Introduction to Optimization, DIKU 2007
Tuesday, February 27
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Program of the day: (Wolsey chapter 7)

- Solving MIP models by branch-and-bound
- Duality
- Design issues in branch-and-bound
- Strong Branching
- Local Branching
- Applications: Knapsack Problem (demo)

Techniques for MIP

- Preprocessing
- Branch-and-bound
- Valid cuts

Development

1960 Breakthrough: branch-and-bound

1970 Small problems ($n < 100$) may be solved. Exponential growth, many important problems cannot be solved.

1983 Crowder, Johnson, Padberg: new algorithm for pure BIP. Sparse matrices up to ($n = 2756$).

1985 Johnson, Kostreva, Sahl: further improvements.

1987 Van Roy, Wolsey: Mixed IP. Up to 1000 binary variables, several continuous variables.

Now Preprocessing, addition of cuts, good branching strategies

Solving IP by enumeration

- Binary IP

$$\begin{array}{ll} \text{maximize} & 2x_1 + 3x_2 - 1x_3 + 5x_4 \\ \text{subject to} & 4x_1 + 1x_2 + 2x_3 + 3x_4 \leq 8 \\ & x_1, x_2, x_3, x_4 \in \{0,1\} \end{array}$$

- Integer IP

$$\begin{array}{ll} \text{maximize} & 2x_1 + 3x_2 - 1x_3 + 5x_4 \\ \text{subject to} & 4x_1 + 1x_2 + 2x_3 + 3x_4 \leq 8 \\ & x_1, x_2, x_3, x_4 \in \mathbb{N}_0 \end{array}$$

- Mixed integer IP

$$\begin{array}{ll} \text{maximize} & 2x_1 + 3x_2 - 1x_3 + 5x_4 \\ \text{subject to} & 4x_1 + 1x_2 + 2x_3 + 3x_4 \leq 8 \\ & x_1, x_2 \in \mathbb{R} \\ & x_3, x_4 \in \{0,1\} \end{array}$$

Elements of Branch-and-bound

Problem

$$\begin{array}{ll} \text{maximize} & cx \\ \text{subject to} & x \in S \end{array}$$

- **Divide and conquer** (Wolsey prop. 7.1)
  $S = S_1 \cup S_2 \cup \ldots \cup S_K$ and $z^k = \max\{cx : x \in S_k\}$

  $$z = \max_{k=1,\ldots,K} z^k$$

  Overlap between S_i and S_j is allowed
  Often: decompose by splitting on decision variable

Elements of Branch-and-bound

- **Upper bound function** (Wolsey prop. 7.2)

$$\bar{z}^k = \sup\{cx : x \in S_k\}$$

then

$$\bar{z} = \max \bar{z}^k$$

is an upper bound on S

- **Lower bound** (so far best solution) $\underline{z}$

- **Upper bound test**

$$\text{if } \bar{z}^k \leq \underline{z} \text{ then } x^* \notin S_k$$

Relaxation (Wolsey 2.1)

$$\max\{cx : x \in S\} \qquad (IP)$$
$$\max\{f(x) : x \in T\} \qquad (RP)$$

RP is a relaxation of IP if

- $S \subseteq T$
- $f(x) \geq cx$ for all $x \in S$

Branch-and-bound

A systematical enumeration technique for solving IP/MIP problems, which apply bounding rules to avoid to examine specific parts of the solution space.

$$\begin{array}{ll} \text{maximize} & cx \\ \text{subject to} & Ax \leq b \\ & x' \geq 0 \\ & x'' \geq 0, \text{integer} \end{array}$$

- Branching tree enumerates all integer variables.
- Once all integer variables are fixed, remaining problem is solved by LP.
- General MIP algorithm does not know structure of problem
- Upper bounds $\bar{z}$ are derived in each node by LP-relaxation.
- If $\bar{z} \leq \underline{z}$ then descendant nodes need not to be examined

Branch-and-bound for MIP (maximization)

Maintain pool of open problems. In each iteration:

