Introduction to Optimization, DIKU 2007
Tuesday, March 13
David Pisinger

Program of the day:

- Repetition: Definition of facets, dimension
- Cover inequalities
- Separation of valid inequalities
- Lifting of inequalities
- Applications: The Traveling Salesman Problem, separation of subtour constraints

Dominance

$$\begin{array}{ll} \text{maximize} & \dots \\ \text{subject to} & 1x_1 + 3x_2 \leq 4 \\ & 2x_1 + 4x_2 \leq 9 \\ & x_1, x_2 \geq 0 \end{array}$$

Multiplying the second inequality with $u = \frac{1}{2}$

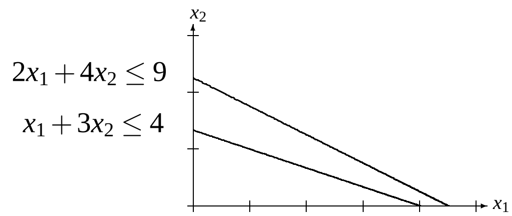
$$1x_1 + 2x_2 \leq \frac{9}{2}$$

First inequality dominates the second.

Dominance:

$$\pi x \leq \pi_0 \qquad \mu x \leq \mu_0$$

$\pi x \leq \pi_0$ *dominates* $\mu x \leq \mu_0$ if there exists $u > 0$ such that $\pi \geq u\mu$ and $\pi_0 \leq u\mu_0$.



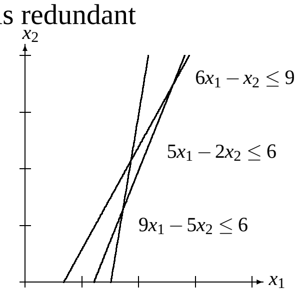
Redundance

$$\begin{array}{ll} \text{maximize} & \dots \\ \text{subject to} & 6x_1 - x_2 \leq 9 \\ & 9x_1 - 5x_2 \leq 6 \\ & 5x_1 - 2x_2 \leq 6 \\ & x_1, x_2 \geq 0 \end{array}$$

Multiplying the first two constraints with $u = (\frac{1}{3}, \frac{1}{3})$

$$5x_1 - 2x_2 \leq 5$$

Last inequality is redundant



Redundance:

$$\pi^i x \leq \pi^i_0, \;\; i = 1, \dots, k$$
$$\mu x \leq \mu_0$$

Inequality $\mu x \leq \mu_0$ is *redundant* if there exists a vector $(u_1, \dots, u_k) \geq 0$ such that

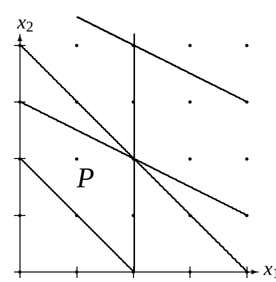
$$(\sum_{i=1}^{k} u_i \pi^i) x_i \leq \sum_{i=1}^{k} u_i \pi^i_0$$

dominates $\mu x \leq \mu_0$

Polyhedra, Facets

Polyhedra $P \subset \mathbb{R}^2$

$$\begin{array}{ll} \text{subject to} & x_1 \leq 2 \\ & x_1 + x_2 \leq 4 \\ & x_1 + 2x_2 \leq 10 \\ & x_1 + 2x_2 \leq 6 \\ & x_1 + x_2 \geq 2 \\ & x_1, x_2 \geq 0 \end{array}$$



- $P \subset \mathbb{R}^2$ and “both directions are present”
- P is full-dimensional.
- The points $(2,0)$, $(1,1)$ and $(2,2)$ are affinely independent points.
- The dimension of P is one less than the number of affinely independent points.

Polyhedra, Facets

Affinely independent

The points $x^1, x^2, \ldots, x^k \in \mathbb{R}^n$ are affinely independent if directions $(x^2 - x^1), \ldots, (x^k - x^1)$ are linearly independent

Dimension

The dimension of P, denoted $\dim(P)$ is one less than the maximum number of affinely independent points in P.

