Simple Production models and lot sizing

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Optimization problems in production planning
Lecture 2
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Outline

- Introduction
- Brief modeling
  - Simple strategies
  - Economic Order Quantity (EOQ)
  - Uncapacitated lot-sizing (ULS)
  - Complex lot-sizing
- ULS
  - Dynamic programming recursion (Wagner-Whitin algorithm)
    - Wagner-Whitin Algorithm Example
  - Reformulations of ULS
    - Facility location
    - Multicommodity flow
    - Shortest path
- Simple lot-sizing (polynomial solvable)
  - Uncapacitated lot-sizing (ULS)
  - Uncapacitated lot-sizing with startup costs (ULSS)
  - Uncapacitated lot-sizing with backlogging (BLS)
  - Constant Capacity lot-sizing (CCLS)
  - Wagner-Whitin costs
- Exercises

Introduction

Production planning:

- When to produce
- How much to produce
- Where to produce (*)

Production process

- Transform raw materials into end products
- Series of transformation steps
- Each step consuming and producing intermediate products
- Raw materials, intermediate and end products may be inventoried

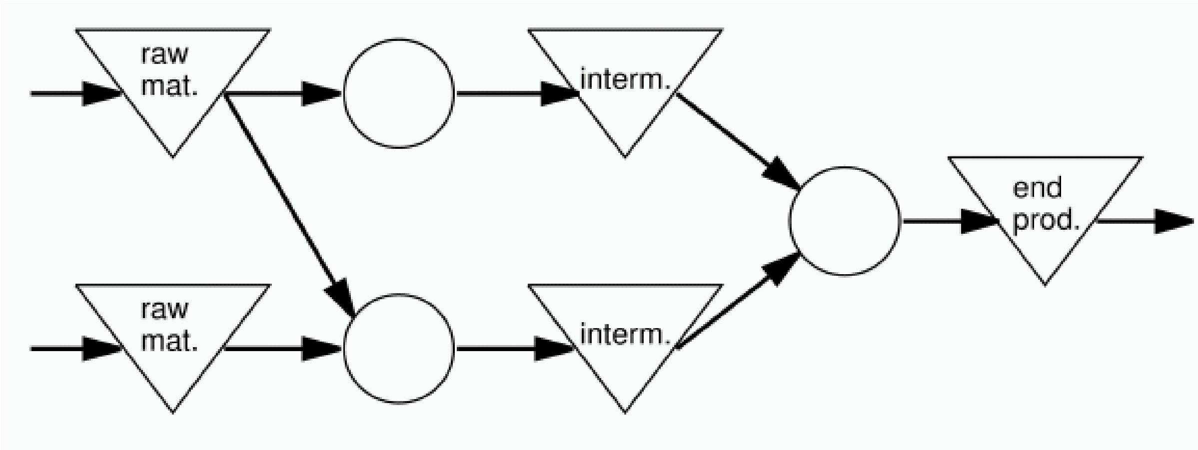

Fig. 1. The production process

Simple strategies

Lot-for-Lot (LFL):

- Order/produce amount required in each period

period	1	2	3	4	5	6	7	8	9	10
demand	2	0	7	6	1	1	1	1	2	2
production	2	0	7	6	1	1	1	1	2	2
setup	1	0	1	1	1	1	1	1	1	1

Fixed Order Period (FOP)

- Order/produce amount required in P period
- When $P = 1$, FOP = LFL

With $P = 2$:

period	1	2	3	4	5	6	7	8	9	10
demand	2	0	7	6	1	1	1	1	2	2
production	2	0	13	0	2	0	2	0	4	0
setup	1	0	1	0	1	0	1	0	1	0

Economic Order Quantity (EOQ)

Produce/buy items in batches

Each item has a fixed production/order cost

Each batch has a fixed cost

Each item has a holding cost (stock)

Constant demand over time

Find optimal size of batch (lot) minimizing total cost

Assumptions:

1. *Production is instantaneous*
2. *Delivery is immediately*
3. *Demand is deterministic*
4. *Demand is constant over time*
5. *Production run incurs fixed setup cost*
6. *Products can be analyzed individually (no interaction between products)*

Economic Order Quantity (EOQ)

Input data:

