

# Solving a transportation problem

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## 1 Solving a transportation problem

Consider the transportation problem with the following data.

	1	2	3	4	5	Supply( $s_i$ )
1	16	16	13	22	17	50
2	14	14	13	19	15	60
3	19	19	20	23	50	50
4	50	12	50	15	11	50
Demand( $d_j$ )	30	20	70	30	60	

Table 1:

We now solve this problem using the transportation method. We have  $m = 4$  sources and  $n = 5$  destinations. The first issue is to construct an initial basic feasible solution. Every basis consists of  $(m + n - 1) = 8$  basic variables. We obtain the initial basis by using the “North-west corner rule”. The result of applying this rule is given in Table 2.

	1	2	3	4	5	Supply( $s_i$ )
1	$30^b$	$20^b$	0	0	0	50
2	0	$0^b$	$60^b$	0	0	60
3	0	0	$10^b$	$30^b$	$10^b$	50
4	0	0	0	0	$50^b$	50
Demand( $d_j$ )	30	20	70	30	60	

Table 2:

The numbers in Table 2 have the following meaning. Each cell indicated the value of the corresponding variable in the basic solution. The numbers with the superscript “b” correspond to basic variables. Hence, the basis constructed with the North west corner rule is given by

$$B^1 = \{(1, 1), (1, 2), (2, 2), (2, 3), (3, 3), (3, 4), (3, 5), (4, 5)\}$$

and the values of the corresponding basic variables are

$$x_{1,1} = 30, x_{1,2} = 20, x_{2,2} = 0, x_{2,3} = 60, x_{3,3} = 10, x_{3,4} = 30, x_{3,5} = 10, x_{4,5} = 50.$$

The spanning tree corresponding to the initial basis  $B^1$  is shown in Fig. 1.

To find the dual basic solution corresponding to  $B^1$ , we set the dual constraints corresponding to each basic variable to equality. Furthermore, since the equality system that defines the transportation problem contains one redundant equality, we fix one of the dual variables to zero (we choose  $v_5 = 0$ ). This gives the system (a)-(i).

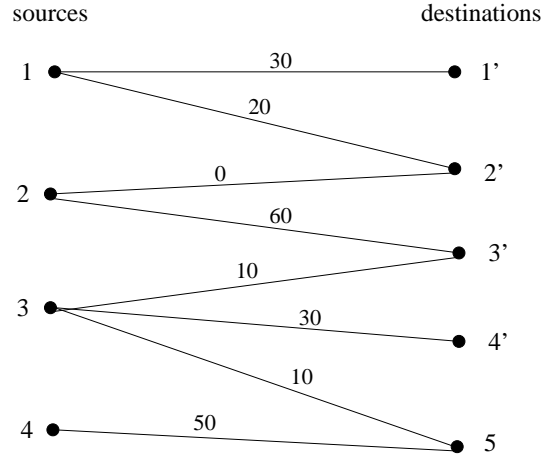


Figure 1: Spanning tree for initial basis  $B^1$

- (a)  $u_1 + v_1 = 16$  (basic variable  $x_{1,1}$ ),
- (b)  $u_1 + v_2 = 16$  (basic variable  $x_{1,2}$ ),
- (c)  $u_2 + v_2 = 14$  (basic variable  $x_{2,2}$ ),
- (d)  $u_2 + v_3 = 13$  (basic variable  $x_{2,3}$ ),
- (e)  $u_3 + v_3 = 20$  (basic variable  $x_{3,3}$ ),
- (f)  $u_3 + v_4 = 23$  (basic variable  $x_{3,4}$ ),
- (g)  $u_3 + v_5 = 50$  (basic variable  $x_{3,5}$ ),
- (h)  $u_4 + v_5 = 11$  (basic variable  $x_{4,5}$ ), and
- (i)  $v_5 = 0$  (basic variable  $x_{1,1}$ ).

Solving the above non-singular system gives the following dual solution.

$$u^{B^1} = (45, 43, 50, 11) \text{ and } v^{B^1} = (-29, -29, -30, -27, 0).$$

We can now calculate the reduced costs corresponding to the basis  $B^1$ . The reduced costs are given by  $r_{i,j}^{B^1} = c_{i,j} - u_i^{B^1} - v_j^{B^1}$ , and are shown in Table 3.