Full-dimensional

$P \subseteq \mathbb{R}^n$ is full-dimensional iff $\dim(P) = n$.
If not full-dimensional, eliminate some variables:

$$\begin{cases} x_1 + x_2 \leq 3 \\ x_1 - x_2 \leq 0 \\ -x_1 + x_2 \leq 0 \end{cases} \Leftrightarrow 2x_1 \leq 3$$

Face

If $\pi x \leq \pi_0$ is a valid inequality of P then F is a face of P

$$F = \{x \in P : \pi x = \pi_0\}$$

Facet

F is a facet of P iff

- F is a face of P
- $\dim(F) = \dim(P) - 1$

Cover inequalities

$$\begin{aligned} &11x_1 + 6x_2 - 6x_3' + 5x_4 + 5x_5 + 4x_6 + x_7 \leq 13 \\ &x \in \{0,1\} \end{aligned}$$

To get positive coefficients we substitute $x_3 = 1 - x_3'$

$$11x_1 + 6x_2 + 6x_3 + 5x_4 + 5x_5 + 4x_6 + x_7 \leq 19$$

Observe

- At most two of x_1, x_2 and x_3 can be 1.
- At most two of x_1, x_2 and x_6 can be 1.
- At most two of x_1, x_5 and x_6 can be 1.
- At most three of x_3, x_4, x_5 and x_6 can be 1.

Cover inequalities

Consider the set

$$X = \left\{ x \in \mathbb{B}^n : \sum_{j=1}^{n} a_j x_j \leq b \right\}$$

We assume that $a_j \geq 0$ and $b \geq 0$.

Cover

A set $C \subseteq N$ is a cover if

$$\sum_{j \in C} a_j > b$$

A set $C \subseteq N$ is a minimal cover if $C \setminus \{j\}$ is not a cover for any $j \in C$

Cover Inequality

If C is a cover the cover inequality

$$\sum_{j \in C} x_j \leq |C| - 1$$

is valid for X.

Cover inequalities

$$11x_1 + 6x_2 + 6x_3 + 5x_4 + 5x_5 + 4x_6 + x_7 \leq 19$$

Family $\mathcal{F}$ of minimal cover inequalities

$$\mathcal{F} = \begin{cases} x_1 + x_2 + x_3 \leq 2 \\ x_1 + x_2 + x_4 \leq 2 \\ x_1 + x_2 + x_5 \leq 2 \\ x_1 + x_2 + x_6 \leq 2 \\ x_1 + x_3 + x_4 \leq 2 \\ x_1 + x_3 + x_5 \leq 2 \\ x_1 + x_3 + x_6 \leq 2 \\ x_1 + x_4 + x_5 \leq 2 \\ x_1 + x_4 + x_6 \leq 2 \\ x_1 + x_5 + x_6 \leq 2 \\ x_2 + x_3 + x_4 + x_5 \leq 3 \\ x_2 + x_3 + x_4 + x_6 \leq 3 \\ x_3 + x_4 + x_5 + x_6 \leq 3 \\ \vdots \end{cases}$$

Separation problem

The separation problem decides whether a LP-solution vector satisfies all constraints of a given family $\mathcal{F}$. If it does not, it must return a violated constraint in $\mathcal{F}$

Separation of cover inequalities

Consider a constraint in a IP model

$$\sum_{i\in I} a_i x_i \le b$$

LP-solution $x = x'$ is fractional. Solve problem

$$\begin{aligned} \gamma = \text{minimize} \quad & \sum_{i\in I}(1-x_i')\delta_i \\ \text{subject to} \quad & \sum_{i\in I} a_i\delta_i \ge b+1 \qquad (1) \\ & \delta_i \in \{0,1\}, \qquad i \in I. \end{aligned}$$

If $\gamma < 1$, let

$$C = \{i \in I : \delta_i = 1\}$$

New inequality

$$\sum_{i\in C} x_i \le |C| - 1 \qquad (2)$$

Example

Consider the IP problem:

$$\begin{aligned} \text{maximize} \quad & 4x_1+5x_2+6x_3+7x_4+8x_5+3x_6+4x_7 \\ \text{subject to} \quad & 11x_1+6x_2+6x_3+5x_4+5x_5+4x_6+x_7 \le 19 \\ & x_1,x_2,x_3,x_4,x_5,x_6,x_7 \in \mathbb{B} \end{aligned}$$