D = demand rate in units
c = unit production cost
A = fixed setup/ordering cost per lot
h = holding cost (e.g. $h = ic$ with interest rate i)
Q = lot size in units (the variable)

(*) Tegning

Combined cost:

$$Y(Q) = \frac{hQ}{2} + \frac{AD}{Q} + cD$$

Objective: *Minimize:* $Y(Q)$

(*) Tegning

Economic Order Quantity (EOQ)

Solution:

- 1 equation with 1 variable
- Find derivative Y' with respect to Q
- Set $Y' = 0$

$$\frac{dY(Q)}{dQ} = \frac{h}{2} - \frac{AD}{Q^2} = 0 \Rightarrow$$

$$Q = \sqrt{\frac{2AD}{h}}$$

Brief modeling: Uncapacitated lot-sizing (ULS)

CORE problem: ULS (aka Dynamic Lot Sizing)

d_t demand at time $t \in T$

d_{st} demand in interval $[s,t] : s,t \in T$

x_t production at time $t \in T$

s_t inventory at end of time $t \in T$

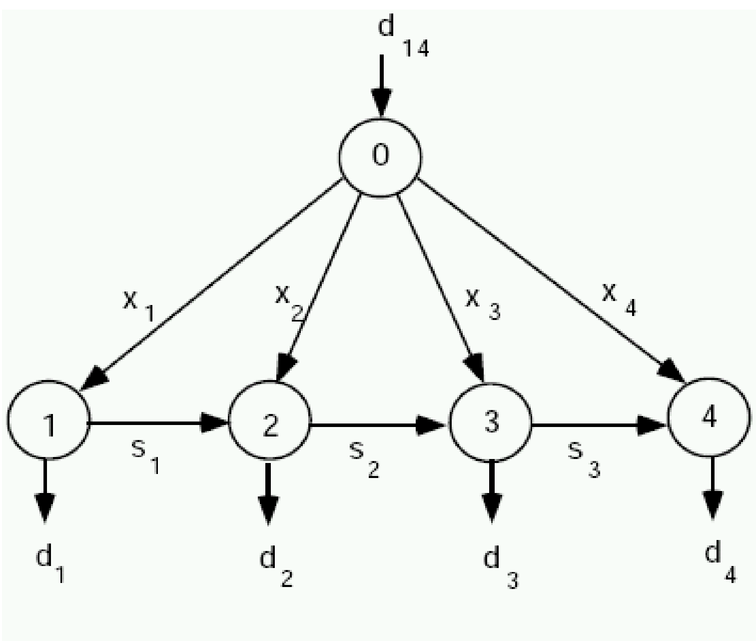

Fig. 3. Uncapacitated lot-sizing network ($n = 4$)

Brief modeling: The model (ULS)

x_t production at time $t \in T$

y_t setup at time $t \in T$

s_t inventory at end of time $t \in T$

d_t demand at time $t \in T$

p_t unit production cost at time $t \in T$

f_t setup cost at time $t \in T$

h_t inventory cost at time $t \in T$

M sufficiently large constant (*big*-M)

$$\min \sum_{t \in T} \left(p_t x_t + f_t y_t + h_t s_t \right) \quad (1)$$

$$\text{s.t.} \quad s_{t-1} + x_t = d_t + s_t \qquad \forall t \in T \quad (2)$$

$$s_0 = s_n = 0 \quad (3)$$

$$x_t \leq M y_t \qquad \forall t \in T \quad (4)$$

$$x_t, s_t \geq 0 \quad (5)$$

$$y_t \in \{0, 1\} \qquad \forall t \in T \quad (6)$$

Brief modeling: Complex lot-sizing

- Capacity on the used resources
- Multiple different items
- Different items dependent on each other
- Multiple resources
- Changeover time and cost on resources
- Backlogging of orders
- Loss of orders
- Profit maximization
- . . .

