Example of the dual simplex method

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1 An example of the dual simplex method

Consider the following linear program:

$$
\begin{aligned}
&\text{Minimize } Z = 7x_1 + 2x_2 + 5x_3 + 4x_4 \\
&\text{subject to} \\
&2x_1 + 4x_2 + 7x_3 + x_4 \geq 5, \\
&8x_1 + 4x_2 + 6x_3 + 4x_4 \geq 8, \\
&3x_1 + 8x_2 + x_3 + 4x_4 \geq 4, \\
&x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0,
\end{aligned}
$$

Adding surplus variables x_5, x_6, x_7 in the three constraints, we obtain:

$$
\begin{aligned}
&\text{Minimize } Z = 7x_1 + 2x_2 + 5x_3 + 4x_4 \\
&\text{subject to} \\
&2x_1 + 4x_2 + 7x_3 + x_4 - x_5 = 5, && (y_1) \\
&8x_1 + 4x_2 + 6x_3 + 4x_4 - x_6 = 8, && (y_2) \\
&3x_1 + 8x_2 + x_3 + 4x_4 - x_7 = 4, && (y_3) \\
&x_i \geq 0 \text{ for } i = 1, 2, \ldots, 7,
\end{aligned}
$$

The dual (D) of (P) is given by:

$$
\begin{aligned}
&\text{Maximize } Z = 5y_1 + 8y_2 + 4y_3 \\
&\text{subject to} \\
&2y_1 + 8y_2 + 3y_3 \leq 7 && (x_1) \\
&4y_1 + 4y_2 + 8y_3 \leq 2 && (x_2) \\
&7y_1 + 6y_2 + y_3 \leq 5 && (x_3) \\
&y_1 + 4y_2 + 4y_3 \leq 4 && (x_4) \\
&y_1 \geq 0 && (x_5) \\
&y_2 \geq 0 && (x_6) \\
&y_3 \geq 0 && (x_7)
\end{aligned}
$$

We see, immediately, that the basis $B^1 = \{5, 6, 7\}$ gives a basic feasible solution to (D). The basic solution y^B to (D) is given by $y^{B^1} = (0, 0, 0)$, and the corresponding basic solution to (P) is given by $x^{B^1} = (0, 0, 0, 0, -5, -8, -4)$, which is not feasible for (P).

It follows that it is natural to use the dual simplex algorithm starting from the basis B^1.

First, write the problem (P) in table form as is shown in Table 1. Next, to find the tableau corresponding to B^1, we "make" the columns in the table corresponding to the basic variables unit vectors. To achieve this, we multiply the last three rows of table 1 with (-1). This gives table 2.

Basic variable	Z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	Right hand side
Z	1	-7	-2	-5	-4	0	0	0	0
x_5	0	2	4	7	1	-1	0	0	5
x_6	0	8	4	6	4	0	-1	0	8
x_7	0	3	8	1	4	0	0	-1	4

Table 1:

Basic variable	Z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	Right hand side
Z	1	-7	-2	-5	-4	0	0	0	0
x_5	0	-2	-4	-7	-1	1	0	0	-5
x_6	0	-8	-4	-6	-4	0	1	0	-8
x_7	0	-3	-8	-1	-4	0	0	1	-4

Table 2:

The information in Table 2 simply states the information needed for the simplex tableau associated with the basis B^1:

$$\begin{aligned}
0 &= Z - 7x_1 - 2x_2 - 5x_3 - 4x_4,\\
-5 &= x_5 - 2x_1 - 4x_2 - 7x_3 - x_4,\\
-8 &= x_6 - 8x_1 - 4x_2 - 6x_3 - 4x_4,\\
-4 &= x_7 - 3x_1 - 8x_2 - x_3 - 4x_4,
\end{aligned}$$

Note that the reduced costs on the non-basic variables x_1, x_2, x_3, x_4 are positive (7, 2, 5 and 4), which indicates that the basis B^1 is optimal for (P), and feasible for (D). However, since the basic variables are negative, the basis B^1 is *not* feasible for (P).

To perform a pivot of the dual simplex algorithm, we write the simplex tableau in the form with dual directions

$$\begin{aligned}
&\text{Maximize } Z_D = b^T y^{B^1} - \sum_{j \in B^1} x_j^{B^1} s_j\\
&\text{s.t.}\\
&\qquad s = s^{B^1} + \sum_{i=1}^{3} \tilde{d}^i s_{j(i)}, \qquad \text{(D')}\\
&\qquad s \geq 0_7.
\end{aligned}$$

where the variables s are the slack variables in the dual constraints. In other words, $s_1 := 7-2y_1-8y_2-3y_3$, $s_2 := 2-4y_1-4y_2-8y_3$, $s_3 := 5-7y_1-6y_2-y_3$, $s_4 := 4-y_1-4y_2-4y_3$, $s_5 := y_1$, $s_6 = y_2$ and $s_7 = y_3$. For the basis B^1, since $y^{B^1} = (0,0,0)$, we have $s^{B^1} = (7,2,5,4,0,0,0)$. Since all slack variables are nonnegative, this indicates that B^1 is dual feasible.

