8E)	$d \geq -2$
8C)	$d \leq \frac{8}{5}$	8F)	$d \geq -\frac{8}{5}$

■

Gomory cuts

Q 9: Assume that (1) had to be solved to integer optimality, i.e. x_1, x_2, x_3 are nonnegative integers. Consider the final Simplex tableau. Derive a Gomory cut from the first constraint in which the basic variable is fractional.

9A)	$x_2 \leq 2$	9D)	$x_2 \leq 1$
9B)	$x_2 \leq \frac{5}{3}$	9E)	$x_1 + x_2 \leq 4$
9C)	$x_1 + x_2 \leq 12$	9F)	$x_2 \leq 9$

■

Cover inequalities

Let

$$K = \text{conv}\left(\{x \in \mathbb{B}^5 \mid 7x_1 + 9x_2 + 6x_3 + 7x_4 + 5x_5 \leq 13\}\right)$$

A minimal cover is $C = \{3, 4, 5\}$, resulting in the cover inequality $x_3 + x_4 + x_5 \leq 2$. We wish to lift this inequality to

$$\alpha x_2 + x_3 + x_4 + x_5 \leq 2$$

Q 10: What is the largest value of α such that the above inequality is valid?

10A)	$\alpha = 0$	10D)	$\alpha = 3$
10B)	$\alpha = 1$	10E)	$\alpha = 4$
10C)	$\alpha = 2$	10F)	$\alpha = 5$

■

Lagrangian Relaxation

Consider the following integer-programming model.

$$\begin{array}{lrcrcrcl} \text{maximize} & 5x_1 & + & 2x_2 & + & x_3 & & \\ \text{subject to} & x_1 & & & - & x_3 & \leq & 4 \\ & & & -x_2 & + & x_3 & \leq & 3 \\ & 2x_1 & + & x_2 & + & 3x_3 & = & 7 \\ & \multicolumn{7}{l}{x_1, x_2, x_3 \geq 0, \mathbb{Z}} \end{array} \tag{3}$$

The optimal solution to the LP-relaxed problem is $x_1 = \frac{7}{2}$, $x_2 = x_3 = 0$.

Q 11: **(text question)** Lagrangian relax the last constraint $2x_1 + x_2 + 3x_3 = 7$ using multiplier λ, and solve the Lagrangian dual problem (i.e. find the value of λ for which the relaxed problem results in the lowest upper bound). ■

Model building

Consider the following production planning problem: We have n time periods $T = \{1,\dots,n\}$. In each time period $t \in T$ we have a demand d_t of items. We may satisfy the demand d_t by producing the necessary items at time t or by producing them at some earlier time $1,\dots,t-1$ and keeping the items at stock. The cost of holding one item on stock from time t to $t+1$ is h_t. The cost of producing one item is c_t. Moreover, we should pay a fixed startup cost f_t is we produce at least one item in time period t. We cannot produce more than the upper limit u_t items at time period t.

Let x_t determine the amount of items produced at time t, and let s_t be the number of items on stock from time t to $t+1$. Moreover, let y_t be a binary variable which indicates whether we produce any items at time t. The initial number of items on stock at time $t=0$ is $s_0 = 0$.

Q 12: **(text question)**. Formulate the problem as a mixed-integer programming model which minimizes the overall production costs while satisfying the demand in each time period. ■

Column Generation

Consider the following *minimum cost multicommodity flow problem*, defined on a weighted oriented graph $G = (V,E,c,\ell)$ where each edge $(i,j) \in E$ has an associated cost c_{ij}, and a *lower bound* ℓ_{ij} on the flow.

A number of commodities $K = \{1,\dots,m\}$ are given by a triple (s_k,t_k,d_k). Here s_k denotes the source of commodity k and t_k denotes the terminal of commodity k, while d_k denotes the number of units to be sent. We are searching for the cheapest possible flow.