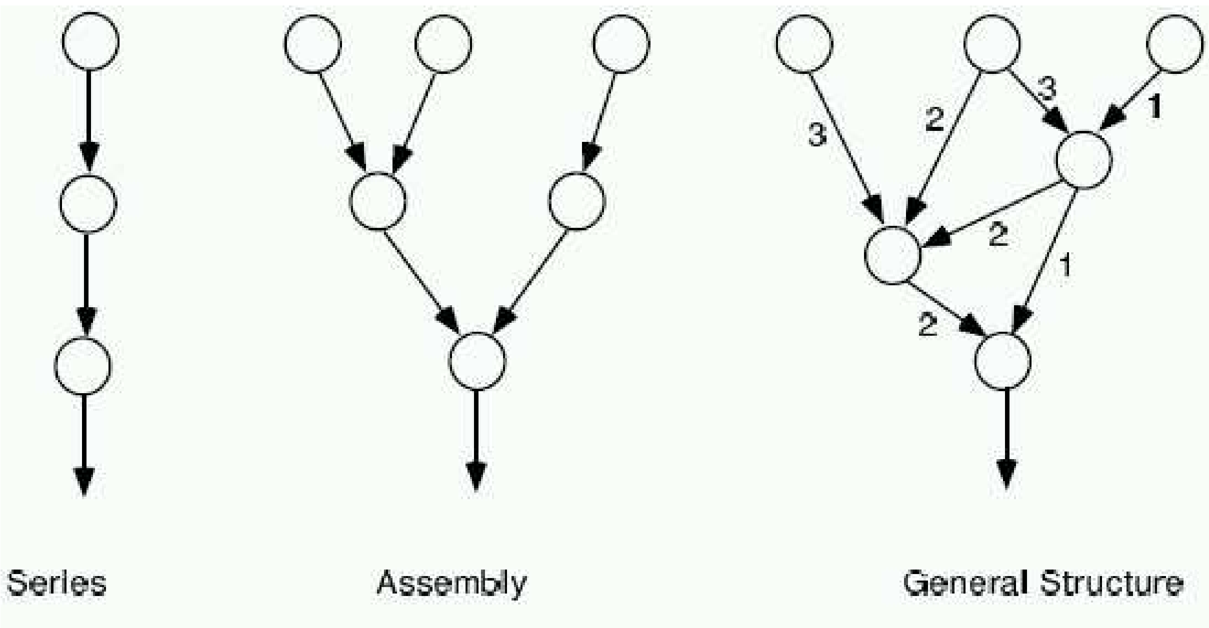

Fig. 4. Types of product structures in multi-level models

Formal model will be presented Wednesday (5th September)

ULS dynamic programming recursion (Wagner-Whitin algorithm)

In the following let

- $d_{kl} = \sum_{t=k}^{l} d_t$
- $s_t = \sum_{k=1}^{t} x_k - d_{1t}$
- $c_t = p_t + \sum_{i=t}^{|T|} h_i$

Then $s_t : \forall t \in T$ can be removed from the objective

Each solution value is now a constant from the original formulation

There exist an optimal solution to ULS where $s_{t-1} x_t = 0, \ \forall t \in T$

So, if last production before period k is in period t then:

- $x_t \geq d_{t,k-1}$
- $x_l = 0, \quad \forall 1 : t < l < k$

ULS dynamic programming recursion (Wagner-Whitin algorithm)

Let $H(k)$ be a minimum cost solution for ULS restricted to periods 1 to k

Following recursion solves ULS (Wagner-Whitin recursion):

$$H(k) = \min_{1 \leq t \leq k} \{H(t-1) + f_t + c_t d_{tk}\} \tag{7}$$

Recursion (7) can clearly be solved in $O(n^2)$ where $n = |T|$

Recursion in book a little bit different

$O(n \lg n)$ running time algorithms exist

Wagner-Whitin algorithm Example

Example from book

RoadHog data:

t	1	2	3	4	5	6	7	8	9	10
d_t	20	50	10	50	50	10	20	40	20	30
p_t	10	10	10	10	10	10	10	10	10	10
f_t	100	100	100	100	100	100	100	100	100	100
h_t	1	1	1	1	1	1	1	1	1	1

Production cost ignored since $p_t = p_k, \quad \forall t, k \in T$

Last production	1	2	3	4	5	6	7	8	9	10
1										
2										
3										
4										
5										
6										
7										
8										
9										
10										
cost										
PP										

Reformulations

Complex lot-sizing problems contain ULS

Different decomposition methods yield ULS as subproblem

Compact reformulations are desired

Facility location (ULS)

Thanks to Krarup (1977)

