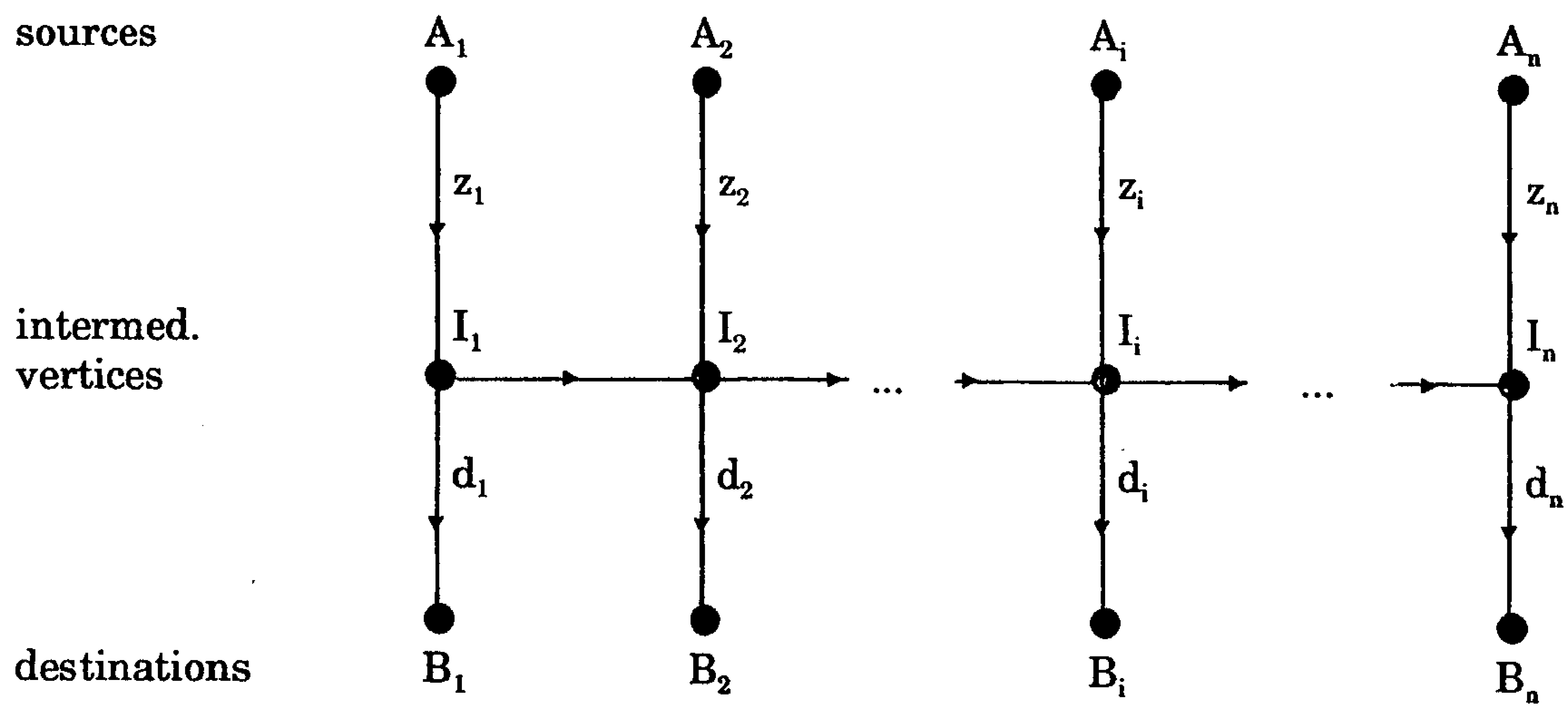
Written exam, 10 June 2003

SOLUTION

A single figure is better than a thousand words! Thus, to facilitate the overview we shall consider the planning problem at hand as a *network flow problem* with

	n sources	$A_1, \dots, A_n$	(production)
	n destinations	$B_1, \dots, B_n$	(demands)
and	n intermediate vertices	$I_1, \dots, I_n$	

The corresponding *directed* network becomes



M1: The objective is to satisfy total demand at least total cost. For week i, i = 1, ... , n, the production costs are $p_i z_i$ and the cost of keeping s_i units in stock from month i to month i+1 is $r_i s_i$. These are the two constituents of the objective function as correctly reflected in answer c).

