

Program of the day

- Overview of course, exercises
- Introduction to Integer Programming
- Modeling (Williams, chapter 9)
- Applications: Opencast mining
- Ladies or tigers

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**Purpose of the course**

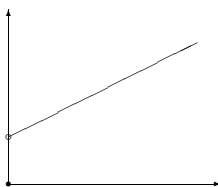
- To learn to build complex models from real life using Mathematical Programming
- To know techniques for solving Mathematical Programming models
- To understand that some problems can be solved efficiently and some cannot
- To learn that the same problem may be formulated in different ways, which are easier/harder to solve
- To know a number of techniques for decreasing solution times (or turn a problem from practically “unsolvable” to “solvable”)

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**Integer Programming**

In first part of course: continuous variables, linear constraints

- Most products are integral (apart from liquids)  
*Airplane production, Tomato Soups*
- Structure of problem leads to IP  
*Graph problems*
- Nonlinear objective functions or constraints occur frequently



- Logical conditions  
*“If I use vegetable oil in the blend, then I must also add 5ml of preservatives”*

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**Integer Programming**

General formulation:

$$\begin{aligned} &\text{maximize} && \sum_{j=1}^n c_j x_j \\ &\text{subject to} && \sum_{j=1}^n a_{1j} x_j \leq b_1 \\ & && \vdots \\ & && \sum_{j=1}^n a_{mj} x_j \leq b_m \\ & && x_j \geq 0, \quad j = 1, \dots, n, \quad x \text{ integer} \end{aligned}$$

where

- $A$  is a  $m \times n$  matrix
- $b$  is a  $m$ -vector
- $c$  is a  $n$ -vector
  
- IP: integer programming model
- ILP: integer *linear* model (all constraints linear)
- PIP: pure integer programming model
- MIP: mixed integer programming model

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## Integer Programming, why?

- IP is much more expressible than LP
- As IP is NP-hard, all NP-hard problems can be formulated as IP-models

But

- IP is not an ideal model
- Many problems cannot be formulated as IP-models in a simple way
- All NP-complete problems are “equivalent” and hence equally good
- IP proposed by Edmonds: Expressibility and LP-bounds

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## Integer Programming

IP powerful method for modeling

- LP easy to solve by e.g. Simplex (polynomial time by interior-point methods).
- General IP is NP-hard
- Many concrete problems may be solved despite NP-hardness
- Specific techniques for individual problems

Special problems

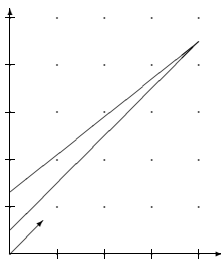
- traveling salesman problem
- project selection
- transportation problem
- assignment problem
- assembly line balancing
- set partitioning problem
- aircrew scheduling
- depot location problem
- sequencing problem
- job-shop scheduling

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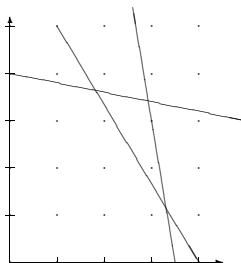
## Hardness of IP

$$\begin{aligned} &\text{maximize } x_1 + x_2 \\ &\text{subject to } -2x_1 + 2x_2 \geq 1 \\ &\quad \quad -8x_1 + 10x_2 \leq 13 \\ &\quad \quad x_1, x_2 \geq 0, \text{ integer} \end{aligned}$$

Solutions are not found in extreme points (or nearby)



Find convex hull



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## Model building

- Indicator variables
- Non-convex problems
- Nonlinear functions
- Logical expressions
- Transformation of “human text” to ILP

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### Indicator variables

- Most important modeling tool!
- $\delta \in \{0, 1\}$
- $\delta = 1$  if and only if some event happens.