If we use the decision variables x_{ij}^k to denote whether commodity $k \in K$ flows along edge $(i,j) \in E$, the problem may be formulated as:

$$\min \sum_{k\in K}\sum_{(i,j)\in E} c_{ij} d_k x_{ij}^k \tag{4}$$

$$\text{s.t} \sum_{i\in V} x_{ij}^k - \sum_{i\in V} x_{ji}^k = 0 \quad k \in K, j \in V \setminus \{s_k,t_k\} \tag{5}$$

$$\sum_{j\in V} x_{s_k,j}^k = 1 \qquad k \in K \tag{6}$$

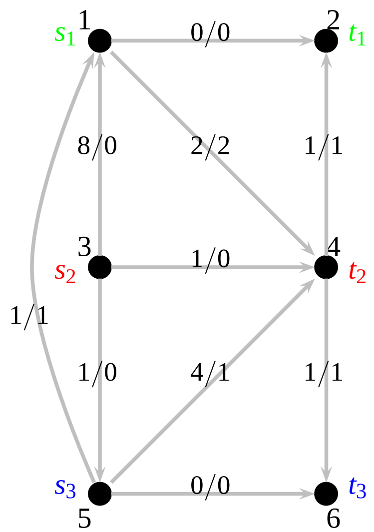
$$\sum_{i\in V} x_{i,t_k}^k = 1 \qquad k \in K \tag{7}$$

$$\sum_{k\in K} d_k x_{ij}^k \geq \ell_{ij} \qquad (i,j) \in E \tag{8}$$

$$0 \leq x_{ij}^k \leq 1 \qquad k \in K, (i,j) \in E \tag{9}$$

There are no upper bounds on the edges. Problems with lower bounds appear frequently in networks where a given amount of float is demanded (e.g. in water pipes to avoid that the water becomes undrinkable).

Q 13: Consider the following instance of the multicommodity flow problem:



<u>edges:</u>

$(i, j, c_{ij}, \ell_{ij})$
$(1,2,0,0)$
$(1,4,2,2)$
$(3,1,8,0)$
$(3,4,1,0)$
$(3,5,1,0)$
$(4,2,1,1)$
$(4,6,1,1)$
$(5,1,1,1)$
$(5,4,4,1)$
$(5,6,0,0)$

<u>commodities:</u>

(s_k, t_k, d_k)
$(1,2,2)$
$(3,4,2)$
$(5,6,2)$

What is the cost z of the cheapest multicommodity flow, if it is assumed that the flow of each commodity may be split arbitrarily?

13A)	$z = 13$	13D)	$z = 20$
13B)	$z = 15$	13E)	$z = 22$
13C)	$z = 27$	13F)	$z = 24$

■

Q 14: What is the cost z of the cheapest multicommodity flow if all d_k items must follow the same path, for each commodity k, i.e. if constraint (9) is replaced with $x_{ij}^k \in \{0,1\}$.

14A)	$z = 13$	14D)	$z = 20$
14B)	$z = 15$	14E)	$z = 22$
14C)	$z = 27$	14F)	$z = 24$

■

Q 15: Let for each commodity k the set R_k denote all simple paths from s_k to t_k through the graph $G = (V, E, c)$. Which of the following paths does not belong to $R_1 \cup R_2 \cup R_3$:

15A)	1→2	cost	0
15B)	1→4→2	cost	3
15C)	3→1→4	cost	10
15D)	3→4	cost	1
15E)	3→5→1→4	cost	4
15F)	3→5→4	cost	5
15G)	3→5→6→4	cost	2
15H)	5→1→4→6	cost	4
15I)	5→4→6	cost	5
15J)	5→6	cost	0

■

Q 16: Consider a given path p for commodity k. In which situation will the path never appear in an optimal solution:

16A) $\ell_{ij} < d_k$ for some edge (i,j) on the path p
16B) $\ell_{ij} > d_k$ for some edge (i,j) on the path p
16C) the path p contains a cycle
16D) we cannot exclude the path