- Facilities represent times for production (open facility $\Rightarrow$ positive production)
- Customers represent demand (customer $t \in T$ has demand of d_t)
- Customers and facilities are located on an one-dimensional line
- Locate facilities to serve customers on an one-way road
- Minimize cost

The model (FL of ULS)

y_s Indicator variable that tells if facility $s \in T$ is used or not (if production at time s)

w_{st} Indicator variable that tells if client $t \in T$ is serviced by facility $s \in T$ (if production at time s is used at time t)

d_t Demand at customer/time $t \in T$

f_s Cost of opening facility $s \in T$ (setup production at time s)

c_s Unit cost of production at facility/time $s \in T$

$$\min \sum_{s=1}^{|T|} \left(f_s y_s + \sum_{t=s}^{|T|} c_s d_t w_{st} \right) \tag{8}$$

$$\text{s.t.} \sum_{s=1}^{t} w_{st} = 1 \qquad \forall t \in T \tag{9}$$

$$w_{st} \leq y_s \qquad 1 \leq s \leq t \leq |T| \tag{10}$$

$$w_{st} \geq 0 \qquad 1 \leq s \leq t \leq |T| \tag{11}$$

$$y_t \in \{0,1\} \qquad \forall t \in T \tag{12}$$

LP relaxation of (8)-(12) always have an optimal solution with y integer

Multicommodity flow (ULS)

- Decompose flow on each arc to one for each destination
- Different commodity to each destination
- Flow on arcs represents production and stocking

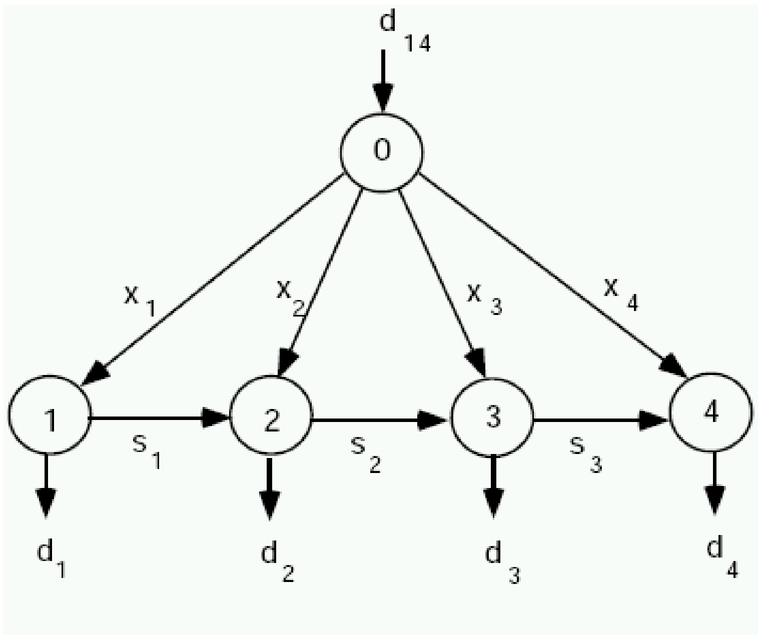

Fig. 3. Uncapacitated lot-sizing network $(n = 4)$

The model (MCF of ULS)

y_i setup at time $i \in T$

x_{it} production of commodity $t \in T$ at time $i \in T$

s_{it} inventory of commodity $t \in T$ at end of time $i \in T$

d_t demand at time $t \in T$

f_i setup cost at time $i \in T$

c_i cost of production at time $i \in T$

δ_{it} is 1 if $i = t$, otherwise 0

$$\min \sum_{i=1}^{|T|} \left(f_i y_i + \sum_{t=i}^{|T|} c_i x_{it} \right) \tag{13}$$

$$\begin{aligned}
\text{s.t.} \quad s_{i-1,t} + x_{it} = \delta_{it} d_t + s_{it} & \qquad 1 \le i \le t \le |T| && (14)\\
s_{0t} = s_{|T|t} = 0 & \qquad 1 \le t \le |T| && (15)\\
x_{it} \le d_t y_i & \qquad 1 \le i \le t \le |T| && (16)\\
x_{it}, s_{it} \ge 0 & \qquad 1 \le i \le t \le |T| && (17)\\
y_t \in \{0,1\} & \qquad \forall t \in T && (18)
\end{aligned}$$