Model:

$$\delta = 1 \Leftrightarrow x > 0$$

$$\delta \in \{0, 1\}, x \geq 0$$

$$\delta = 1 \Rightarrow x > 0$$

$\delta = 1 \Rightarrow x \geq \varepsilon$      $\varepsilon$  level for  $x$  regarded as 0  
 $x - \varepsilon\delta \geq 0, \delta \in \{0, 1\}$

$$x > 0 \Rightarrow \delta = 1$$

$\delta = 0 \Rightarrow x = 0$   
 $x - M\delta \leq 0, \delta \in \{0, 1\}$      $M$  upper bound on  $x$

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### Indicator variables

Logical implications  $X \Leftrightarrow Y$

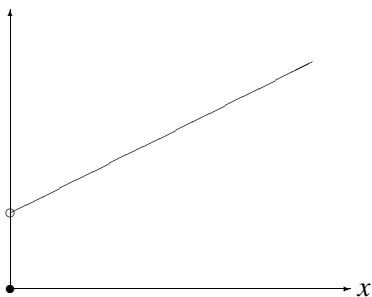
X	Y	$X \Rightarrow Y$	$X \Leftrightarrow Y$	$\neg X \Rightarrow \neg Y$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	F
F	F	T	T	T

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### Fixed-charge problem

cost function

$$f(x) = \begin{cases} ax + b & \text{if } x > 0 \\ 0 & \text{if } x = 0 \end{cases}$$



Model:

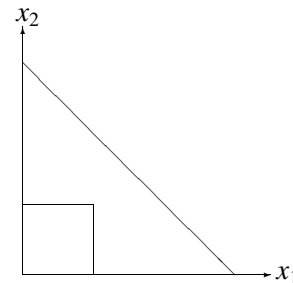
$$\begin{cases} \text{minimize} & ax + \delta b \\ \text{subject to} & x - M\delta \leq 0 \\ & x - \varepsilon\delta \geq 0 \\ & \delta \in \{0, 1\}, x \geq 0 \end{cases}$$

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### Non-convex problems

constraints:

$$\begin{aligned} x_1 + x_2 &\leq b \\ x_1 \geq 1 \text{ or } x_2 &\geq 1 \\ x_1, x_2 &\geq 0 \end{aligned}$$



Modeling tool

$$\delta = 1 \Rightarrow x \geq \varepsilon$$

Two indicator variables  $\delta_1, \delta_2$ :

$$\begin{cases} x_1 + x_2 &\leq b \\ \delta_1 + \delta_2 &\geq 1 \\ x_1 - 1\delta_1 &\geq 0 \\ x_2 - 1\delta_2 &\geq 0 \\ x_1, x_2 &\geq 0, \\ \delta_1, \delta_2 &\in \{0, 1\} \end{cases}$$

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## Indicator variables

“if  $A$  is included in the blend then  $B$  is included in the blend”

can be modeled by using constraints

$x_A > 0 \Rightarrow \delta = 1$	$x_A - M\delta \leq 0, \delta \in \{0, 1\}$ $M$ upper bound on $x_A$
$\delta = 1 \Rightarrow x_B > 0$	$x_B - \epsilon\delta \geq 0, \delta \in \{0, 1\}$ $\epsilon$ level for $x_B$ regarded as 0

### Example

Assume that  $x_A$  and  $x_B$  are proportions in blend i.e.  $x_A + x_B = 1$ .

$$M = 1 \quad \epsilon = 0.01$$

Formulation:

$$\begin{cases} x_A - \delta & \leq 0 \\ x_B - 0.01\delta & \geq 0 \\ \delta & \in \{0, 1\} \end{cases}$$

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## Indicator variables for linear inequalities

### Example

“If resources needed for production of  $x_1, x_2$  and  $x_3$  are below the limit of one truck, then use the other truck for some other purpose.”

General form

$$\sum_{j=1}^n a_j x_j \leq b \Leftrightarrow \delta = 1, \delta \in \{0, 1\}$$

- $\sum_{j=1}^n a_j x_j \leq b \Leftrightarrow \delta = 1$  has the MIP formulation

$$\sum_{j=1}^n a_j x_j + M\delta \leq M + b$$

where  $M$  is upper bound on  $\sum_{j=1}^n a_j x_j - b$

$$\delta = 1: \sum_{j=1}^n a_j x_j \leq b$$

$$\delta = 0: \sum_{j=1}^n a_j x_j - b \leq M$$

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## Indicator variables for linear inequalities

- $\sum_{j=1}^n a_j x_j \leq b \Rightarrow \delta = 1$  has the MIP formulation

$$\sum_{j=1}^n a_j x_j - (m - \epsilon)\delta \geq b + \epsilon$$

where  $m$  is lower bound on  $\sum_{j=1}^n a_j x_j - b$ .

