Program of the day:
- Repetition: Definition of facets, dimension
- Cover inequalities
- Separation of valid inequalities
- Lifting of inequalities
- Applications: The Traveling Salesman Problem, separation of subtour constraints

Dominance

\[
\begin{align*}
\text{maximize} & \quad \ldots \\
\text{subject to} & \quad 1x_1 + 3x_2 \leq 4 \\
& \quad 2x_1 + 4x_2 \leq 9 \\
& \quad x_1, x_2 \geq 0
\end{align*}
\]

Multiplying the second inequality with \( u = \frac{1}{2} \)

\[
1x_1 + 2x_2 \leq \frac{9}{2}
\]

First inequality dominates the second.

Dominance:

\[
\pi x \leq \pi_0 \\
\mu x \leq \mu_0
\]

\( \pi x \leq \pi_0 \) dominates \( \mu x \leq \mu_0 \) if there exists \( u > 0 \) such that \( \pi \geq u \mu \) and \( \pi_0 \leq u \mu_0 \).

Redundance

\[
\begin{align*}
\text{maximize} & \quad \ldots \\
\text{subject to} & \quad 6x_1 - x_2 \leq 9 \\
& \quad 9x_1 - 5x_2 \leq 6 \\
& \quad 5x_1 - 2x_2 \leq 6 \\
& \quad x_1, x_2 \geq 0
\end{align*}
\]

Multiplying the first two constraints with \( u = \left( \frac{1}{3}, \frac{1}{5} \right) \)

\[
5x_1 - 2x_2 \leq 5
\]

Last inequality is redundant

Polyhedra, Facets

Polyhedron \( P \subset \mathbb{R}^2 \)

\[
\begin{align*}
\text{subject to} & \quad x_1 + x_2 \leq 2 \\
& \quad x_1 + 2x_2 \leq 4 \\
& \quad x_1 + 2x_2 \leq 10 \\
& \quad x_1 + x_2 \geq 6 \\
& \quad x_1, x_2 \geq 0
\end{align*}
\]

- \( P \subset \mathbb{R}^2 \) and “both directions are present”
- \( P \) is full-dimensional.
- The points \((2,0), (1,1) \) and \((2,2)\) are affinely independent points.
- The dimension of \( P \) is one less than the number of affinely independent points.
Polyhedra, Facets

Affinely independent
The points \( x^1, x^2, \ldots, x^k \in \mathbb{R}^n \) are affinely independent if directions \( (x^2 - x^1), \ldots, (x^k - x^1) \) are linearly independent.

Dimension
The dimension of \( P \), denoted \( \dim(P) \), is one less than the maximum number of affinely independent points in \( P \).

Full-dimensional
\( P \subseteq \mathbb{R}^n \) is full-dimensional iff \( \dim(P) = n \).

If not full-dimensional, eliminate some variables:
\[
\begin{align*}
x_1 + x_2 & \leq 3 \\
x_1 - x_2 & \leq 0 \iff 2x_1 \leq 3 \\
-x_1 + x_2 & \leq 0
\end{align*}
\]

Face
If \( \pi x \leq \pi_0 \) is a valid inequality of \( P \) then \( F \) is a face of \( P \)
\[
F = \{ x \in P : \pi x = \pi_0 \}
\]

Facet
\( F \) is a facet of \( P \) iff
- \( F \) is a face of \( P \)
- \( \dim(F) = \dim(P) - 1 \)

Cover inequalities
Consider the set
\[
X = \left\{ x \in \mathbb{B}^n : \sum_{j=1}^{n} a_j x_j \leq b \right\}
\]
We assume that \( a_j \geq 0 \) and \( b \geq 0 \).

Cover
A set \( C \subseteq N \) is a cover if
\[
\sum_{j \in C} a_j > b
\]
A set \( C \subseteq N \) is a minimal cover if \( C \setminus \{ j \} \) is not a cover for any \( j \in C \).

Cover Inequality
If \( C \) is a cover the cover inequality
\[
\sum_{j \in C} x_j \leq |C| - 1
\]
is valid for \( X \).