- If infeasible, backtrack
- Solve LP-relaxation, getting $\bar{x}$ and $\bar{z}$
- If $\bar{z} \leq \underline{z}$ then backtrack
- If all x are integral: update $\underline{z}$, backtrack
- Choose a fractional variable $\bar{x}_i = d$
- Branch on

$$\bar{x}_i \leq \lfloor d \rfloor \qquad \bar{x}_i \geq \lceil d \rceil$$

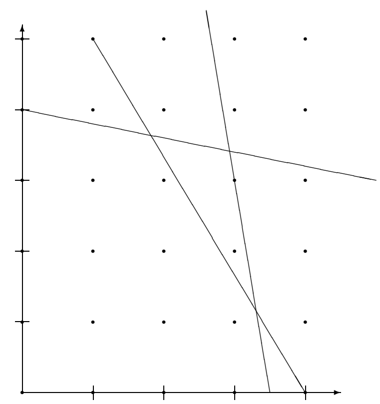
Where

- $\underline{z}$ is so far best solution (incumbent solution)
- $\bar{z}$ is upper bound at node
- $\bar{x}$ is LP-solution to current problem

Branch-and-bound for MIP

Example:

$$\begin{array}{lrcrcr} \text{maximize} & x_1 & + & x_2 & & \\ \text{subject to} & x_1 & + & 5x_2 & \leq & 20 \\ & 5x_1 & + & 3x_2 & \leq & 20 \\ & 6x_1 & + & x_2 & \leq & 21 \\ & x_1, & & x_2 & \geq & 0, \text{ integer} \end{array}$$



Branch on most fractional variable, best-first search

Root node

- LP-solution $x_1 = \frac{20}{11} = 1.8181, x_2 = \frac{40}{11} = 3.6363$.
- Lower bound $z = -\infty$.
- Two nodes: $x_2 \leq 3$ and $x_2 \geq 4$ with upper bounds $\bar{z} = 5.2$ and $\bar{z} = 4$.

Node 1

- Add constraint $x_2 \leq 3$, getting LP-solution $x_1 = \frac{11}{5} = 2.2$ and $x_2 = 3$.
- Two nodes: $x_1 \leq 2$ and $x_1 \geq 3$ with upper bounds $\bar{z} = 5$ and $\bar{z} = \frac{14}{3} = 4.6667$.

Node 2

- Add constraint $x_1 \leq 2$, getting LP-solution $x_1 = 2$ and $x_2 = 3$. Upper bound $\bar{z} = 5$. Feasible solution $\underline{z} = 5$.

Node 3

- Add constraint $x_1 \geq 3$, getting LP-solution $x_1 = 3$ and $x_2 = \frac{5}{3} = 1.6667$. Upper bound $\bar{z} = 4.6667 < \underline{z}$.

Node 4

- Add constraint $x_2 \geq 4$, getting LP-solution $x_1 = 0$ and $x_2 = 4$. Upper bound $\bar{z} = 4 < \underline{z}$.

Design issues

$$\begin{array}{ll} \text{maximize} & cx \\ \text{subject to} & x \in S \end{array}$$

Pruning rules (Wolsey 7.2)

- Prune by optimality $z^k = \max\{cx : x \in S_k\}$
- Prune by bound $\bar{z}_k \leq \underline{z}$
- Prune by infeasibility $S_k = \emptyset$

Branching rules (Wolsey 7.4)

- most fractional variable j i.e. $x_j - \lfloor x_j \rfloor$ close to $\frac{1}{2}$
- least fractional variable
- greedy approach

Selecting next problem

- Depth-first-search
  (quickly find solution, small changes in LP, space)
- Best-first-search
  (greedy approach)

Duality

Branch-and-bound, economics: upper bound on LP.