$$\delta = 0 \Rightarrow \sum_{j=1}^n a_j x_j \geq b + \epsilon$$

$$\delta = 0: \sum_{j=1}^n a_j x_j \geq b + \epsilon$$

$$\delta = 1: \begin{cases} \sum_{j=1}^n a_j x_j - m + \epsilon \geq b + \epsilon \\ \sum_{j=1}^n a_j x_j - b \geq m \end{cases}$$

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## Indicator variables for inequalities, example

Logical condition

$$\begin{aligned} 2x_1 + 3x_2 \leq 1 &\Leftrightarrow \delta = 1 \\ \delta &\in \{0, 1\} \\ 0 \leq x_1 \leq 1, \quad 0 \leq x_2 \leq 1 \end{aligned}$$

We find

$$\begin{aligned} M &= \text{u.b.}(\sum_{j=1}^n a_j x_j - b) \\ &= \text{u.b.}(2x_1 + 3x_2 - 1) = 4 \end{aligned}$$

and

$$\begin{aligned} m &= \text{l.b.}(\sum_{j=1}^n a_j x_j - b) \\ &= \text{l.b.}(2x_1 + 3x_2 - 1) = -1 \end{aligned}$$

choose  $\epsilon = 0.01$ , i.e. constraint broken when  $2x_1 + 3x_2 \geq 1.01$

Constraints

$$\begin{aligned} 2x_1 + 3x_2 + 4\delta &\leq 4 + 1 \\ 2x_1 + 3x_2 - (-1 - 0.01)\delta &\geq 1 + 0.01 \end{aligned}$$

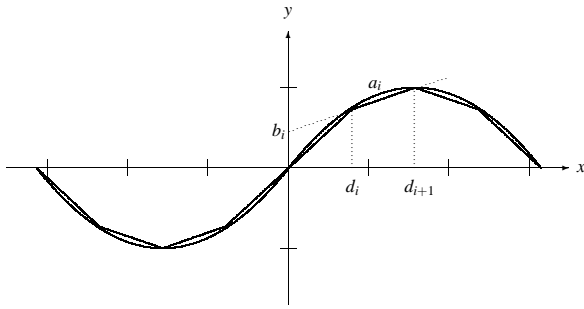
Which results in model:

$$\begin{cases} 2x_1 + 3x_2 + 4\delta \leq 5 \\ 2x_1 + 3x_2 + 1.01\delta \geq 1.01 \\ \delta \in \{0, 1\} \\ 0 \leq x_1 \leq 1, \\ 0 \leq x_2 \leq 1 \end{cases}$$

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## Nonlinear functions

Frequently, the objective function or some of the constraints may contain nonlinear functions.



Approx. nonlinear function by piecewise linear function

- Split into  $m$  intervals
- For each interval  $[d_i, d_{i+1}]$

$$d_i \leq x \leq d_{i+1} \Leftrightarrow y = a_i x + b_i$$

- Model as

$$\begin{cases} d_i \leq x & \Leftrightarrow \delta_1 = 1 \\ x \leq d_{i+1} & \Leftrightarrow \delta_2 = 1 \\ \delta_1 + \delta_2 = 2 & \Leftrightarrow \delta = 1 \\ y = a_i x + b_i & \Leftrightarrow \delta = 1 \end{cases}$$

- Many intervals  $m$ , better precision but much harder to solve!

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## Logical conditions and 0-1 variables

- If no depot is sited here then it will not be possible to supply any of the customers from the depot.
- If we manufacture product A then we must also manufacture product B or at least one of products C and D.
- If we do not place an electronic module in this position, then no wires can be connected into this position.

Introduce an indicator variable  $\delta_i \in \{0, 1\}$  with each condition  $X_i$

$$\boxed{\text{condition } X_i \text{ is true} \Leftrightarrow \delta_i = 1}$$

In this way we may formulate:

$X_1 \vee X_2$	$\delta_1 + \delta_2 \geq 1$
$X_1 \wedge X_2$	$\delta_1 = 1, \delta_2 = 1$
$X_1 \Rightarrow X_2$	$\delta_1 - \delta_2 \leq 0$
$X_1 \Leftrightarrow X_2$	$\delta_1 - \delta_2 = 0$

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## Transformation to linear form

Write up the text in ordinary mathematical form

$$(\sin(x_1) \leq \frac{1}{2} \vee x_1 x_2 \leq x_3) \Rightarrow (x_3 = 1 \vee x_2 + x_1 \leq 1)$$