Cover inequalities
\[
11x_1 + 6x_2 - 6x'_3 + 5x_4 + 5x_5 + 4x_6 + x_7 \leq 13
\]
\( x \in \{0, 1\} \)

To get positive coefficients we substitute \( x_3 = 1 - x'_3 \)
\[
11x_1 + 6x_2 + 6x_3 + 5x_4 + 5x_5 + 4x_6 + x_7 \leq 19
\]

Observe
- At most two of \( x_1, x_2 \) and \( x_3 \) can be 1.
- At most two of \( x_1, x_2 \) and \( x_6 \) can be 1.
- At most two of \( x_1, x_5 \) and \( x_6 \) can be 1.
- At most three of \( x_3, x_4, x_5 \) and \( x_6 \) can be 1.

Separation problem
The separation problem decides whether a LP-solution vector satisfies all constraints of a given family \( \mathcal{F} \). If it does not, it must return a violated constraint in \( \mathcal{F} \).
Consider a constraint in a IP model
\[ \sum_{i \in I} a_i x_i \leq b \]
LP-solution \( x = x' \) is fractional. Solve problem
\[ \gamma = \min \sum_{i \in I} (1 - x'_i) \delta_i \]
subject to \( \sum_{i \in I} a_i \delta_i \geq b + 1 \) \hspace{1cm} (1)
\( \delta_i \in \{0, 1\}, \quad i \in I. \)
If \( \gamma < 1, \) let
\[ C = \{ i \in I : \delta_i = 1 \} \]
New inequality
\[ \sum_{i \in C} x_i \leq |C| - 1 \] \hspace{1cm} (2)

**C is a violated cover**

- It is a cover since
  \[ \sum_{i \in C} a_i = \sum_{i \in I} a_i \delta_i \geq b + 1 > b \]
- Assume that we remove items \( j \) with \( 1 - x'_j = 0 \) from \( C \) as long as \( \sum_{i \in C} a_i > b \). Then \( C \) is a minimal cover, since if we were able to remove an item \( j \) from \( C \) and still have a cover, then we would have a solution to (1) with smaller objective function.
- It is a violated inequality since assume that (2) actually was satisfied for the current solution \( x' \). Then we can choose \( \delta_k = 1 \) for \( k \in C \) as a solution to (1). This is a valid solution (due to the definition of \( C \)), and it has objective value
  \[ \sum_{i \in I} (1 - x'_i) \delta_i = \sum_{i \in C} (1 - x'_i) = |C| - \sum_{i \in C} x'_i \geq |C| - |C| + 1 = 1 \]
  But this violates the assumption saying that \( \gamma < 1. \)

**Example**

Consider the IP problem:
\[ \text{maximize} \quad 4x_1 + 5x_2 + 6x_3 + 7x_4 + 8x_5 + 3x_6 + 4x_7 \]
subject to \( 11x_1 + 6x_2 + 6x_3 + 5x_4 + 5x_5 + 4x_6 + x_7 \leq 19 \)
\( x_1, x_2, x_3, x_4, x_5, x_6, x_7 \in \mathbb{B} \)
LP-solution: \( x_2 = \frac{1}{3}, x_3 = 1, x_4 = 1, x_5 = 1, x_7 = 1 \)
Separation problem
\[ \text{minimize} \quad 1\delta_1 + \frac{2}{3}\delta_2 + 0\delta_3 + 0\delta_4 + 0\delta_5 + 1\delta_6 + 0\delta_7 \]
subject to \( 11\delta_1 + 6\delta_2 + 6\delta_3 + 5\delta_4 + 5\delta_5 + 4\delta_6 + 1\delta_7 \geq 20 \)
\( \delta_1, \delta_2, \delta_3, \delta_4, \delta_5, \delta_6, \delta_7 \in \mathbb{B} \)
Solution: \( \delta_2 = 1, \delta_3 = 1, \delta_4 = 1, \delta_5 = 1 \) with \( \gamma = \frac{2}{3} < 1 \)
Cover
\[ C = \{2, 3, 4, 5\} \]
Most violated cover inequality
\[ x_2 + x_3 + x_4 + x_5 \leq 3 \]

**Cover inequalities**
\[ 11x_1 + 6x_2 + 6x_3 + 5x_4 + 5x_5 + 4x_6 + x_7 \leq 19 \]
Minimal cover inequality
\[ x_3 + x_4 + x_5 + x_6 \leq 3 \]
Extended cover inequalities for \( C = \{3, 4, 5, 6\} \)
\[ x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \leq 3 \]
which dominates the first-mentioned