LP-solution: $x_2 = \frac{1}{3}, x_3 = 1, x_4 = 1, x_5 = 1, x_7 = 1$

Separation problem

$$\begin{aligned} \text{minimize} \quad & 1\delta_1+\tfrac{2}{3}\delta_2+0\delta_3+0\delta_4+0\delta_5+1\delta_6+0\delta_7 \\ \text{subject to} \quad & 11\delta_1+6\delta_2+6\delta_3+5\delta_4+5\delta_5+4\delta_6+1\delta_7 \ge 20 \\ & \delta_1,\delta_2,\delta_3,\delta_4,\delta_5,\delta_6,\delta_7 \in \mathbb{B} \end{aligned}$$

Solution: $\delta_2 = 1$, $\delta_3 = 1$, $\delta_4 = 1$, $\delta_5 = 1$. with $\gamma = \frac{2}{3} < 1$

Cover

$$C = \{2,3,4,5\}$$

Most violated cover inequality

$$x_2 + x_3 + x_4 + x_5 \le 3$$

C is a violated cover

- It is a *cover* since
$$\sum_{i\in C} a_i = \sum_{i\in I} a_i\delta_i \ge b+1 > b$$
- Assume that we remove items j with $1 - x_j' = 0$ from C as long as $\sum_{i\in C} a_i > b$. Then C is a *minimal* cover, since if we were able to remove an item j from C and still have a cover, then we would have a solution to (1) with smaller objective function.
- It is a *violated* inequality since assume that (2) actually was satisfied for the current solution x'. Then we can choose $\delta_k = 1$ for $k \in C$ as a solution to (1). This is a valid solution (due to the definition of C), and it has objective value
$$\sum_{i\in I}(1-x_i')\delta_i = \sum_{i\in C}(1-x_i') = |C| - \sum_{i\in C} x_i' \ge |C| - |C| + 1 = 1$$
But this violates the assumption saying that $\gamma < 1$.

Cover inequalities

$$11x_1+6x_2+6x_3+5x_4+5x_5+4x_6+x_7 \le 19$$

Minimal cover inequality

$$x_3 + x_4 + x_5 + x_6 \le 3$$

Extended cover inequalities for $C = \{3,4,5,6\}$

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \le 3$$

which dominates the first-mentioned

Extended cover inequalities

If C is a cover for X, the extended cover inequality

$$\sum_{j\in E(C)} x_j \le |C| - 1$$

is valid, where

$$E(C) = C \cup \{j \in N : a_j \ge a_i \text{ for all } i \in C\}$$

Lifting Cover Inequalities

$$11x_1+6x_2+6x_3+5x_4+5x_5+4x_6+x_7 \le 19 \qquad (3)$$

Have found cover inequality

$$x_3+x_4+x_5+x_6 \le 3$$

What is the value of α_1 such that

$$\alpha_1 x_1+x_3+x_4+x_5+x_6 \le 3 \qquad (4)$$

is valid for all $x \in X$?

Constraint (4) must be valid whenever (3) is valid.
Most difficult to satisfy (4) when $x_2 = x_7 = 0$.

- $x_1 = 0$ then (4) is valid.
- $x_1 = 1$ then we demand that

  $$\alpha_1+x_3+x_4+x_5+x_6 \le 3$$

  whenever

  $$11+6x_3+5x_4+5x_5+4x_6 \le 19$$

  Maximize $x_3+x_4+x_5+x_6$ subject to the second inequality

  $$\begin{aligned} \gamma = \text{maximize} \;\; & x_3+x_4+x_5+x_6 \\ \text{subject to} \;\; & 11+6x_3+5x_4+5x_5+4x_6 \le 19 \\ & x_i \in \{0,1\} \end{aligned}$$

  a Knapsack Problem with solution $\gamma = 1$.
  Thus $\alpha_1 = 3-\gamma = 2$.