Stepwise transformation

- Arithmetic functions are replaced by piecewise linear approximations of the functions.
- Products of decision variables are transformed into products of binary variables. Products of binary variables may easily be expressed as logical constraints, and thus put on binary form.
- Relations are transformed into linear inequalities with boolean variables.

$$(ax \leq b) \Leftrightarrow (\delta = 1)$$

- Boolean logics are transformed into linear form.

$$(B_1 \vee B_2) \Leftrightarrow (\delta' = 1)$$

- The resulting expression should be true  $\delta_{all} = 1$

- Domains of variables are defined.

$$\delta_i \in \{0, 1\}$$

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## Transformation of general constraints to linear form

Step 3:

Relation	ILP-constraints
$Ax \leq b$	$Ax + (\delta - 1)M \leq b, \quad Ax + \delta M \geq b + \epsilon$
$Ax < b$	$Ax + (\delta - 1)M \leq b - \epsilon, \quad Ax + \delta M \geq b$
$Ax > b$	$Ax + (1 - \delta)M \geq b + \epsilon, \quad Ax - \delta M \leq b$
$Ax \geq b$	$Ax + (1 - \delta)M \geq b, \quad Ax - \delta M \leq b - \epsilon$
$Ax = b$	$Ax \geq b \wedge Ax \leq b,$

Step 4:

Relation	Meaning	ILP-constraints
$B_1 \vee B_2$	$\delta = 1 \Leftrightarrow \delta_1 = 1 \vee \delta_2 = 1$	$\delta - \delta_1 - \delta_2 \leq 0, \quad \delta_1 + \delta_2 - 2\delta \leq 0$
$B_1 \wedge B_2$	$\delta = 1 \Leftrightarrow \delta_1 = 1 \wedge \delta_2 = 1$	$2\delta - \delta_1 - \delta_2 \leq 0, \quad \delta_1 + \delta_2 - \delta \leq 1$
$B_1 \Rightarrow B_2$	$\delta = 1 \Leftrightarrow (\delta_1 = 1 \Rightarrow \delta_2 = 1)$	$\delta_1 - \delta_2 + \delta \leq 1, \quad \delta_1 - \delta_2 + 2\delta \geq 1$
$B_1 \Leftrightarrow B_2$	$\delta = 1 \Leftrightarrow (\delta_1 = 1 \Leftrightarrow \delta_2 = 1)$	use: $(B_1 \Rightarrow B_2) \wedge (B_2 \Rightarrow B_1)$
$\neg B_1$	$\delta = 1 \Leftrightarrow \neg(\delta_1 = 1)$	$\delta = 1 - \delta_1$

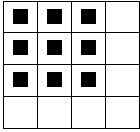
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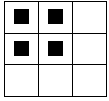
## Opencast mining

The optimal solution

- Level 1 (surface)



- Level 2 (25 ft depth)



- Level 3 (50 ft depth)



- Level 4 (75 ft depth)



Net profit is 17.500 pounds.

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## Ladies or tigers (Smullyan 1982)

Prisoner in castle, meets nine doors:

- Lady (immediately marry her)
- Tiger (immediately eaten)

Prefers to marry lady than be eaten

Statements on the nine doors are:

Door 1: The lady is in an odd-numbered room

Door 2: This room is empty

Door 3: Either sign 5 is right or sign 7 is wrong

Door 4: Sign 1 is wrong

Door 5: Either sign 2 or sign 4 is right

Door 6: Sign 3 is wrong

Door 7: The lady is not in room 1

Door 8: This room contains a tiger and room 9 is empty

Door 9: This room contains a tiger and sign 6 is wrong

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## Ladies or tigers (Smullyan 1982)

In addition

- Only one lady
- Each other room: Tiger or empty
- Sign on door of lady is true
- Sign on door of every tiger is false
- Sign on door of empty room can be either true or false.

No unique solution until the prisoner is told whether or not room eight is empty. Then unique solution

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## Ladies or tigers (Smullyan 1982)

Define subscripts  $i = 1, \dots, 9$  and  $j = 1, \dots, 3$  where (1 lady, 2 tiger, 3 empty) and as above variables are

$$x_{i,j} = \begin{cases} 1 & \text{if door } i \text{ hides prize } j \\ 0 & \text{otherwise} \end{cases}$$

$$t_i = \begin{cases} 1 & \text{if statement on door } i \text{ is true} \\ 0 & \text{otherwise} \end{cases}$$

“Either-or” means “ordinary or”

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