	1	2	3	4	5	$u_i$
1	0	0	-2	4	-28	45
2	0	0	0	3	-28	43
3	-2	-2	0	0	0	50
4	68	30	69	31	0	11
$v_j$	-29	-29	-30	-27	0	

Table 3: Reduced costs from basis  $B^1$

From Table 3, we note that not all the reduced costs are non-negative, which means that the current solution is *not* optimal. We next choose a non-basic variable with a negative reduced cost

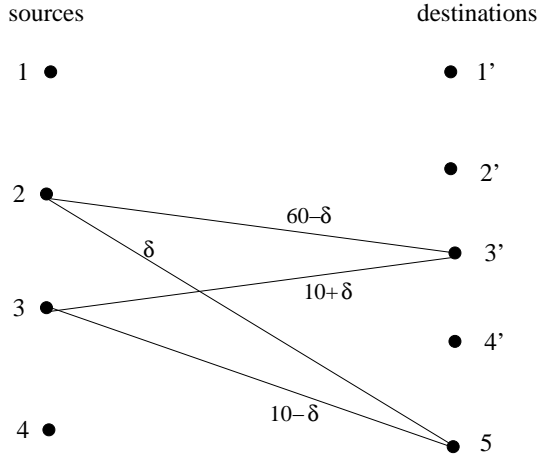


Figure 2: Cycle created by adding edge (2,5) to the spanning tree

to enter the basis. We choose the variable  $x_{2,5}$ . Adding the edge (2,5) to the spanning tree of Fig. 1 creates the cycle of Fig. 2.

Changing the value  $x_{2,5}$  from its current value of zero to  $x_{2,5} = \delta \geq 0$  forces the changes in the basic variables on the cycle of Fig. 2 indicated in Fig. 2. We would like the value of  $\delta$  to be as large as possible without making any of the current basic variable negative. From Fig. 2, it is clear that the largest possible value of  $\delta$  is  $\delta^* = 10$ , and that the basic variable  $x_{3,5}$  is the restricting variable. Therefore, the variable  $x_{3,5}$  exits the basis, and  $x_{2,5}$  enters the basis. Making the changes following from setting  $x_{2,5}$  to the value 10, and making the appropriate changes in the basic solution creates the basic solution in Table 4.

	1	2	3	4	5	Supply
1	$30^b$	$20^b$	0	0	0	50
2	0	$0^b$	$50^b$	0	$10^b$	60
3	0	0	$20^b$	$30^b$	0	50
4	0	0	0	0	$50^b$	50
Demand	30	20	70	30	60	

Table 4:

Hence, the new basis is given by

$$B^2 = \{(1, 1), (1, 2), (2, 2), (2, 3), (2, 5), (3, 3), (3, 4), (4, 5)\}$$

and the values of the corresponding basic variables are

$$x_{1,1} = 30, x_{1,2} = 20, x_{2,2} = 0, x_{2,3} = 50, x_{2,5} = 10, x_{3,3} = 20, x_{3,4} = 30, x_{4,5} = 50.$$

The spanning tree corresponding to the basis  $B^2$  is given in Fig. 3.

The next step is then to compute the dual solution corresponding to the basis  $B^2$ . As before, we set dual inequalities corresponding to basic variables to equalities, and fix the dual variable  $v_5$  to zero. This gives the following non-singular system.

- (a)  $u_1 + v_1 = 16$  (basic variable  $x_{1,1}$ ),
- (b)  $u_1 + v_2 = 16$  (basic variable  $x_{1,2}$ ),