■

We want to use a *path formulation* of the multicommodity flow problem. Let $a^k_{r,ij}$ denote whether a path $r \in R_k$ contains the edge (i,j), and let $\hat{c}^k_r$ denote the cost of path $r \in R_k$ when sending d_k units of the commodity. Lad x^k_r for each path $r \in R_k$ denote whether the path is used for commodity k. In this way the model becomes:

$$\min \quad \sum_{k \in K} \sum_{r \in R_k} \hat{c}^k_r x^k_r \tag{10}$$

$$\text{s.t.} \sum_{k \in K} \sum_{r \in R_k} d_k a^k_{r,ij} x^k_r \geq \ell_{ij} \quad (i,j) \in E \tag{11}$$

$$\sum_{r \in R_k} x^k_r \geq 1 \qquad k \in K \tag{12}$$

$$0 \leq x^k_r \leq 1 \qquad k \in K, r \in R_k \tag{13}$$

Here (11) ensures that the lower bound ℓ_{ij} of each edge is respected, while (12) ensure that the demanded amount is sent from s_k to t_k. We will solve the problem with delayed column generation. At a given moment of the column generation we have a restricted master problem with the following four paths each containing two units

commodity k	path	cost
1	1→4→2	6
2	3→4	0
3	5→1→4→6	8
4	5→4→6	10

This leads to the following LP-model corresponding to constraints (10) - (13):

$$\begin{array}{lllllllll}
\min & 6x^1_1 & + \; 0x^2_1 & + \; 8x^3_1 & + \; 10x^3_2 & & & \\
\text{s.t.} & & & & & \geq 0 & & (1,2) \\
& 2x^1_1 & & + \; 2x^3_1 & & \geq 2 & & (1,4) \\
& & & & & \geq 0 & & (3,1) \\
& & 2x^2_1 & & & \geq 0 & & (3,4) \\
& & & & & \geq 0 & & (3,5) \\
& 2x^1_1 & & & & \geq 1 & & (4,2) \\
& & & 2x^3_1 & + \; 2x^3_2 & \geq 1 & & (4,6) \\
& & & 2x^3_1 & & \geq 1 & & (5,1) \\
& & & & 2x^3_2 & \geq 1 & & (5,4) \\
& & & & & \geq 0 & & (5,6) \\
& & & & & & & \\
& x^1_1 & & & & \geq 1 & & \\
& & x^2_1 & & & \geq 1 & & \\
& & & x^3_1 & + \; x^3_2 & \geq 1 & & \\
\multicolumn{6}{l}{0 \leq x^1_1, x^2_1, x^3_1, x^3_2 \leq 1} & &
\end{array} \tag{14}$$

Let y_{ij} denote the dual variable associated with constraint (11) for edge (i,j), and let $\bar{y}_k$ denote the dual variable associated with constraint (12) for commodity k. The dual variables for the above model (14) are $y_{51}=4$, $y_{54}=5$ (other $y_{ij}=0$), while $\bar{y}_1=6$, $\bar{y}_2=2$, $\bar{y}_3=0$.

Q 17: What is the reduced cost of path 3→5→1→4 (using Taha's definition of reduced cost, i.e. the reverse sign of Wolsey's definition)

17A)	0	17D)	6
17B)	2	17E)	8
17C)	4	17F)	10

■

In the following iterations, we add paths 1→2 and 3→5→1→4 to the model (14). After having solved the problem to LP-optimality, we find the dual variables $y_{42}=3$, $y_{54}=1$, while remaining $y_{ij}=0$, and $\bar{y}_1=0$, $\bar{y}_2=2$, $\bar{y}_3=8$.

Q 18: Which of the following valid paths has positive reduced cost (using Taha's definition of reduced cost)

- 18A) 1→2
- 18B) 1→4→2
- 18C) 3→1→4
- 18D) 3→4
- 18E) 3→5→1→4
- 18F) 3→5→4
- 18G) 3→5→6→4
- 18H) 5→1→4→6
- 18I) 5→4→6
- 18J) 5→6

■

After adding the path to the problem and solving the LP-relaxation, we get the dual variables $y_{42}=3$, $y_{51}=3$, $y_{54}=5$ (other $y_{ij}=0$), and $\bar{y}_1=0$, $\bar{y}_2=2$, $\bar{y}_3=0$.

Q 19: (**text question**)

a) Show that for the considered instance, the LP-solution of the simple formulation (4) – (9) and the path formulation (10) – (13) give the same solution value.

b) Show that this holds in general for any instance.