**Extended cover inequalities**
If \( C \) is a cover for \( X \), the extended cover inequality
\[ \sum_{j \in E(C)} x_j \leq |C| - 1 \]
is valid, where
\[ E(C) = C \cup \{ j \in N : a_j \geq a_i \text{ for all } i \in C \} \]
Lifting Cover Inequalities

11x_1 + 6x_2 + 6x_3 + 5x_4 + 5x_5 + 4x_6 + x_7 \leq 19 \quad (3)

Have found cover inequality

x_3 + x_4 + x_5 + x_6 \leq 3

What is the value of \( \alpha_1 \) such that

\[ \alpha_1 x_1 + x_3 + x_4 + x_5 + x_6 \leq 3 \] \quad (4)

is valid for all \( x \in X \)?

Constraint (4) must be valid whenever (3) is valid. Most difficult to satisfy (4) when \( x_2 = x_7 = 0 \).

- \( x_1 = 0 \) then (4) is valid.
- \( x_1 = 1 \) then we demand that

\[ \alpha_1 + x_3 + x_4 + x_5 + x_6 \leq 3 \]

whenever

\[ 11 + 6x_3 + 5x_4 + 5x_5 + 4x_6 \leq 19 \]

Maximize \( x_3 + x_4 + x_5 + x_6 \) subject to the second inequality

\[ \gamma = \text{maximize} \quad x_3 + x_4 + x_5 + x_6 \]

subject to \( 11 + 6x_3 + 5x_4 + 5x_5 + 4x_6 \leq 19 \)

\[ x_j \in \{0, 1\} \]

Thus \( \alpha_1 = 3 - \gamma = 2 \).

Strength of cover inequalities (Balas)

- Order the variables so that \( a_1 \geq a_2 \geq \ldots \geq a_n \)
- Let \( C = \{j_1, j_2, \ldots, j_n\} \) be a cover where \( j_1 < j_2 < \ldots < j_n \)
- Let \( p = \min \{j : j \in N \setminus E(C)\} \)
- The cover inequality

\[ \sum_{j \in E(C)} x_j \leq |C| - 1 \]

is a facet of \( \text{conv}(X) \) if one of the following holds

- \( C = N \)
- \( E(C) = N \) and \( \sum_{j \in C \setminus \{j_1, j_2\}} a_j + a_1 \leq b \)
- \( C = E(C) \) and \( \sum_{j \in C \setminus \{j_1, j_2\}} a_j + a_p \leq b \)
- \( C \subset E(C) \subset N \) and \( \sum_{j \in C \setminus \{j_1, j_2\}} a_j + a_1 \leq b \) and \( \sum_{j \in C \setminus \{j_1, j_2\}} a_j + a_p \leq b \)

Symmetric Traveling Salesman Problem

One of most famous and most applicable optimization problems

Given a finite number of “cities” along with the cost of travel between each pair of them, find the cheapest way of visiting all the cities and returning to your starting point.

Recently Applegate, Bixby, Chvatal, Cook solved USA13509 and Germany15112

Symmetric Traveling Salesman Problem

- Set of \( V \) cities
- To each edge \( e \in E \) is associated a cost \( c_e \)
- Visit each city exactly once
- Minimize travel cost
- \( x_e = 1 \) if edge \( e \) is used

Model 1

\[
\begin{align*}
\min & \quad \sum_{e \in E} c_e x_e \\
\text{s.t.} & \quad \sum_{e \in \delta(j)} x_e = 2, \quad j \in V \\
& \quad \sum_{e \in E(S)} x_e \leq |S| - 1, \quad S \subset V, S \neq V \\
& \quad x_e \in \{0, 1\}
\end{align*}
\]

Model 2

\[
\begin{align*}
\min & \quad \sum_{e \in E} c_e x_e \\
\text{s.t.} & \quad \sum_{e \in \delta(j)} x_e = 2, \quad j \in V \\
& \quad \sum_{e \in E(S)} x_e \geq 2, \quad S \subset V, S \neq V \\
& \quad x_e \in \{0, 1\}
\end{align*}
\]

degree constraint, subtour elimination constraint
Symmetric Traveling Salesman Problem