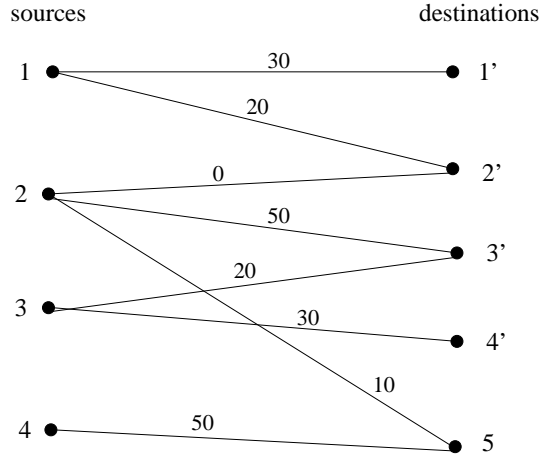


Figure 3: Spanning tree corresponding to the basis  $B^2$

- (c)  $u_2 + v_2 = 14$  (basic variable  $x_{2,2}$ ),
- (d)  $u_2 + v_3 = 13$  (basic variable  $x_{2,3}$ ),
- (e)  $u_2 + v_5 = 15$  (basic variable  $x_{2,5}$ ),
- (f)  $u_3 + v_3 = 20$  (basic variable  $x_{3,3}$ ),
- (g)  $u_3 + v_4 = 23$  (basic variable  $x_{3,4}$ ),
- (h)  $u_4 + v_5 = 11$  (basic variable  $x_{4,5}$ ), and
- (i)  $v_5 = 0$  (fixing  $v_5$  to zero).

Solving this system gives the following dual basic solution  $(u^{B^2}, v^{B^2})$ .

$$u^{B^2} = (17, 15, 22, 11) \text{ and } v^{B^2} = (-1, -1, -2, 1, 0).$$

We can now compute the reduced costs according to the formula  $r_{i,j}^{B^2} = c_{i,j} - u_i^{B^2} - v_j^{B^2}$ . This gives the reduced costs in Table 5.

	1	2	3	4	5	$u_i$
1	0	0	-2	4	0	17
2	0	0	0	3	0	15
3	-2	-2	0	0	28	22
4	40	2	41	3	0	11
$v_j$	-1	-1	-2	1	0	

Table 5:

Again, we see that not all the reduced costs are non-negative. Hence, the basis  $B^2$  is *not* optimal. We next choose a non-basic variable with a negative reduced cost to enter the basis. We choose the variable  $x_{1,3}$  with a reduced cost  $r_{1,3}^{B^2} = -2$ . Adding the edge  $(1,3)$  to the spanning tree of Fig. 3 creates the cycle indicated in Fig. 4. Changing the value of the variable  $x_{1,3}$  from its current value of zero to the value  $x_{1,3} = \delta \geq 0$  forces the changes in the remaining basic variables indicated in Fig. 4.

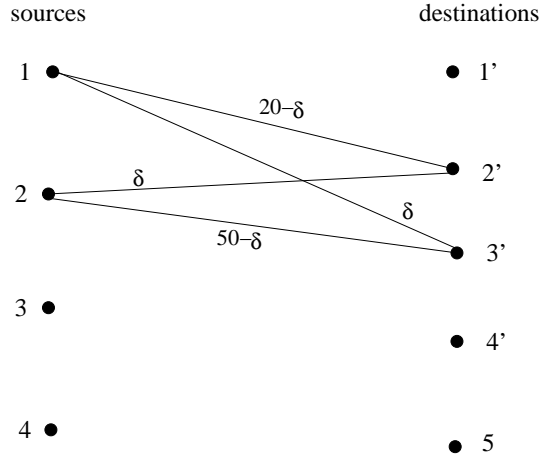


Figure 4: Cycle created by adding edge (1,3) to spanning tree from  $B^2$

It follows from Fig. 4 that the variable  $x_{1,2}$  is the restricting variable ( $x_{1,2}$  is the first basic variable to become negative when  $\delta$  is increased from zero). Hence, we have  $\delta^* = 20$ , and the new basis becomes

$$B^3 = \{(1, 1), (1, 3), (2, 2), (2, 3), (2, 5), (3, 3), (3, 4), (4, 5)\}$$

and the values of the corresponding basic variables are

$$x_{1,1} = 30, x_{1,3} = 20, x_{2,2} = 20, x_{2,3} = 30, x_{2,5} = 10, x_{3,3} = 20, x_{3,4} = 30, x_{4,5} = 50.$$

The new basic solution corresponding to  $B^3$  is given in Table 6.

	1	2	3	4	5	Supply
1	$30^b$	0	$20^b$	0	0	50
2	0	$20^b$	$30^b$	0	$10^b$	60
3	0	0	$20^b$	$30^b$	0	50
4	0	0	0	0	$50^b$	50
Demand	30	20	70	30	60	

Table 6:

The spanning tree corresponding to the basis  $B^3$  is given in Fig. 5.

The next step is to figure out whether the basis  $B^3$  is optimal or not. To do this, we first need to compute the dual basic solution corresponding to  $B^3$ . The equality system that needs to be solved is given below.