$$\begin{array}{lrcrcrcrcr} \text{maximize} & 4x_1 & + & x_2 & + & 5x_3 & + & 3x_4 & & \\ \text{subject to} & x_1 & - & x_2 & - & x_3 & + & 3x_4 & \leq & 1 \\ & 5x_1 & + & x_2 & + & 3x_3 & + & 8x_4 & \leq & 55 \\ & -x_1 & + & 2x_2 & + & 3x_3 & - & 5x_4 & \leq & 3 \\ & & & & & & & x_1,x_2,x_3,x_4 & \geq & 0 \end{array} \tag{1}$$

Multiplying the second constraint by two

$$10x_1 + 2x_2 + 6x_3 + 16x_4 \leq 110$$

thus $z^* \leq 110$.
Linear combination of some constraints: second and third constraint

$$4x_1 + 3x_2 + 6x_3 + 3x_4 \leq 58$$

thus $z^* \leq 58$.
In general *any* linear combination.
multipliers y_1, y_2, y_3, demand $y_1, y_2, y_3 \geq 0$

$$\begin{array}{l} y_1(x_1 - x_2 - x_3 + 3x_4) + \\ y_2(5x_1 + x_2 + 3x_3 + 8x_4) + \\ y_3(-x_1 + 2x_2 + 3x_3 - 5x_4) \leq y_1 + 55y_2 + 3y_3 \end{array}$$

which is equivalent to

$$\begin{array}{l} (y_1 + 5y_2 - y_3)x_1 + \\ (-y_1 + y_2 + 2y_3)x_2 + \\ (-y_1 + 3y_2 + 3y_3)x_3 + \\ (3y_1 + 8y_2 + 5y_3)x_4 \leq y_1 + 55y_2 + 3y_3 \end{array} \tag{2}$$

coefficients must exceed those in (1):

$$\begin{array}{rcrcrcr} y_1 & + & 5y_2 & - & y_3 & \geq & 4 \\ -y_1 & + & y_2 & + & 2y_3 & \geq & 1 \\ -y_1 & + & 3y_2 & + & 3y_3 & \geq & 5 \\ 3y_1 & + & 8y_2 & + & 5y_3 & \geq & 3 \\ & & & & y_1,y_2,y_3 & \geq & 0 \end{array}$$

minimize the right-hand side of (2).
dual problem:

$$\begin{array}{lrcrcrcr} \text{minimize} & y_1 & + & 55y_2 & + & 3y_3 & & \\ \text{subject to} & y_1 & + & 5y_2 & - & y_3 & \geq & 4 \\ & -y_1 & + & y_2 & + & 2y_3 & \geq & 1 \\ & -y_1 & + & 3y_2 & + & 3y_3 & \geq & 5 \\ & 3y_1 & + & 8y_2 & + & 5y_3 & \geq & 3 \\ & & & & & y_1,y_2,y_3 & \geq & 0 \end{array}$$

Duality

primal problem.

$$\begin{aligned} \text{maximize} \quad & \sum_{j=1}^{n} c_j x_j \\ \text{subject to} \quad & \sum_{j=1}^{n} a_{ij} x_j \leq b_i \quad i = 1, \ldots, m \\ & x_j \geq 0 \quad j = 1, \ldots, n \end{aligned}$$

associated dual problem

$$\begin{aligned} \text{minimize} \quad & \sum_{i=1}^{m} b_j y_i \\ \text{subject to} \quad & \sum_{i=1}^{m} a_{ij} y_i \geq c_j \quad j = 1, \ldots, n \\ & y_i \geq 0 \quad i = 1, \ldots, m \end{aligned}$$

Weak duality

For every primal feasible solution $(x_1, \ldots, x_n)$
for every dual feasible solution $(y_1, \ldots, y_m)$:

$$\sum_{j=1}^{n} c_j x_j \leq \sum_{j=1}^{n} \left(\sum_{i=1}^{m} a_{ij} y_i \right) x_j = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij} x_j \right) y_i \leq \sum_{i=1}^{m} b_i y_i$$

Deriving bounds efficiently

- At each branching node we add one constraint
- New LP-problems need to be solved
- Can we reuse some computations ?