Subtour LP

\[ \begin{align*}
\text{min} & \quad \sum_{e \in E} c_e x_e \\
\text{s.t.} & \quad \sum_{e \in \delta(j)} x_e = 2, \quad j \in V \\
& \quad \sum_{e \in \delta(S)} x_e \geq 2, \quad S \subset V, S \neq V \\
& \quad 0 \leq x_e \leq 1
\end{align*} \]

exponentially many constraints

Cutting plane algorithm:

1. solve problem without subtour elimination constraints getting \( x_e^* \)
2. if \( x_e^* \) is a Hamilton cycle, stop
3. solve separation problem obtaining a valid inequality
   \[ \sum_{e \in \delta(S)} x_e \geq 2 \]
   such that
   \[ \sum_{e \in \delta(S)} x_e^* < 2 \]
4. add the valid inequality to the problem and repeat

Separation problem:

- capacitated network \((V, E, d)\)
- \( d_e = x_e^* \)
- find min cut in graph
- optimal solution has value less than 2 iff violated constraint exists
- min-cut can be found in \( O(nm\log n) \) time where \( n = |V| \) and \( m = |E| \).
- try each pair of nodes, i.e. run min-cut \( n(n-1)/2 \) times

\[ \begin{pmatrix}
-4 & 3 & 3 & 5 & 2 & 5 \\
-5 & 3 & 3 & 4 & 7 \\
-4 & -4 & 6 & 0 & 4 \\
-4 & -4 & -4 & 6 & 0 \\
-5 & -5 & -5 & -5 & 8 \\
-3 & -3 & -3 & -3 & 3 \\
\end{pmatrix} \]

\[ c_e = \begin{pmatrix}
0 & 5 & 3 & 6 & 8 & 5 \\
0 & 0 & 5 & 6 & 8 & 5 \\
0 & 0 & 0 & 6 & 8 & 5 \\
0 & 0 & 0 & 0 & 6 & 8 \\
0 & 0 & 0 & 0 & 0 & 6 \\
0 & 0 & 0 & 0 & 0 & 3 \\
\end{pmatrix} \]
Symmetric Traveling Salesman Problem

\begin{align*}
\text{minimize} & \quad + 4 x_{12} + 3 x_{13} + 3 x_{14} + 5 x_{15} + 2 x_{16} + 5 x_{17} + 5 x_{23} + 3 x_{24} + 3 x_{25} + 4 x_{26} + 7 x_{27} + 4 x_{34} + 6 x_{35} + 6 x_{36} + 4 x_{37} + 4 x_{45} + 4 x_{46} + 6 x_{47} + 5 x_{56} + 8 x_{57} + 3 x_{67} \\
\text{subject to} & \quad x_{12} + x_{13} + x_{14} + x_{15} + x_{16} + x_{17} = 2 \\
& \quad x_{12} + x_{23} + x_{24} + x_{25} + x_{26} + x_{27} = 2 \\
& \quad x_{13} + x_{23} + x_{34} + x_{35} + x_{36} + x_{37} = 2 \\
& \quad x_{14} + x_{24} + x_{34} + x_{45} + x_{46} + x_{47} = 2 \\
& \quad x_{15} + x_{25} + x_{35} + x_{45} + x_{56} + x_{57} = 2 \\
& \quad x_{16} + x_{26} + x_{36} + x_{46} + x_{56} + x_{67} = 2 \\
& \quad x_{17} + x_{27} + x_{37} + x_{47} + x_{57} + x_{67} = 2 \\
& \quad x_{13} + x_{23} + x_{34} + x_{36} + x_{37} + x_{56} + x_{57} = 2 \\
& \quad x_{12} + x_{23} + x_{26} + x_{27} + x_{14} + x_{34} + x_{46} + x_{47} + x_{15} + x_{35} + x_{46} + x_{57} = 2 \\
& \quad x_{12} + x_{13} + x_{14} + x_{15} + x_{16} + x_{17} = 2 + 2 + x_{23} + x_{24} + x_{25} + x_{26} + x_{27} + x_{34} + x_{35} + x_{36} + x_{37} + x_{45} + x_{46} + x_{47} + x_{56} + x_{57} + x_{67} = 2 \\
\text{binary} & \quad x_{12}, x_{13}, x_{14}, x_{15}, x_{16}, x_{17}, x_{23}, x_{24}, x_{25}, x_{26}, x_{27}, x_{34}, x_{35}, x_{36}, x_{37}, x_{45}, x_{46}, x_{47}, x_{56}, x_{57}, x_{67}
\end{align*}