Strength of cover inequalities (Balas)

- Order the variables so that $a_1 \ge a_2 \ge \ldots \ge a_n$
- Let $C = \{j_1, j_2, \ldots, j_n\}$ be a cover where $j_1 < j_2 < \ldots < j_n$
- Let $p = \min\{j : j \in N \setminus E(C)\}$
- The cover inequality

  $$\sum_{j \in E(C)} x_j \le |C|-1$$

  is a facet of $\text{conv}(X)$ if one of the following holds

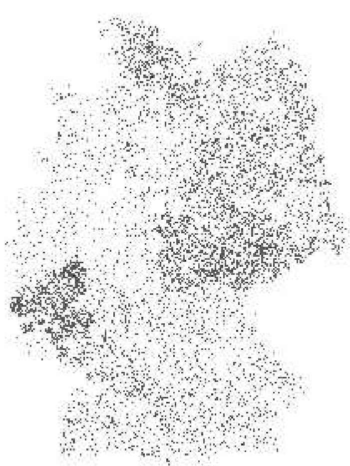
  · $C = N$
  · $E(C) = N$ and $\sum_{j \in C\setminus\{j_1,j_2\}} a_j + a_1 \le b$
  · $C = E(C)$ and $\sum_{j \in C\setminus\{j_1\}} a_j + a_p \le b$
  · $C \subset E(C) \subset N$ and $\sum_{j \in C\setminus\{j_1,j_2\}} a_j + a_1 \le b$ and $\sum_{j \in C\setminus\{j_1\}} a_j + a_p \le b$

Symmetric Traveling Salesman Problem

One of most famous and most applicable optimization problems

> Given a finite number of "cities" along with the cost of travel between each pair of them, find the cheapest way of visiting all the cities and returning to your starting point.

Recently Applegate, Bixby, Chvatal, Cook solved USA13509 and Germany15112



Symmetric Traveling Salesman Problem

- Set of V cities
- To each edge $e \in E$ is associated a cost c_e
- Visit each city exactly once
- Minimize travel cost
- $x_e = 1$ if edge e is used

Model 1

$$\begin{aligned} \min \;\; & \sum_{e \in E} c_e x_e \\ \text{s.t.} \;\; & \sum_{e \in \delta(j)} x_e = 2 && , \; j \in V \\ & \sum_{e \in E(S)} x_e \le |S|-1 && , \; S \subset V, S \neq V \\ & x_e \in \{0,1\} \end{aligned}$$

Model 2

$$\begin{aligned} \min \;\; & \sum_{e \in E} c_e x_e \\ \text{s.t.} \;\; & \sum_{e \in \delta(j)} x_e = 2 && , \; j \in V \\ & \sum_{e \in \delta(S)} x_e \ge 2 && , \; S \subset V, S \neq V \\ & x_e \in \{0,1\} \end{aligned}$$

degree constraint, subtour elimination constraint

Symmetric Traveling Salesman Problem

Subtour LP

$$
\begin{aligned}
\min \quad & \sum_{e \in E} c_e x_e \\
\text{s.t.} \quad & \sum_{e \in \delta(j)} x_e = 2 \quad , \; j \in V \\
& \sum_{e \in \delta(S)} x_e \geq 2 \quad , \; S \subset V, S \neq V \\
& 0 \leq x_e \leq 1
\end{aligned}
$$

exponentially many constraints

Cutting plane algorithm:

1 solve problem without subtour elimination constraints getting x_e^*

2 if x_e^* is a Hamilton cycle, stop

3 solve separation problem obtaining a valid inequality

$$\sum_{e \in \delta(S)} x_e \geq 2$$

such that

$$\sum_{e \in \delta(S)} x_e^* < 2$$

4 add the valid inequality to the problem and repeat

Symmetric Traveling Salesman Problem

Separation problem:

- capacitated network (V, E, d)
- $d_e = x_e^*$
- find min cut in graph
- optimal solution has value less than 2 iff violated constraint exists
- min-cut can be found in $O(nm \log n)$ time where $n = |V|$ and $m = |E|$.
- try each pair of nodes, i.e. run min-cut $n(n-1)/2$ times

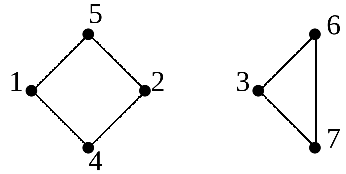
$$
c_e = \begin{pmatrix}
- & 4 & 3 & 3 & 5 & 2 & 5 \\
- & - & 5 & 3 & 3 & 4 & 7 \\
- & - & - & 4 & 6 & 0 & 4 \\
- & - & - & - & 4 & 4 & 6 \\
- & - & - & - & - & 5 & 8 \\
- & - & - & - & - & - & 3 \\
- & - & - & - & - & - & -
\end{pmatrix}
$$