M2: Additional constraints? A *flow conservation equation* must apply for each of the intermediate vertices,

Vertex I_i: **inflow** $= s_{i-1} + z_i = s_i + d_i =$ **outflow**

or

$$s_{i-1} + z_i - s_i = d_i, \quad i = 1, \dots, n$$

as in answer d). Note that $z_i \leq d_i + s_i$ (answer a)) is necessary but not sufficient. a) is automatically satisfied by all solutions satisfying d) and is therefore redundant.

$n = 4$	$d_1 = 7,\ \ r_1 = 3$	$d_2 = 9,\ \ r_2 = 6$	$d_3 = 8,\ \ r_3 = 4$	$d_4 = 5$
$p_1 = 2$	$c_{11} =$ $(2+0)7 = 14$	$c_{12} =$ $(2+3)9 = 45$	$c_{13} =$ $(2+3+6)8 = 88$	$c_{14} =$ $(2+3+6+4)5 = 75$
$p_2 = 5$	$c_{21} =$	$c_{22} =$ $(5+0)9 = 45$	$c_{23} =$ $(5+6)8 = 88$	$c_{24} =$ $(5+6+4)5 = 75$
$p_3 = 12$	$c_{31} =$	$c_{32} =$	$c_{33} =$ $(12+0)8 = 96$	$c_{34} =$ $(12+4)5 = 80$
$p_4 = 14$	$c_{41} =$	$c_{42} =$	$c_{43} =$	$c_{44} =$ $(14+0)5 = 70$

The minimum value of the objective function is 14+45+88+70 = **217**.

Alternatively, **LP*** can be viewed as a *balanced Transportation Problem*:

	7	9	8	5	69-(7+9+8+5) = 40
7+9+8+5 = 29	2	5	11	15	0
9+8+5 = 22	-	5	11	15	0
8+5 = 13	-	-	12	16	0
5 = 5	-	-	-	14	0

The numbers above are self-explanatory and it is easily verified that the same optimal solution obtains.

ii) It is seen that $z_3 = 0$ and $z_4 = 5$ in all optimal solutions. As to z_1 and z_2 there are 4 alternate solutions, all reflecting the decomposition principle used:

z_1	7	16	15	24
z_2	17	8	9	0

However, if that principle is abandoned, we have in total 18 alternate solutions:

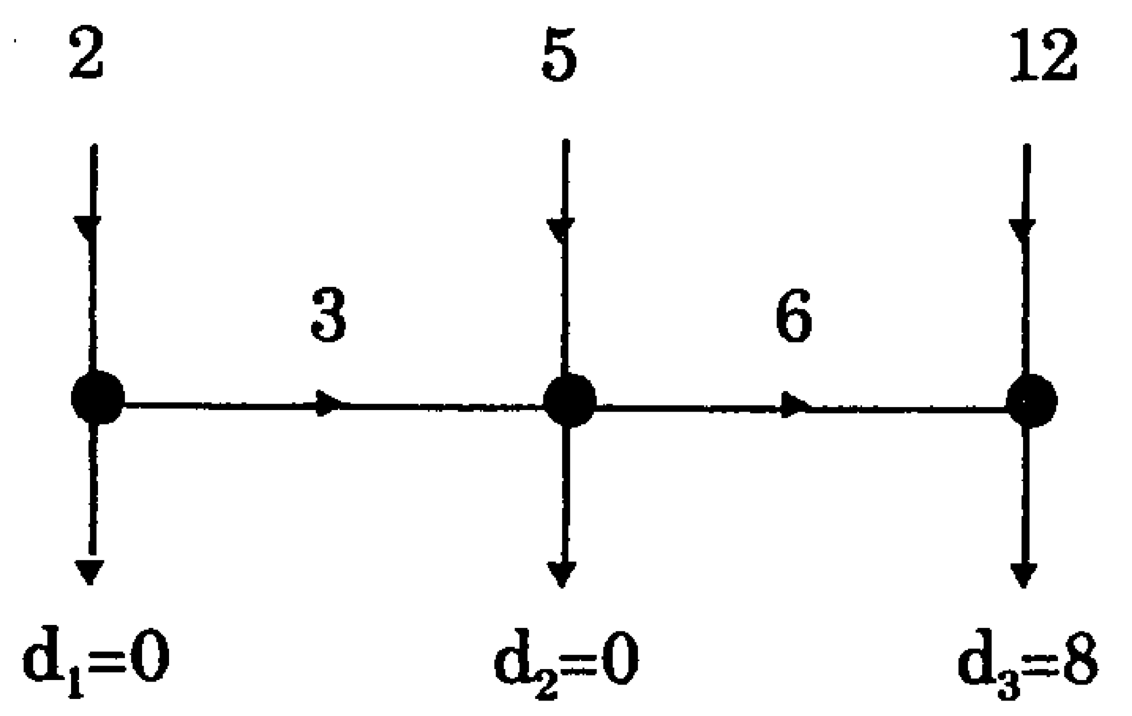
$$z_1 = 7+\delta, \quad z_2 = 17-\delta, \quad \delta = 0,1, \dots, 17$$

