Introduction to Optimization:

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Your assignment

20 different questions Q1-Q20 are posed on the subsequent pages. Q1-Q8 and Q11-Q18 are *multiple choice questions*. For each of these, the only correct answer is one of the answers proposed. To answer a specific question, you are requested without further explanation to write, for example, "7.b" as your answer to question Q7. Q9-Q10 and Q19-Q20 are ordinary *text questions*. Each correct answer to a multiple choice question gives 4 points, to a text question gives 9 points. The maximum score is thus 100 points.

Note: only the last 10 questions are available

Planning the study

A DIKU-student is working in company XYZ to make some money for his studying. During the low-activity hours between 14 and 16, he is planning the courses for the next semester. Looking in the study-plan (*lektionskataloget*) he finds five different courses A, B, C, D and E which seem to be interesting, since they deal with algorithms and combinatorial optimization. However, there are some complicating constraints as follows:

- Courses A and B are both at Friday morning, and thus cannot be followed at the same time.
- To follow courses B or C it is necessary also to follow course D.
- Obviously he cannot follow a course without having the corresponding text book. The cost of books for the individual courses is given in the table below, and the student cannot afford books for more than 1100 kr.
- Some courses are easier than others. Having spoken with some older students, he finds out, that the minimum number of weekly hours needed to pass the exam are as listed in the following table.

course	A	B	C	D	E
book price	300 kr	200 kr	800 kr	400 kr	450 kr
hours preparation	10 h	15 h	13 h	8 h	12 h

- The courses A,B,C are quite mathematically oriented, while the courses D,E are practically oriented. If the student is using more than 25 hours a week on the mathematical courses, he would like also to use 10 hours on the practical courses.
- The student wishes to follow in a satisfactory way most possible courses.

Let $\delta_i \in \{0,1\}$ indicate whether course i is taken, and $x_i \geq 0$ be the corresponding number of hours used for preparation. The objective function is obviously

$$\text{maximize } \delta_A + \delta_B + \delta_C + \delta_D + \delta_E$$

Q 11 How would you ensure that $\delta_i = 1$ only if the student follows the course using the appropriate number of hours for preparation

11.a) $\delta_A \geq x_A, \quad \delta_B \geq x_B, \quad \delta_C \geq x_C, \quad \delta_D \geq x_D, \quad \delta_E \geq x_E.$

11.b) $10\delta_A \leq x_A, \quad 15\delta_B \leq x_B, \quad 13\delta_C \leq x_C, \quad 8\delta_D \leq x_D, \quad 12\delta_E \leq x_E.$

11.c) $\delta_A \leq 10x_A, \quad \delta_B \leq 15x_B, \quad \delta_C \leq 13x_C, \quad \delta_D \leq 8x_D, \quad \delta_E \leq 12x_E.$

11.d) $x_A - 80\delta_A \geq 0, \quad x_B - 80\delta_B \geq 0, \quad x_C - 80\delta_C \geq 0, \quad x_D - 80\delta_D \geq 0, \quad x_E - 80\delta_E \geq 0.$

11.e) $x_A + 10\delta_A \geq 0, \quad x_B + 15\delta_B \geq 0, \quad x_C + 13\delta_C \geq 0, \quad x_D + 8\delta_D \geq 0, \quad x_E + 12\delta_E \geq 0.$

□

To finish the model, the student writes down the following constraints

$$\begin{array}{ll}
30\delta_A + 20\delta_B + 80\delta_C + 40\delta_D + 45\delta_E \le 110 & (1)\\
x_D + x_E \ge \frac{10}{25}(x_A + x_B + x_C) & (2)\\
\delta_B \le \delta_D & (3)\\
\delta_C \le \delta_D & (4)\\
x_A + x_B + x_C - 144\delta \le 24 & (5)\\
x_D + x_E - 10\delta \ge 0 & (6)\\
\delta_A + \delta_B \le 1 & (7)\\
\delta_A, \delta_B, \delta_C, \delta_D, \delta_E, \delta \in \{0,1\} & (8)\\
x_A, x_B, x_C, x_D, x_E \ge 0 & (9)
\end{array}$$

Q 12 Unfortunately his model is not correct. Which inequality or inequalities should be removed to get a proper formulation of his problem?