- (a)  $u_1 + v_1 = 16$  (basic variable  $x_{1,1}$ ),
- (b)  $u_1 + v_3 = 13$  (basic variable  $x_{1,3}$ ),
- (c)  $u_2 + v_2 = 14$  (basic variable  $x_{2,2}$ ),
- (d)  $u_2 + v_3 = 13$  (basic variable  $x_{2,3}$ ),
- (e)  $u_2 + v_5 = 15$  (basic variable  $x_{2,5}$ ),
- (f)  $u_3 + v_3 = 20$  (basic variable  $x_{3,3}$ ),

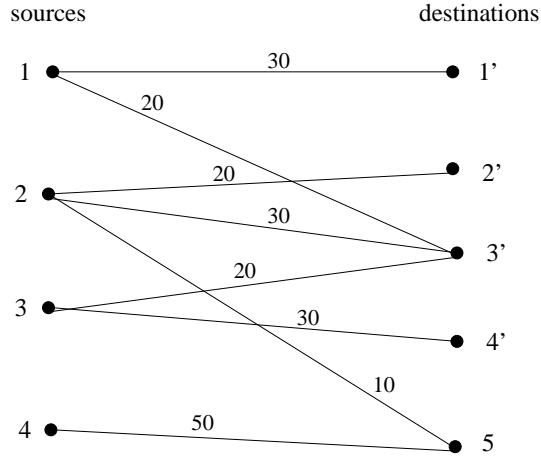


Figure 5: Spanning tree corresponding to the basis  $B^3$

- (g)  $u_3 + v_4 = 23$  (basic variable  $x_{3,4}$ ),
- (h)  $u_4 + v_5 = 11$  (basic variable  $x_{4,5}$ ), and
- (i)  $v_5 = 0$  (fixing the dual variable  $v_5$  to zero).

Solving the above non-singular system gives the following dual basic solution

$$u^{B^3} = (15, 15, 22, 11) \text{ and } v^{B^3} = (1, -1, -2, 1, 0)$$

We can now calculate the reduced costs corresponding to  $B^3$  according to the formula  $r_{i,j}^{B^3} = c_{i,j} - u_i^{B^3} - v_j^{B^3}$ . This gives the reduced costs given in Table 7.

	1	2	3	4	5	$u_i$
1	0	2	0	6	2	15
2	-2	0	0	3	0	15
3	-4	-2	0	0	28	22
4	38	2	41	3	0	11
$v_j$	1	-1	-2	1	0	

Table 7:

Again, we see that not all reduced costs are non-negative, so the basis  $B^3$  is not optimal. We choose the variable  $x_{3,1}$  with a reduced cost of  $r_{3,1}^{B^3} = -4$  to enter the basis. Adding the edge  $(3, 1)$  to the spanning tree of Fig. 5 gives the cycle indicated in Fig. 6.

Changing the value of the variable  $x_{3,1}$  from its current value of zero to the value  $x_{3,1} = \delta \geq 0$  forces the changes in the values of the other basic variables indicated in Fig. 6. As can be seen from the figure, the variable  $x_{3,3}$  is the restricting basic variable, and therefore  $x_{3,3}$  leaves the basis. We have  $\delta^* = 20$ . The new basis  $B^4$  is given by

$$B^4 = \{(1, 1), (1, 3), (2, 2), (2, 3), (2, 5), (3, 1), (3, 4), (4, 5)\}$$

and the values of the corresponding basic variables are

$$x_{1,1} = 10, x_{1,3} = 40, x_{2,2} = 20, x_{2,3} = 30, x_{2,5} = 10, x_{3,1} = 20, x_{3,4} = 30, x_{4,5} = 50.$$

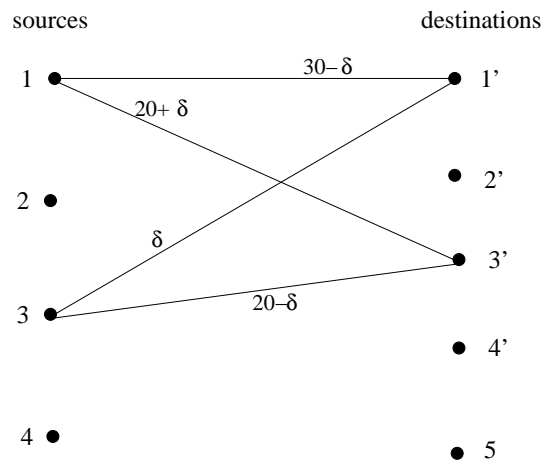


Figure 6: Cycle obtained by adding edge (3,1) to spanning tree corresponding to  $B^3$