■

Answers

Answer 1 Simplex

Iteration 0:

basic	x_1	x_2	x_3	x_4	solution
z	-3	-2	-2	0	0
x_4	2	4	0	1	10
x_3	2	-2	1	0	5

x_1 entering variable, x_3 leaving variable

Iteration 1:

basic	x_1	x_2	x_3	x_4	solution
z	0	-5	$-\frac{1}{2}$	0	$\frac{15}{2}$
x_4	0	6	-1	1	5
x_1	1	-1	$\frac{1}{2}$	0	$\frac{5}{2}$

x_2 entering variable, x_4 leaving variable

Iteration 2:

basic	x_1	x_2	x_3	x_4	solution
z	0	0	$-\frac{4}{3}$	$\frac{5}{6}$	$\frac{35}{3}$
x_2	0	1	$-\frac{1}{6}$	$\frac{1}{6}$	$\frac{5}{6}$
x_1	1	0	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{10}{3}$

x_3 entering variable, x_1 leaving variable

Iteration 3:

basic	x_1	x_2	x_3	x_4	solution
z	4	0	0	$\frac{3}{2}$	25
x_2	$\frac{1}{2}$	1	0	$\frac{1}{4}$	$\frac{5}{2}$
x_3	3	0	1	$\frac{1}{2}$	10

hence the optimal solution is

$$x_2 = \frac{5}{2}, x_3 = 10$$

basis and inverse basis is:

$$A_B = \begin{pmatrix} 4 & 0 \\ -2 & 1 \end{pmatrix} \qquad A_B^{-1} = \frac{1}{4}\begin{pmatrix} 1 & 0 \\ 2 & 4 \end{pmatrix}$$

■

Answer 2

$$\begin{aligned} \text{minimize} \quad & 10y_1 + 5y_2 \\ \text{subject to} \quad & 2y_1 + 2y_2 \geq 3 \\ & 4y_1 - 2y_2 \geq 2 \\ & y_2 \geq 2 \\ & y_1 \geq 0, y_2 \in \mathbb{R} \end{aligned} \tag{15}$$

■

Answer 3 We use method 1 from Taha section 4.2.3. The start variables in Simplex are (x_4, x_3). The final coefficients are $(\frac{3}{2}, 2)$. To these we should add the original objective coefficients $(0, 2)$ getting $(y_1, y_2) = (\frac{3}{2}, 2)$. ■

Answer 4 From the final Simplex tableau we can see that the reduced cost of variable x_1 is $\overline{c}_1 = 4$. This means that the objective value is decreased by 4 units for each increase of x_1 by one. This means that we get a loss of $2 \cdot 4 = 8$. ■

Answer 5 Feasibility range. Assume that we add $(d_1, 0)$ to the right-hand-sides. The final basic tableau in Simplex will then become

$$x_b + A_B^{-1} A_N x_N = A_B^{-1}(b + d)$$

which becomes

$$\begin{pmatrix} x_2 \\ x_3 \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 1 & 0 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} 10 + d_1 \\ 5 \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 10 + d_1 \\ 40 + 2d_1 \end{pmatrix}$$

current solution remains feasible as long as $x_2, x_3 \geq 0$ hence

$$10 + d_1 \geq 0, \qquad 40 + 2d_1 \geq 0$$

i.e. $d_1 \geq -10$. ■

Answer 6 The dual price y_2 says how much the objective is increased/decreased when we increase/decrease the right-hand side by one unit. In our case we get the marginal value $-1 \cdot y_2 = -2$. ■

Answer 7 The reduced cost of product x_4 is founds as $\overline{c}_4 = 3y_1 + 1y_2 - c_4$. The variable will enter basis if the reduced cost is negative, i.e. $\overline{c}_4 = 3y_1 + 1y_2 - c_4 < 0$. From this we find $\frac{9}{2} + 2 - c_4 < 0$ or $c_4 > \frac{13}{2}$. ■

Answer 8 The reduced costs of the modified problem are

$$\overline{c}'_N = \overline{c}_N + d_B \overline{A}_N - d_N$$

where $\overline{A}_N$ refers to the A-matrix in the final Simplex tableau. In our case we get

$$\begin{pmatrix} \overline{c}'_1 \\ \overline{c}'_4 \end{pmatrix} = \begin{pmatrix} 4 \\ \frac{3}{2} \end{pmatrix} + \begin{pmatrix} d \\ d \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ 3 & \frac{1}{2} \end{pmatrix} - \begin{pmatrix} d \\ 0 \end{pmatrix}$$

current solution remains optimal as long as $\overline{c}'_1 \geq 0$ and $\overline{c}'_4 \geq 0$ i.e.