12.a) Inequality (1) should be removed.

12.b) Inequalities (5) and (6) should be removed.

12.c) Inequality (7) should be removed.

12.d) Inequalities (3) and (4) should be removed.

12.e) Inequality (2) should be removed.

12.f) Inequalities (8) and (9) should be removed.

□

Solving the (now correct) model to LP-optimality gave a fractional solution

$$\delta_A = 0,\ \delta_B = 1,\ \delta_C = \frac{1}{16},\ \delta_D = 1,\ \delta_E = 1$$

To tighten the formulation, the student wants to derive a cover inequality from the constraint

$$30\delta_A + 20\delta_B + 80\delta_C + 40\delta_D + 45\delta_E \le 110$$

In order to separate the most violated cover inequality he solves a knapsack problem.

Q 13 What is the correct form of this knapsack problem.

13.a)

$$\begin{array}{rl}
\gamma = \min & x'_A + x'_B + x'_C + x'_D + x'_E\\
\text{s.t.} & x'_A + x'_B + x'_C + x'_D + x'_E \ge 3\\
& x'_A, x'_B, x'_C, x'_D, x'_E \in \{0,1\}
\end{array}$$

13.b)

$$\begin{array}{rl}
\gamma = \min & 0x'_A + 1x'_B + \frac{1}{16}x'_C + 1x'_D + 1x'_E\\
\text{s.t.} & 30x'_A + 20x'_B + 80x'_C + 40x'_D + 45x'_E \ge 111\\
& x'_A, x'_B, x'_C, x'_D, x'_E \in \{0,1\}
\end{array}$$

13.c)

$$\begin{array}{rl}
\gamma = \min & 1x'_A + 0x'_B + \frac{15}{16}x'_C + 0x'_D + 0x'_E\\
\text{s.t.} & 30x'_A + 20x'_B + 80x'_C + 40x'_D + 45x'_E \ge 111\\
& x'_A, x'_B, x'_C, x'_D, x'_E \in \{0,1\}
\end{array}$$

13.d)

$$\begin{array}{rl}
\gamma = \min & x'_A + x'_B + x'_C + x'_D + x'_E\\
\text{s.t.} & 30x'_A + 20x'_B + 80x'_C + 40x'_D + 45x'_E \ge 111\\
& x'_A, x'_B, x'_C, x'_D, x'_E \in \{0,1\}
\end{array}$$

13.e)

$$\begin{array}{rl}
\gamma = \max & \delta_A + \delta_B + \delta_C + \delta_D + \delta_E\\
\text{s.t.} & 30\delta_A + 20\delta_B + 80\delta_C + 40\delta_D + 45\delta_E \le 110\\
& \delta_A, \delta_B, \delta_C, \delta_D, \delta_E \in \{0,1\}
\end{array}$$

□

When solving the knapsack problem in the previous question, there may be several equivalent solutions. In this case, you should choose the solution with most x'_i set to one.

Q 14 What is the derived cover inequality.

14.a) $\delta_B + \delta_C + \delta_D \le 2$

14.b) $\delta_A + \delta_B + \delta_C + \delta_D + \delta_E \le 2$

14.c) $\delta_B + \delta_D + \delta_E \le 3$

14.d) $\delta_B + \delta_C + \delta_D + \delta_E \le 3$

14.e) No valid cover inequality can be derived.