$$\begin{aligned} \text{maximize} \quad & cx \\ \text{subject to} \quad & Ax \leq b \\ & a'x \leq b' \\ & x \geq 0 \end{aligned}$$

- The previous solution is not feasible!
- Simplex needs feasible solutions in every step
- Consider dual problem

$$\begin{aligned} \text{minimize} \quad & yb + y'b' \\ \text{subject to} \quad & yA + y'a' \geq c \\ & y, y' \geq 0 \end{aligned}$$

- When primal problem gets additional constraint $a'x \leq b'$ the dual problem gets one more variable y'
- The same y is feasible to dual problem ($y' = 0$)
- Same basis solution (for dual problem) can be used
- Normally, only a few steps are needed to find new LP-optimum

The use of interior-point algorithms

- Simplex runs in exponential time (worst-case)
- Interior-point algorithms solve LP-problem in polynomial time
- May be useful for solving MIP problems, if degenerate problem
- Use interior-point to find LP-relaxation at root node
- Derive dual solution (Complementary slackness)
- Use dual simplex at other branching nodes

Design issues

Relaxation (Wolsey 2.1)

$$\max\{cx : x \in S\} \qquad (IP)$$
$$\max\{f(x) : x \in T\} \qquad (RP)$$

RP is a relaxation of IP if

- $S \subseteq T$
- $f(x) \geq cx$ for all $x \in S$

Which constraints should be relaxed

- Quality of bound (tightness of relaxation)
- Remaining problem can be solved efficiently
- Constraints difficult to formulate mathematically
- Constraints which are too expensive to write up

Strong branching

Applegate, Bixby, Chvatal, and Cook (1995) for TSP
Linderoth, Savelsbergh (1999) for MIP

Assume binary MIP to be maximized

- Normal branch-and-bound: choose a subproblem, choose a variable to branch at, create two new subproblems. (*sample*)
- If we decide to branch on a variable which has limited or no effect on the LP-bound on subsequent nodes, we have essentially doubled the total work.
- Strong branching exploits a set of candidate variables specified by the user (*several samples*)
- For each candidate variable, test both branches, evaluate upper bounds by solving LP-relaxation (not necessarily to optimality)
- Choose the best variable for branching, and create two new subproblems

Strong branching

Which variable should we choose?

- The ones for which the upper bound of both subproblems is decreased most
- The ones for which the upper bound on average is decreased most

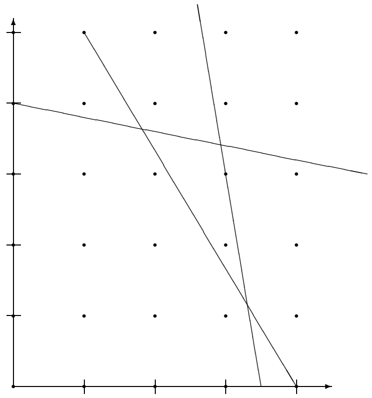
Improving performance

- The samples are only used as a heuristic, hence we do not need to find exact lower bounds
- Dual simplex with a limited number of iterations.

Strong branching, example

$$\begin{array}{lrcrcl} \text{maximize} & x_1 & + & x_2 & & \\ \text{subject to} & x_1 & + & 5x_2 & \leq & 20 \\ & 5x_1 & + & 3x_2 & \leq & 20 \\ & 6x_1 & + & x_2 & \leq & 21 \\ & x_1, & & x_2 & \geq & 0, \ \text{integer} \end{array}$$

LP-solution: $x_1 = 1.8181$, $x_2 = 3.6363$, $\bar{z} = 5.4545$



Only two variables → sample both

- $x_1 \geq 2$: $\bar{z} = 5.3333$
  $x_1 \leq 1$: $\bar{z} = 4.8$
- $x_2 \geq 4$: $\bar{z} = 5.2$
  $x_2 \leq 3$: $\bar{z} = 4$

Branching x_2: better upper bounds for both branches

Local branching

Fischetti, Lodi (2003)