$$4 + \frac{5}{2}d \geq 0, \qquad \frac{3}{2} + \frac{3}{4}d \geq 0$$

which is equivalent to

$$d \geq -\frac{8}{5}, \qquad d \geq -2$$

hence the correct answer is $d \geq -\frac{8}{5}$. ■

Answer 9 Variable x_2 is the only variable having fractional value in the final Simplex tableau. The corresponding equation is

$$\frac{1}{2}x_1 + x_2 + \frac{1}{4}x_4 = \frac{5}{2}$$

The corresponding Gomory cut is

$$\frac{1}{2}x_1 + \frac{1}{4}x_4 \geq \frac{1}{2}$$

substituting $x_4 = 10 - 2x_1 - 4x_2$ we get

$$\frac{1}{2}x_1 + \frac{1}{4}(10 - 2x_1 - 4x_2) \geq \frac{1}{2}$$

which is equivalent to $x_2 \leq 2$. The current solution value is $x_1 = 0$, $x_2 = \frac{5}{2}$, $x_3 = 10$, which violates the constraint. ■

Let

$$K = \text{conv}\left(\{x \in \mathbb{B}^5 \mid 7x_1 + 9x_2 + 6x_3 + 8x_4 + 4x_5 \leq 13\}\right)$$

A minimal cover is $C = \{3,4\}$, resulting in the cover inequality $x_3 + x_4 \leq 1$. We wish to lift this inequality to

$$\alpha x_2 + x_3 + x_4 \leq 1$$

Answer 10 We have the inequality

$$7x_1 + 9x_2 + 6x_3 + 7x_4 + 5x_5 \leq 13$$

and the minimal cover $C = \{3,4,5\}$. To find the biggest α such that $\alpha x_2 + x_3 + x_4 + x_5 \leq 2$ is a valid inequality, we solve the problem:

$$\begin{array}{ll} \max & x_3 + x_4 + x_5 \\ \text{s.t.} & 9 + 6x_3 + 7x_4 + 5x_5 \leq 13 \\ & x_3, x_4, x_5 \in \{0,1\} \end{array}$$

which has the solution $\gamma = 0$. So the largest value of α is $\alpha = 2 - \gamma = 2$. ■

Answer 11 After Lagrangian relaxation, the problem becomes

$$\begin{array}{lrcrcrcl} \text{maximize} & (5-2\lambda)x_1 & + & (2-\lambda)x_2 & + & (1-3\lambda)x_3 & + & 7\lambda \\ \text{subject to} & x_1 & & & - & x_3 & \leq & 4 \\ & & & -x_2 & + & x_3 & \leq & 3 \\ & \multicolumn{7}{l}{x_1, x_2, x_3 \geq 0, \mathbb{Z}} \end{array} \tag{16}$$

It is easily seen that the remaining constraints are TU, hence the optimal choice of Lagrangian multiplier λ corresponds to the dual variable associated with the original problem.

The LP-relaxed dual problem is

$$\begin{array}{lrrrcl} \text{minimize} & 4y_1 & +3y_2 & +7y_3 & & \\ \text{subject to} & y_1 & & +2y_3 & \geq & 5 \\ & & -y_2 & +y_3 & \geq & 2 \\ & -y_1 & +y_2 & +3y_3 & \geq & 1 \\ & \multicolumn{5}{l}{y_1, y_2 \geq 0, y_3 \in \mathbb{R}} \end{array} \tag{17}$$

To find the dual solution to the LP-relaxation of (3) we use complementary slackness:

Since the two first constraints in (3) are not tight, the corresponding dual variables are $y_1 = y_2 = 0$. Moreover, we should have the same optimal solution value, hence

$$4y_1 + 3y_2 + 7y_3 = 5x_1 + 2x_2 + x_3$$

meaning that $7y_3 = 5\frac{7}{2}$ hence $y_3 = \frac{5}{2}$. The optimal value of λ is $\lambda = y_3 = \frac{5}{2}$. ■

Answer 12 We get the model:

$$\begin{array}{lll} \text{minimize} & \sum_{t\in T}(c_t x_t + h_t s_t + f_t y_t) & \\ \text{subject to} & s_{t-1} + x_t = s_t + d_t & t \in T \\ & x_t \le u_t y_t & t \in T \\ & x_t, s_t \ge 0 & t \in T \\ & y_t \in \{0,1\} & t \in T \end{array}$$

The first constraint says that the amount of items on stock at time $t-1$ plus the amount of items produced at time t equals the amount of items at stock at time t plus the demand at time t.