Symmetric Traveling Salesman Problem

```
minimize
+ 4 x12 + 3 x13 + 3 x14 + 5 x15 + 2 x16 + 5 x17
        + 5 x23 + 3 x24 + 3 x25 + 4 x26 + 7 x27
                + 4 x34 + 6 x35 + 0 x36 + 4 x37
                        + 4 x45 + 4 x46 + 6 x47
                                + 5 x56 + 8 x57
                                        + 3 x67

subject to
      x12 + x13 + x14 + x15 + x16 + x17 = 2
x12 +       x23 + x24 + x25 + x26 + x27 = 2
x13 + x23 +       x34 + x35 + x36 + x37 = 2
x14 + x24 + x34 +       x45 + x46 + x47 = 2
x15 + x25 + x35 + x45 +       x56 + x57 = 2
x16 + x26 + x36 + x46 + x56 +       x67 = 2
x17 + x27 + x37 + x47 + x57 + x67       = 2

binary
    x12 x13 x14 x15 x16 x17
        x23 x24 x25 x26 x27
            x34 x35 x36 x37
                x45 x46 x47
                    x56 x57
                        x67
end
```



Symmetric Traveling Salesman Problem

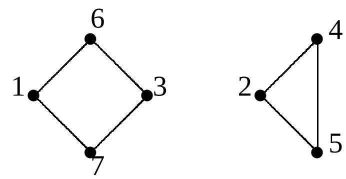
```
minimize
+ 4 x12 + 3 x13 + 3 x14 + 5 x15 + 2 x16 + 5 x17
        + 5 x23 + 3 x24 + 3 x25 + 4 x26 + 7 x27
                + 4 x34 + 6 x35 + 0 x36 + 4 x37
                        + 4 x45 + 4 x46 + 6 x47
                                + 5 x56 + 8 x57
                                        + 3 x67

subject to
      x12 + x13 + x14 + x15 + x16 + x17 = 2
x12 +       x23 + x24 + x25 + x26 + x27 = 2
x13 + x23 +       x34 + x35 + x36 + x37 = 2
x14 + x24 + x34 +       x45 + x46 + x47 = 2
x15 + x25 + x35 + x45 +       x56 + x57 = 2
x16 + x26 + x36 + x46 + x56 +       x67 = 2
x17 + x27 + x37 + x47 + x57 + x67       = 2

x13 + x23 + x34 + x35 +
x16 + x26 + x46 + x56 +
x17 + x27 + x47 + x57 >= 2

binary
    x12 x13 x14 x15 x16 x17
        x23 x24 x25 x26 x27
            x34 x35 x36 x37
                x45 x46 x47
                    x56 x57
                        x67

end
```



Symmetric Traveling Salesman Problem

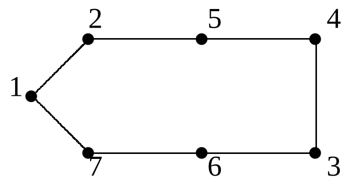
```
minimize
+ 4 x12 + 3 x13 + 3 x14 + 5 x15 + 2 x16 + 5 x17
        + 5 x23 + 3 x24 + 3 x25 + 4 x26 + 7 x27
                + 4 x34 + 6 x35 + 0 x36 + 4 x37
                        + 4 x45 + 4 x46 + 6 x47
                                + 5 x56 + 8 x57
                                        + 3 x67

subject to
      x12 + x13 + x14 + x15 + x16 + x17 = 2
x12 +       x23 + x24 + x25 + x26 + x27 = 2
x13 + x23 +       x34 + x35 + x36 + x37 = 2
x14 + x24 + x34 +       x45 + x46 + x47 = 2
x15 + x25 + x35 + x45 +       x56 + x57 = 2
x16 + x26 + x36 + x46 + x56 +       x67 = 2
x17 + x27 + x37 + x47 + x57 + x67       = 2

x13 + x23 + x34 + x35 +
x16 + x26 + x46 + x56 +
x17 + x27 + x47 + x57 >= 2

x12 + x23 + x26 + x27 +
x14 + x34 + x46 + x47 +
x15 + x35 + x56 + x57 >= 2

binary
    x12 x13 x14 x15 x16 x17
        x23 x24 x25 x26 x27
            x34 x35 x36 x37
                x45 x46 x47
                    x56 x57
                        x67
end
```



Prize Collecting Traveling Salesman Problem

- Set of N cities.
- Salesman starts in city 1.
- To each edge e is associated a cost c_e
- To each node j is associated a profit f_j
- Visit *at least* two other cities
- Maximize profit – cost.