The basis matrix, and its inverse, are given by

$$A_{B^1} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \text{ and } A_{B^1}^{-1} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

The values $j(1)$, $j(2)$ and $j(3)$ are defined as follows. Since $A_{B^1}^{-1}a_{.5} = (1,0,0)$, $A_{B^1}^{-1}a_{.6} = (0,1,0)$ and $A_{B^1}^{-1}a_{.7} = (0,0,1)$, we have $j(1) = 5$, $j(2) = 6$ and $j(3) = 7$.

To obtain the dual directions $\tilde{d}^i$ for $i = 1,2,3$, we need to calculate the updated simplex tableau coefficients $\bar{a}_{.1} = A_{B^1}^{-1}a_{.1}$, $\bar{a}_{.2} = A_{B^1}^{-1}a_{.2}$, $\bar{a}_{.3} = A_{B^1}^{-1}a_{.3}$ and $\bar{a}_{.4} = A_{B^1}^{-1}a_{.4}$ for the non-basic variables x_1, x_2, x_3 and x_4. We obtain $\bar{a}_{.1} = (-2,-8,-3)$ $\bar{a}_{.2} = (-4,-4,-8)$, $\bar{a}_{.3} = (-7,-6,-1)$ and $\bar{a}_{.4} = (-1,-4,-4)$. The dual directions are given by

$$\tilde{d}^i_k := \begin{cases} 1, & \text{if } k \in B^1 \text{ and } k = j(i), \\ 0, & \text{if } k \in B^1 \text{ and } k \neq j(i), \\ \bar{a}_{i,k}, & \text{if } k \in N^1. \end{cases} \qquad (1)$$

for $i = 1,2,3$, where $N^1 := \{1,2,3,4\}$ denotes the non-basic variables. The dual directions can then be calculated to be $\tilde{d}^1 = (-2,-4,-7,-1,1,0,0)$, $\tilde{d}^2 = (-8,-4,-6,-4,0,1,0)$ and $\tilde{d}^3 = (-3,-8,-1,-4,0,0,1)$

The problem (D') for the basis B^1 can now be written as follows.

$$\text{Maximize } Z' = 0 + 5s_5 + 8s_6 + 4s_7,$$

$$\text{subject to}$$

$$\begin{pmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \\ s_5 \\ s_6 \\ s_7 \end{pmatrix} = \begin{pmatrix} 7 \\ 2 \\ 5 \\ 4 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -2 \\ -4 \\ -7 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} s_5 + \begin{pmatrix} -8 \\ -4 \\ -6 \\ -4 \\ 0 \\ 1 \\ 0 \end{pmatrix} s_6 + \begin{pmatrix} -3 \\ -8 \\ -1 \\ -4 \\ 0 \\ 0 \\ 1 \end{pmatrix} s_7.$$

$$s \geq 0_7.$$

To perform a dual pivot, we need to select an entering dual slack variable, or equivalently, an exiting variable in the problem (P) (the variable in (P) corresponding to this slack variable). We select s_6 (or equivalently x_6, since x_6 violates feasibility in (P) the most (8)). To find the entering variable in (P), or the leaving slack variable in (D), we need to solve the problem of maximizing $\delta \geq 0$ such that

$$s^{B^1} + \delta\tilde{d}^2 \geq 0_7.$$

The optimal value δ^* of δ is given by the formula (we have $j(2) = 6$)

$$\delta^* = \min\{-\frac{s_j^{B^1}}{\bar{a}_{2,j}} : j \in N^1 \text{ and } \bar{a}_{2,j} < 0\},$$

or equivalently

$$\delta^* = \min\{-\frac{7}{(-8)}, -\frac{2}{(-4)}, -\frac{5}{(-6)}, -\frac{4}{(-4)}\} = \frac{1}{2}.$$

We have that s_2 is a variable that becomes zero first (achieves the minimum). Since the variable in (P) that corresponds to s_2 is the variable x_2, this means x_2 enters the basis in (P). The new basis is therefore $B^2 = \{2,5,7\}$.

Observe that *all* the information for performing this pivot is available from the row of Table 2 corresponding to x_6 and the row corresponding to Z. This is very important. Deciding that x_6 leaves the basis was based on the fact that the number -8 in the right hand side column is smaller than the other right hand sides (-5 and -4). The values of the dual slack variables for the non-basic variables are the same as the reduced costs on the corresponding variables in (P), and these are given in the row corresponding to Z (the numbers 7, 2, 5 and 4). Finally, the values $\bar{a}_{2,1} = -8$, $\bar{a}_{2,2} = -4$, $\bar{a}_{2,3} = -6$ and $\bar{a}_{2,4} = -4$ are given in the row of Table 2 corresponding to x_6.