Finally, if "fractional" devices IT03 are allowed to be manufactured and kept in stock, we have infinitely many alternate solutions:

$$z_1 = 7+\delta, \quad z_2 = 17-\delta, \quad 0 \le \delta \le 17$$

iii) If $p_4 = 14$ is replaced by $p_4 = 14+\Delta$, only the "c_{44} entry" is affected. It appears that the optimal solution remains optimal for $\Delta<1$. For $\Delta=1$, the demand d_4 can be

M3: For the given instance, the corresponding network is



Let q_{ij} be the per unit cost of supplying the demand in week j from the production set up in week i, $i \leq j$. We find: $q_{13} = 2+3+6 = 11$, $q_{23} = 5+6 = 11$, $q_{33} = 12$. The correct answer is thus $d_3 \times \min\{q_{13}, q_{23}, q_{33}\} = 8 \times 11 = 88$ as in answer c).

This question is actually meant to show that the entire problem is *decomposable* in general and optimally solvable "by inspection": nothing more is needed than repeat the above argument for each week having a positive demand.

M4: (1) is indeed true since we have n linearly independent constraints, cf. M2. It follows then that (3) is true as well whereas (2) is sheer nonsense.
Hence, c) is correct.

M5: Since c_{ij} must equal $q_{ij}d_j$, we find for all i,j, $i \leq j$:

$n = 4$	$d_1 = 7$, $r_1 = 3$	$d_2 = 9$, $r_2 = 6$	$d_3 = 8$, $r_3 = 4$	$d_4 = 5$
$p_1 = 2$	$c_{11} =$ $(2+0)7 = 14$	$c_{12} =$ $(2+3)9 = 45$	$c_{13} =$ $(2+3+6)8 = 88$	$c_{14} =$ $(2+3+6+4)5 = 75$
$p_2 = 5$	$c_{21} =$	$c_{22} =$ $(5+0)9 = 45$	$c_{23} =$ $(5+6)8 = 88$	$c_{24} =$ $(5+6+4)5 = 75$
$p_3 = 12$	$c_{31} =$	$c_{32} =$	$c_{33} =$ $(12+0)8 = 96$	$c_{34} =$ $(12+4)5 = 80$
$p_4 = 14$	$c_{41} =$	$c_{42} =$	$c_{43} =$	$c_{44} =$ $(14+0)5 = 70$

Thus, $c_{14} + c_{24} + c_{34} + c_{44} = 75+75+80+70 = 300$ as in answer c). Note that the correct value of the sum must be divisible by $d_4 = 5$. A clever shortcut is therefore to realize that "300" is the only number among those listed satisfying that property.

T1:

i) The optimal solution(s) can be derived from the entries in the C-matrix representing the column mimima as shown in **bold** below:

satisfied from the production made in weeks 1,2,4 at a unit cost of 15. For $\Delta>1$ it is no longer profitable to set up a production in week 4.

T2:

LP-Y, shown below with the constraints numbered by [1]-[7], appears to optimally solvable by inspection

$$\max \quad 7y_1 + 9y_2 + 8y_3 + 5y_4$$

$$\begin{array}{rrrrll}
y_1 & & & & \leq 2 & [1] \\
 & y_2 & & & \leq 5 & [2] \\
 & & y_3 & & \leq 12 & [3] \\
 & & & y_4 & \leq 14 & [4] \\
-y_1 & + y_2 & & & \leq 3 & [5] \\
 & -y_2 & + y_3 & & \leq 6 & [6] \\
 & & -y_3 & + y_4 & \leq 4 & [7]
\end{array}$$

y_1, y_2, y_3, y_4 free

Since all variables have positive coefficients in the objective function, a first idea is to make all variables as large as possible. Disregarding the upper bounds on each of the 4 variables (constraints [1]-[4]) the structure of the remaining 3 constraints supports that idea: the larger y_k is, the larger value can be assigned to y_{k+1}, k=1,2,3. Hence, an optimal solution is