Introduce variables

- $x_e = 1$ if edge e is used.
- $y_j = 1$ if node j is visited.

Formulation

$$\begin{array}{lll} \max & \sum_{j\in N} f_j y_j - \sum_{e\in E} c_e x_e & \\ \text{s.t.} & \sum_{e\in\delta(j)} x_e = 2y_j & , j \in N \\ & \sum_{e\in E(S)} x_e \le \sum_{i\in S\setminus\{k\}} y_i & , k\in S, S\subseteq N\setminus\{1\} \\ & y_1 = 1 & \\ & x_e \in \{0,1\}, y_j \in \{0,1\} & \end{array}$$

Separation for generalized subtour constraints

Assume that we solve the ILP-problem

$$\begin{array}{lll} \max & \sum_{j\in N} f_j y_j - \sum_{e\in E} c_e x_e & \\ \text{s.t.} & \sum_{e\in\delta(j)} x_e = 2y_j & , j \in N \\ & y_1 = 1 & \\ & x \in \{0,1\}, y \in \{0,1\} & \end{array} \qquad (5)$$

getting a solution (x^*, y^*). How do we find a violated GSE constraint?

- $N' = N \setminus 1$
- $E' = E \setminus \{\delta(1)\}$
- $z_i = 1$ iff $i \in S$

A constraint for (k, S) is violated if

$$\sum_{e\in E'(S)} x_e^* > \sum_{i\in S\setminus\{k\}} y_i^*$$

This can be formulated as a maximization problem

$$\begin{array}{ll} \gamma = \max & \sum_{e\in E'} x_e^* z_i z_j - \sum_{i\in N'\setminus\{k\}} y_i^* z_i \\ \text{s.t.} & z_k = 1 \\ & z \in \{0,1\} \end{array}$$

Separation for generalized subtour constraints

The quadratic 0-1 program

$$\begin{array}{ll} \gamma = \max & \sum_{e=(i,j)\in E'} x_e^* z_i z_j - \sum_{i\in N'\setminus\{k\}} y_i^* z_i \\ \text{s.t.} & z_k = 1 \\ & z \in \{0,1\} \end{array}$$

can be reformulated using

$$w_{(i,j)} = 1 \;\Leftrightarrow\; z_i = 1 \text{ and } z_j = 1$$

but since we maximize only

$$w_{(i,j)} = 1 \;\Rightarrow\; z_i = 1 \text{ and } z_j = 1$$

is needed

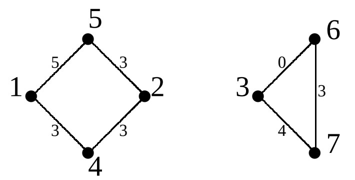
$$\begin{array}{lll} \gamma = \max & \sum_{e=(i,j)\in E'} x_e^* w_e - \sum_{i\in N'\setminus\{k\}} y_i^* z_i & \\ \text{s.t.} & w_{(i,j)} \le z_i & , (i,j)\in E' \\ & w_{(i,j)} \le z_j & , (i,j)\in E' \\ & z_k = 1 & \\ & w \in \{0,1\}, z \in \{0,1\} & \end{array}$$

This formulation is TU and thus can be solved in polynomial time

Separation for generalized subtour constraints

$$f = (2,4,1,3,7,1,7) \text{ and}$$

$$c_e = \begin{pmatrix} - & 4 & 3 & 3 & 5 & 2 & 5 \\ - & - & 5 & 3 & 3 & 4 & 7 \\ - & - & - & 4 & 6 & 0 & 4 \\ - & - & - & - & 4 & 4 & 6 \\ - & - & - & - & - & 5 & 8 \\ - & - & - & - & - & - & 3 \\ - & - & - & - & - & - & - \end{pmatrix}$$



The LP-relaxation of (5) gives the routes

$$(1,5,2,4) \text{ and } (3,6,7)$$

The separation algorithm returns

$$x_{36} + x_{37} + x_{67} \leq y_3 + y_7$$

which cuts off the subtour $(3,6,7)$.