The second constraint says that if $x_t > 0$ then $y_t = 1$. Moreover, the constraint ensures that $x_t \le u_t$. ■

Answer 13 We send one unit along each of the following paths

path	cost
1 →2	0
1 →4 →2	3
3 →4	1
3 →5 →1 →4	4
5 →6	0
5 →4 →6	5

resulting in an overall cost of 13. ■

Answer 14 We send two units along each of the following paths

path	cost
1 →4 →2	3
3 →5 →1 →4	4
5 →4 →6	5

resulting in an overall cost of 24. ■

Answer 15 The path G) 3→5→6→4 is illegal, since 6→4 has reverse orientation. ■

Answer 16 We cannot exclude the path in any of the cases.

Even if $\ell_{ij} > d_k$ we cannot exclude the path, since the lower bound ℓ_{ij} shall be satisfied by the sum of all paths.

To ensure the lower bound ℓ_{ij} on each edge it may be necessary to have cycles in the graph. ■

Answer 17 The reduced costs of the paths are:

path	cost	$a_{ij}y_{ij}$	$\bar{y}_i$	reduced cost
1 →2	$2\cdot 0$		6	6
1 →4 →2	$2\cdot 3$		6	0
3 →1 →4	$2\cdot 10$		2	-18
3 →4	$2\cdot 1$		2	0
3 →5 →1 →4	$2\cdot 4$	$2\cdot 4$	2	2
3 →5 →4	$2\cdot 5$		2	-8
5 →1 →4 →6	$2\cdot 4$	$2\cdot 4$		0
5 →4 →6	$2\cdot 5$	$2\cdot 5$		0
5 →6	$2\cdot 0$			0

The reduced cost of path 3→5→1→4 is hence 2. ■

Answer 18 The reduced costs of the paths are:

path	cost	$a_{ij}y_{ij}$	$\bar{y}_i$	reduced cost
1 →2	$2\cdot 0$			0
1 →4 →2	$2\cdot 3$	$2\cdot 3$		0
3 →1 →4	$2\cdot 10$			-20
3 →4	$2\cdot 1$			-2
3 →5 →1 →4	$2\cdot 4$			-8
3 →5 →4	$2\cdot 5$	$2\cdot 1$		-8
5 →1 →4 →6	$2\cdot 4$		8	0
5 →4 →6	$2\cdot 5$	$2\cdot 1$	8	0
5 →6	$2\cdot 0$		8	8

Hence, path 5→6 has positive reduced cost. ■

Answer 19

a) We have the following dual variables: $y_{42} = 3, y_{51} = 3, y_{54} = 5$ (other $y_{ij} = 0$), and $\bar{y}_1 = 0, \bar{y}_2 = 2, \bar{y}_3 = 0$.

   A check similar to the above shows that no paths have positive reduced cost, hence the column generation terminates. The objective value can be found from the dual solution as $3+3+5+2 = 13$. This corresponds to the objective value of the simple formulation.

b) For a given solution to the simple formulation, we may split every $s_k{\rightarrow}t_k$ flow into individual sub-paths (this is always possible due to flow conservation, and it is the main principle of the Ford-Fulkerson algorithm for Maximum-flow). Each of these sub-paths will be present in the path formulation, since from question 16, we know that no paths will be excluded in the path formulation.

   Every solution to the path formulation is also a solution to the simple formulation by simply setting $x^k_{ij} = \sum_{(i,j)\in r} x^k_r$, (i.e. x^k_{ij} is equal to the sum of all path flows on edge (i, j)). Such a solution is valid, since the sum of the path flows respects the lower bounds, and flow conservation is satisfied.

■