Separation for generalized subtour constraints

Assume that we solve the ILP-problem

\begin{align*}
\max \sum_{j \in N} f_j y_j - \sum_{e \in E} c_e x_e \\
\text{s.t.} & \quad \sum_{e \in \delta(j)} x_e = 2y_j, \quad j \in N \\
& \quad y_1 = 1 \\
& \quad x \in \{0, 1\}, y \in \{0, 1\}
\end{align*}

getting a solution \((x^*, y^*)\). How do we find a violated GSE constraint?

- \(N' = N \setminus 1\)
- \(E' = E \setminus \{\delta(1)\}\)
- \(z_i = 1\) iff \(i \in S\)

A constraint for \((k, S)\) is violated if

\[ \sum_{e \in E'(S)} x_e^* > \sum_{i \in S \setminus \{k\}} y_i^* \]

This can be formulated as a maximization problem

\[ \gamma = \max \sum_{e \in E'} x_e^* z_e - \sum_{i \in N \setminus \{k\}} y_i^* z_i \]

s.t. \(z_k = 1\) \(z \in \{0, 1\}\)

Prize Collecting Traveling Salesman Problem

- Set of \(N\) cities.
- Salesman starts in city 1.
- To each edge \(e\) is associated a cost \(c_e\)
- To each node \(j\) is associated a profit \(f_j\)
- Visit at least two other cities
- Maximize profit - cost.

Introduce variables

- \(x_e = 1\) if edge \(e\) is used.
- \(y_j = 1\) if node \(j\) is visited.

Formulation

\begin{align*}
\max & \quad \sum_{j \in N} f_j y_j - \sum_{e \in E} c_e x_e \\
\text{s.t.} & \quad \sum_{e \in \delta(j)} x_e = 2y_j, \quad j \in N \\
& \quad \sum_{e \in E(S)} x_e \leq \sum_{i \in S \setminus \{k\}} y_i, \quad k \in S, S \subseteq N \setminus \{1\} \\
& \quad y_1 = 1 \\
& \quad x \in \{0, 1\}, y \in \{0, 1\}
\end{align*}

Separation for generalized subtour constraints

The quadratic 0-1 program

\[ \gamma = \max \sum_{e=(i,j) \in E'} x_e^* z_{ij} - \sum_{i \in N \setminus \{k\}} y_i^* z_i \\
\text{s.t.} \quad \sum_{e \in E'(S)} x_e = 1 \quad z \in \{0, 1\} \]

can be reformulated using

\[ w_{(i,j)} = 1 \iff z_i = 1 \text{ and } z_j = 1 \]

but since we maximize only

\[ w_{(i,j)} = 1 \implies z_i = 1 \text{ and } z_j = 1 \]

is needed

\[ \gamma = \max \sum_{e=(i,j) \in E'} x_e^* w_e - \sum_{i \in N \setminus \{k\}} y_i^* z_i \\
\text{s.t.} \quad w_{(i,j)} \leq z_i, \quad (i, j) \in E', \quad w_{(i,j)} \leq z_j, \quad (i, j) \in E' \\
& \quad z_k = 1 \\
& \quad w \in \{0, 1\}, z \in \{0, 1\} \]

This formulation is TU and thus can be solved in polynomial time.
Separation for generalized subtour constraints

\[ f = (2, 4, 1, 3, 7, 7) \text{ and } c_e = \begin{pmatrix}
-4 & 3 & 3 & 5 & 2 & 5 \\
-3 & 5 & 3 & 4 & 7 \\
-5 & -4 & 4 & 0 & 4 \\
-4 & -4 & -4 & 4 & 6 \\
-4 & -4 & -4 & 5 & 8 \\
-4 & -4 & -4 & -4 & 3 \\
\end{pmatrix} \]

The LP-relaxation of (5) gives the routes

(1, 5, 2, 4) and (3, 6, 7)

The separation algorithm returns

\[ x_{36} + x_{37} + x_{67} \leq y_3 + y_7 \]

which cuts off the subtour (3, 6, 7).