LP relaxation of (13)-(18) always have an optimal solution with y integer

Shortest path (ULS)

- Solution consists of selecting some nodes (node 1, 3 and 7 in Fig. 5 – productions)
- Can be expressed in directed acyclic graph $G(A,V)$
  - One node $v \in V$ for each production/demand/time $t \in T$
  - One arc $(u,v) \in A$ for each $u,v \in V : u < v$

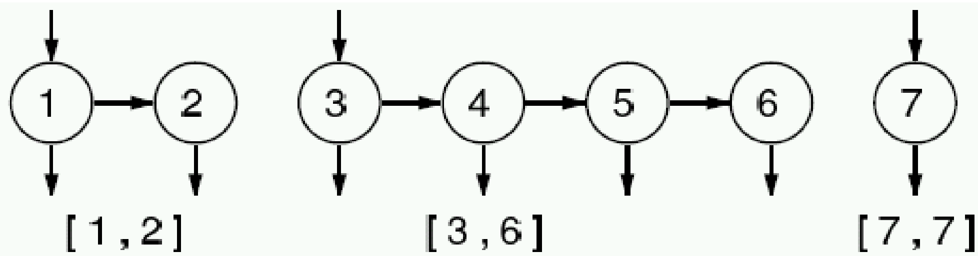

Fig. 5. An extreme solution as a succession of regeneration intervals

Selecting arc $(u,v) \in A$ represents serving demands $[u, v-1]$ with production from time $u \in T$ and setting $s_{v-1} = 0$

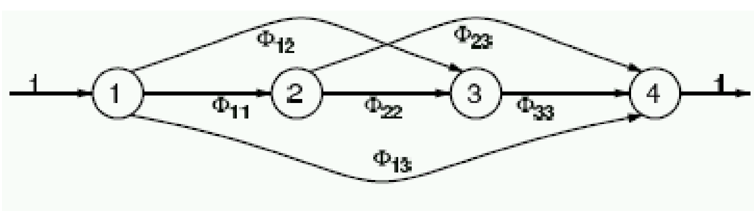

Fig. 6. Shortest Path Formulation of ULS $(n = 3)$

Φ_{uv} reflects the cost of producing and inventorying in this interval

The model (SP of ULS)

y_t setup at time $t \in T$

Φ_{it} production for interval $[i, t-1]$ done at time $i \in T$ (Phi)

d_{tk} accumulated demand in interval $[t, k]$

f_t setup cost at time $t \in T$

c_t cost of production at time $t \in T$

$$\min \sum_{t=1}^{|T|} \left(f_t y_t + \sum_{k=t}^{|T|} c_t d_{tk} \Phi_{tk} \right) \tag{19}$$

$$\text{s.t.} \sum_{t \in T} \Phi_{1t} = 1 \tag{20}$$

$$\sum_{i=1}^{t} \Phi_{it} - \sum_{l=t+2}^{|T|} \Phi_{t+1,l} = 0 \qquad 1 \leq t \leq |T|-1 \tag{21}$$

$$\sum_{t \in T} \Phi_{t|T|} = 1 \tag{22}$$

$$\sum_{k=t}^{|T|} \Phi_{tk} \leq y_t \qquad \forall t \in T \tag{23}$$

$$1 \geq \Phi_{tk} \geq 0 \qquad 1 \leq t \leq k \leq |T| \tag{24}$$

$$y_t \in \{0, 1\} \qquad \forall t \in T \tag{25}$$

LP relaxation of (20)-(25) always have an optimal solution with y integer – NOTE: (19) not necessary

Simple lot-sizing (polynomial solvable)

Three different linear formulations for ULS shown, all with $O(n^2)$ variables, but no more than $O(n^2)$ constraints

Interested in formulations with $O(n)$ variables, but potentially an exponential number of constraints

Only $O(n)$ constraints needed to describe any extreme point/solution

Solution: (1)-(6) combined with separation

Exercises

1) (23) can be removed from shortest path reformulation by making some adjustments to (19)–(25). Give the new model.
2) Give 'real world' restrictions that cannot be handled by the models? (Make some up or look on the world wide Internet)