$y_1 = 2$ [1],

$y_2 \leq 5$ [2], $\quad y_2 \leq 3 + y_1 = 5$ [6]

$\Rightarrow \quad y_2 = 5,$

$y_3 \leq 12$ [3], $\quad y_3 \leq 6 + y_2 = 11$ [6]

$\Rightarrow \quad y_3 = 11,$

$y_4 \leq 14$ [4], $\quad y_4 \leq 4 + y_3 = 15$ [7]

$\Rightarrow \quad y_4 = 14$

with a maximum value of the objective function equalling **217**.

T3:

i) Did **LP-Y** come from a clear sky or is "**T2**, ii)" in any way related to some of the previous questions?

Good clues are the rediscovery of the number "217" and the designation of the 7 variables. With variables z_1,z_2,z_3,z_4, s_1,s_2,s_3, the dual **LP-ZS** of **LP-Y** reads

	z_1	z_2	z_3	z_4	s_1	s_2	s_3		
obj.fct.		2	5	12	14	3	6	4	(min)
	1				-1			= 7	
		1			1	-1		= 9	
			1			1	-1	= 8	
				1			1	= 5	

All variables nonnegative

which is recognized as the instance called **LP***, cf. M5.

ii) Already done: see **T1** i)!

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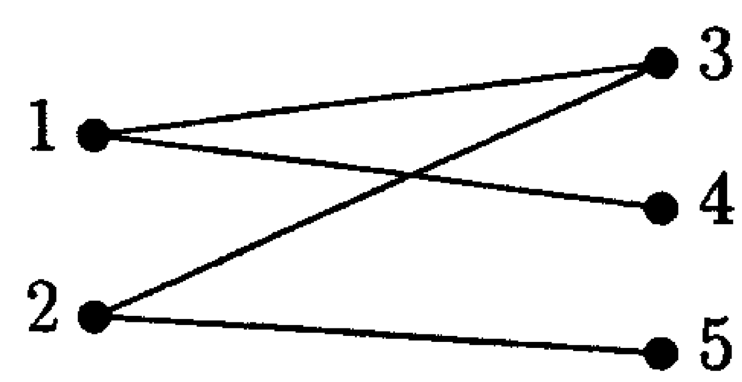
Answers

- M10:



Rows: vertices 1,2,3,4,5 **Columns:** edges in order top-down

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Hence the correct answer is c).

- T6: We prove the stated by use of proposition 3.2 page 39 in Wolsey. The partitioning of the columns into two sets M_1, M_2 is chosen as the partitioning of the vertices in the bipartite graph.

- M11: Adding slack variables $s_1, s_2 \geq 0$ we get the model:

$$\begin{array}{llllllllll} \text{maximize} & 4x_1 & + & 2x_2 & + & x_3 & & & & \\ \text{subject to} & 14x_1 & + & 10x_2 & + & 11x_3 & + & s_1 & & = 32 \\ & -10x_1 & + & 8x_2 & + & 9x_3 & & & - s_2 & = 0 \\ & x_1, x_2, x_3 \in \mathbb{Z}_+ & & & & & & & & \end{array}$$

To find the simplex tableau we use the equation

$$Bx_B + Nx_N = b \qquad \Leftrightarrow \qquad x_B + B^{-1}Nx_N = B^{-1}b$$

where $x_B = (x_1, x_2)$, $x_N = (x_3, s_1, s_2)$ and

$$B^{-1} = \frac{1}{212}\begin{pmatrix} 8 & -10 \\ 10 & 14 \end{pmatrix} \qquad b = \begin{pmatrix} 32 \\ 0 \end{pmatrix} \qquad N = \begin{pmatrix} 11 & 1 & 0 \\ 9 & 0 & -1 \end{pmatrix}$$

Inserting the stated we get the simplex tableau

$$x_1 + \tfrac{1}{212}(-2x_3 + 8s_1 + 10s_2) = \tfrac{1}{212}256$$
$$x_2 + \tfrac{1}{212}(236x_3 + 10s_1 - 14s_2) = \tfrac{1}{212}320$$