	1	2	3	4	5	Supply
1	$10^b$	0	$40^b$	0	0	50
2	0	$20^b$	$30^b$	0	$10^b$	60
3	$20^b$	0	0	$30^b$	0	50
4	0	0	0	0	$50^b$	50
Demand	30	20	70	30	60	

Table 8:

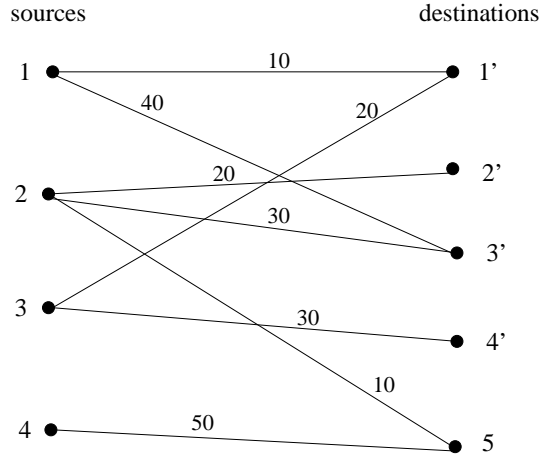


Figure 7: Spanning tree corresponding to the basis  $B^4$

The basic solution corresponding to  $B^4$  is given in Table 8.

The spanning tree corresponding to  $B^4$  is given in Fig. 7.

The next step is then to compute the dual basic solution corresponding to  $B^4$ . The equality system that needs to be solved is given below.

- (a)  $u_1 + v_1 = 16$  (basic variable  $x_{1,1}$ ),
- (b)  $u_1 + v_3 = 13$  (basic variable  $x_{1,3}$ ),
- (c)  $u_2 + v_2 = 14$  (basic variable  $x_{2,2}$ ),
- (d)  $u_2 + v_3 = 13$  (basic variable  $x_{2,3}$ ),
- (e)  $u_2 + v_5 = 15$  (basic variable  $x_{2,5}$ ),
- (f)  $u_3 + v_1 = 19$  (basic variable  $x_{3,1}$ ),
- (g)  $u_3 + v_4 = 23$  (basic variable  $x_{3,4}$ ),
- (h)  $u_4 + v_5 = 11$  (basic variable  $x_{4,5}$ ), and
- (i)  $v_5 = 0$  (fixing  $v_5$  to zero).

Solving the above system gives the following dual basic solution

$$u^{B^4} = (15, 15, 18, 11) \text{ and } v^{B^4} = (1, -1, -2, 5, 0)$$

From the dual basic solution, the reduced costs can be calculated based on the formula  $r_{i,j}^{B^4} = c_{i,j} - u_i^{B^4} - v_j^{B^4}$ . The reduced costs for the basis  $B^4$  are given in Table 9.

As can be seen from Table 9, we still do not have that all reduced costs are non-negative, so the basis  $B^4$  is not optimal. We therefore choose a non-basic variable with a negative reduced cost. We choose the variable  $x_{2,1}$  with a reduced cost of  $r_{2,1}^{B^4} = -2$ . Adding the edge  $(2, 1)$  to the spanning tree of Fig. 7 gives the cycle in Fig. 8.

Changing the value of the variable  $x_{2,1}$  from its current value of zero to  $x_{2,1} = \delta \geq 0$  forces the changes in the values of the basic variables indicated in Fig. 8. As can be seen from Fig. 8, the maximum possible value of  $\delta$  is  $\delta^* = 10$ , and the most restrictive basic variable is  $x_{1,1}$ . Therefore  $x_{1,1}$  leaves the basis, and  $x_{2,1}$  enters the basis. Hence, the new basis is given by

	1	2	3	4	5	$u_i$
1	0	2	0	2	2	15
2	-2	0	0	-1	0	15
3	0	2	4	0	32	18
4	38	2	41	-1	0	11
$v_j$	1	-1	-2	5	0	

Table 9:

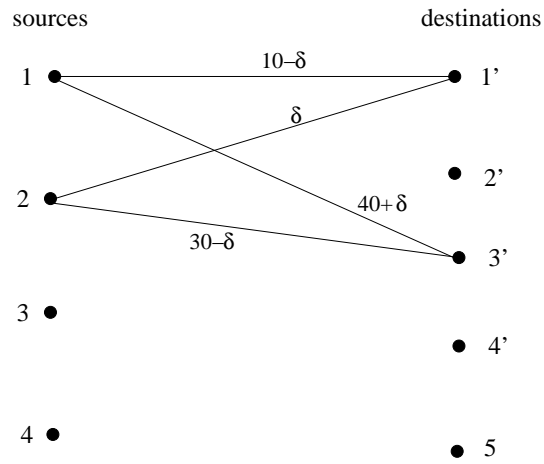


Figure 8: Cycle created by adding edge (2,1) to spanning tree from  $B^4$

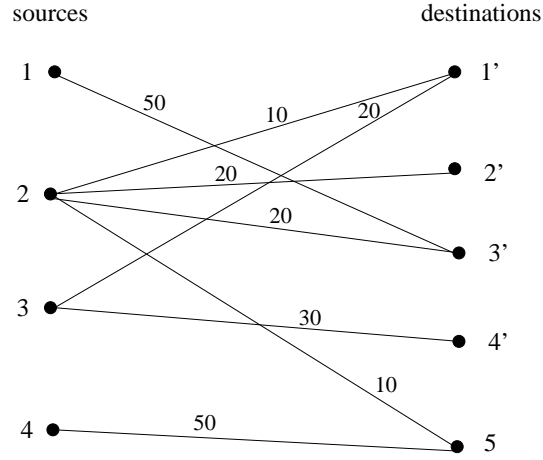


Figure 9: Spanning tree corresponding to the basis  $B^5$

$$B^5 = \{(1, 3), (2, 1), (2, 2), (2, 3), (2, 5), (3, 1), (3, 4), (4, 5)\}$$

and the values of the corresponding basic variables are

$$x_{1,3} = 50, x_{2,1} = 10, x_{2,2} = 20, x_{2,3} = 20, x_{2,5} = 10, x_{3,1} = 20, x_{3,4} = 30, x_{4,5} = 50.$$

The values of the basic variables in the basic solution corresponding to  $B^5$  are given in Table 10.

	1	2	3	4	5	Supply
1	0	0	$50^b$	0	0	50
2	$10^b$	$20^b$	$20^b$	0	$10^b$	60
3	$20^b$	0	0	$30^b$	0	50
4	0	0	0	0	$50^b$	50
Demand	30	20	70	30	60	

Table 10:

The spanning tree corresponding to the basis  $B^5$  is given in Fig. 9.

We now find the dual basic solution corresponding to  $B^5$ . The equality system that needs to be solved is given below.

- (a)  $u_1 + v_3 = 13$  (basic variable  $x_{1,3}$ ),
- (b)  $u_2 + v_1 = 14$  (basic variable  $x_{2,1}$ ),
- (c)  $u_2 + v_2 = 14$  (basic variable  $x_{2,2}$ ),
- (d)  $u_2 + v_3 = 13$  (basic variable  $x_{2,3}$ ),
- (e)  $u_2 + v_5 = 15$  (basic variable  $x_{2,5}$ ),
- (f)  $u_3 + v_1 = 19$  (basic variable  $x_{3,1}$ ),
- (g)  $u_3 + v_4 = 23$  (basic variable  $x_{3,4}$ ),

(h)  $u_4 + v_5 = 11$  (basic variable  $x_{4,5}$ ), and

(i)  $v_5 = 0$  (fixing  $v_5$  to zero).

Solving the above non-singular system results in the following dual basic solution

$$u^{B^5} = (15, 15, 20, 11) \text{ and } v^{B^5} = (-1, -1, -2, 3, 0)$$

From this dual basic solution, we can calculate the reduced costs on the variables based on the formula  $r_{i,j}^{B^5} = c_{i,j} - u_i^{B^5} - v_j^{B^5}$ . The reduced costs are given in Table 11.

	1	2	3	4	5	$u_i$
1	2	2	0	4	2	15
2	0	0	0	1	0	15
3	0	0	2	0	30	20
4	40	2	41	1	0	11
$v_j$	-1	-1	-2	3	0	

Table 11:

We now see that all reduced costs are non-negative. This means that  $B^5$  is an optimal basis, and that the corresponding basic solution is optimal for the transportation problem.