- Important to have good incumbent solution
- 2-optimal solution for TSP, QAP, KP works well
- In general: if we have a good feasible solution $\hat{x}$ we do not want to change it too much
- At most k variables may change their value from $\hat{x}$
- Restrict search to k-optimal solutions

Example

$$\begin{array}{ll} \text{maximize} & 4x_1 + 5x_2 + 6x_3 + 7x_4 + 8x_5 \\ \text{subject to} & 3x_1 + 4x_2 + 5x_3 + 6x_4 + 7x_5 \leq 10 \\ & x_1, x_2, x_3, x_4, x_5 \in \{0, 1\} \end{array}$$

Greedy solution: $\hat{x}_1 = 1$, $\hat{x}_2 = 1$, $\hat{x}_3 = 0$, $\hat{x}_4 = 0$, $\hat{x}_5 = 0$.
Restrict to 2-opt

$$(1 - x_1) + (1 - x_2) + x_3 + x_4 + x_5 \leq 2$$

we get the constraint

$$-x_1 - x_2 + x_3 + x_4 + x_5 \leq 0$$

Other branch demands more than 2 changes

$$(1 - x_1) + (1 - x_2) + x_3 + x_4 + x_5 \geq 3$$

Optimal solution: $x_1 = 0$, $x_2 = 1$, $x_3 = 0$, $x_4 = 1$, $x_5 = 0$.

Local branching

Assume that a feasible solution $\hat{x}$ has been found

- Left branch $\Delta(x,\hat{x}) \leq k$
- Right branch $\Delta(x,\hat{x}) \geq k+1$

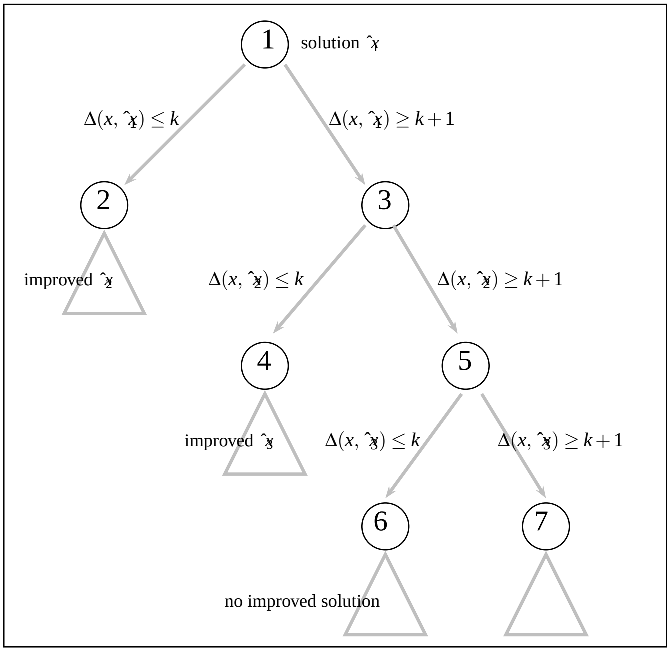
Where

$$\Delta(x,\hat{x}) = \sum_{j\in N} |x_j - \hat{x}_j| = \sum_{\{j\in N|\hat{x}_j=1\}} (1-x_j) + \sum_{\{j\in N|\hat{x}_j=0\}} x_j$$

How large should we choose k?

$$k \approx 10$$

Local branching, exact algorithm



Example: Knapsack Problem

Given n items and a knapsack

- Item j has the weight w_j
- Profit of item j is p_j
- The capacity of the knapsack is c

$$\begin{aligned} \text{maximize} \quad & \sum_{j=1}^{n} p_j x_j \\ \text{subject to} \quad & \sum_{j=1}^{n} w_j x_j \leq c \\ & x_j \in \{0,1\}, \quad j = 1,\ldots,n. \end{aligned}$$

Important problem

- Budgeting
- Transportation
- Subproblem (e.g. separation of valid inequalities)