To obtain the new simplex tableau corresponding to B^2, we "massage" table 2 to make x_2 the basic variable in the row of Table 2 corresponding to x_6. This means that we must make the column of the table corresponding to x_2 a unit vector with Gaussian elimination, where the value "1" is placed in the row corresponding to x_6.

First multiply the row of Table 2 corresponding to x_6 with $-\frac{1}{4}$. This gives Table 3.

Basic variable	Z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	Right hand side
Z	1	-7	-2	-5	-4	0	0	0	0
x_5	0	-2	-4	-7	-1	1	0	0	-5
x_6	0	2	1	$1\frac{1}{2}$	1	0	$-\frac{1}{4}$	0	2
x_7	0	-3	-8	-1	-4	0	0	1	-4

Table 3:

Next, multiply the row corresponding to x_6 with 8, and add the result to the row corresponding to x_7. This gives Table 4.

Basic variable	Z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	Right hand side
Z	1	-7	-2	-5	-4	0	0	0	0
x_5	0	-2	-4	-7	-1	1	0	0	-5
x_6	0	2	1	$1\frac{1}{2}$	1	0	$-\frac{1}{4}$	0	2
x_7	0	15	0	11	4	0	-2	1	12

Table 4:

We now multiply the row of table 4 corresponding to x_6 with 4, and add the result to the row corresponding to x_5. This gives Table 5.

Basic variable	Z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	Right hand side
Z	1	-7	-2	-5	-4	0	0	0	0
x_5	0	6	0	-1	3	1	-1	0	3
x_6	0	2	1	$1\frac{1}{2}$	1	0	$-\frac{1}{4}$	0	2
x_7	0	15	0	11	4	0	-2	1	12

Table 5:

Finally, multiply the row of Table 5 corresponding to x_6 with 2, and add the result to the row of Table 5 corresponding to Z, and lastly declare x_2 to be the basic variable in the row where x_6 was the basic variable before. This gives Table 6.

The new basic solution for (P) can now be read from the right hand side column of Table 6 as $x^{B^2} = (0, 2, 0, 0, 3, 0, 12)^T$. The reduced costs on the non-basic variables can be read from the row corresponding to Z ($r_1^{B^2} = 3$, $r_3^{B^2} = 2$, $r_4^{B^2} = 2$ and $r_6^{B^2} = \frac{1}{2}$). Hence, the basis B^2 is optimal.

Basic variable	Z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	Right hand side
Z	1	-3	0	-2	-2	0	$-\frac{1}{2}$	0	4
x_5	0	6	0	-1	3	1	-1	0	3
x_2	0	2	1	$1\frac{1}{2}$	1	0	$-\frac{1}{4}$	0	2
x_7	0	15	0	11	4	0	-2	1	12

Table 6:

We now describe exactly how a pivot can be done by manipulating the tables only (without necessarily calculating the dual directions).

(i) In the initial table representing the first basic solution, identify a right hand side (basic variable) which is negative (the variable x_6 with the value -8 in our case). This variable leaves the basis for (P).

(ii) In the row of the table corresponding to that specific row, identify the updated simplex tableau coefficients on the non-basic variables (the numbers $\bar{a}_{2,1} = -8$, $\bar{a}_{2,2} = -4$, $\bar{a}_{2,3} = -6$ and $\bar{a}_{2,4} = -4$ in our case). Also, identify the reduced costs on the non-basic variables in the row corresponding to Z, which also give the values of the dual slack variables (the numbers $r_1^{B^1} = s_1^{B_1} = 7$, $r_2^{B^1} = s_2^{B_1} = 2$, $r_3^{B^1} = s_3^{B_1} = 5$ and $r_4^{B^1} = s_4^{B_1} = 4$ in our case).

(iii) Compute the following minimum, and the non-basic variable $j \in N$ that achieves the minimum:

$$\delta^* = \min\{-\frac{s_j^B}{\bar{a}_{\bar{i},j}} : j \in N \text{ and } \bar{a}_{\bar{i},j} < 0\}$$

(in our case the ratio:

$$\delta^* = \min\{-\frac{s_1^{B^1}}{\bar{a}_{2,1}}, -\frac{s_1^{B^1}}{\bar{a}_{2,2}}, -\frac{s_1^{B^1}}{\bar{a}_{2,3}}, -\frac{s_1^{B^1}}{\bar{a}_{2,4}}\} = \frac{1}{2}$$

and the non-basic variable x_2). The non-basic variable that achieves the minimum leaves the basis.

(iv) Update the basis by performing Gaussian elimination on the table to obtain the new simplex tableau. Continue if necessary, from this new simplex tableau.