□

Pasta production

Company XYZ is also producing excellent Italian dishes. For these dishes it is an art to choose the correct type of pasta for a given sauce. There is only have a limited quantity b of the individual pasta types, thus when planning the dishes, a taste-index c is given for a given combination of sauce and pasta, while the matrix A gives the quantity of pasta used for the given dish. This can be formulated as the following linear model

$$\begin{array}{rl}
\max & cx\\
\text{s.t.} & Ax \le b\\
& x \ge 0
\end{array}$$

where c and x are vectors of length n, b is a vector of length m, and A is an $n \times m$ matrix.

Q 15 Assume that b is a vector of integers. Regardless of the values of b and c, for which one of the following matrices A will all basic solutions be integer valued. All blank entries in the matrices are zero.

$$A_1 = \begin{pmatrix} & 1 & & 1 & \\ 1 & & 1 & & \\ & & & & 1 \\ & 1 & & -1 & \\ & & & & 1 \end{pmatrix} \quad A_2 = \begin{pmatrix} 1 & & & -1 & 1 \\ & & & -1 & -1 \\ & 1 & 1 & & \\ & 1 & & & \\ & & & & 1 \end{pmatrix} \quad A_3 = \begin{pmatrix} 1 & & & & 1 \\ & & 1 & -1 & \\ & & 1 & 1 & \\ 1 & 1 & & & \\ & & -1 & & 1 \end{pmatrix}$$

$$A_4 = \begin{pmatrix} & 1 & & & 1 \\ 1 & & & 1 & \\ & 1 & -1 & & \\ & 1 & & & 1 \\ -1 & & 1 & & \end{pmatrix} \quad A_5 = \begin{pmatrix} 1 & & & & 1 \\ -1 & -1 & & & -1 \\ & 1 & & 2 & \\ & & -1 & & \\ & & 1 & & \end{pmatrix} \quad A_6 = \begin{pmatrix} 5 & & & & \\ & 5 & & & \\ & & 5 & & \\ & & & 5 & \\ & & & & 5 \end{pmatrix}$$

15.a) Matrix A_1.

15.b) Matrix A_2.

15.c) Matrix A_3.

15.d) Matrix A_4.

15.e) Matrix A_5.

15.f) Matrix A_6.

15.g) None of the above.

□

Gomory cut

Only a few people know that Gomory actually started his career in Company XYZ. For several years he worked in the kitchen, where he cut out the pizzas. One day after having cut out several hundreds of pizzas he went home and developed *Gomory's cutting plane algorithm*. Gomory's career in company XYZ however ended when he started to arrange the olives at all integer coordinates, and cutting out the pizzas such that no integer points were excluded. The last pizza with olives at integer coordinates corresponded to the following optimization problem:

$$\begin{array}{rrcrcr} z = \max & 4x_1 & - & x_2 & & \\ \text{s.t.} & 7x_1 & - & 2x_2 & \leq & 14 \\ & & & x_2 & \leq & 3 \\ & 2x_1 & - & 2x_2 & \leq & 3 \\ & x_1, x_2 \geq 0 & & & & \end{array}$$

We add slack variable $x_3 \geq 0$ to the first constraint, $x_4 \geq 0$ to the second constraint, and $x_5 \geq 0$ to the last constraint. Running some iterations of the Simplex algorithm, the following equations appear

$$\begin{aligned} &z = \tfrac{59}{7} - \tfrac{4}{7}x_3 - \tfrac{1}{7}x_4 \\ &x_1 + \tfrac{1}{7}x_3 + \tfrac{2}{7}x_4 = \tfrac{20}{7} \\ &x_2 + x_4 = 3 \\ &-\tfrac{2}{7}x_3 + \tfrac{10}{7}x_4 + x_5 = \tfrac{23}{7} \end{aligned}$$

Q 16 What is the optimal LP-solution $x = (x_1, x_2, x_3, x_4, x_5)$ to the above problem

16.a) $x = (\frac{20}{7}, 3, 0, 0, \frac{23}{7})$

16.b) $x = (\frac{59}{7}, 0, 0, -\frac{4}{7}, -\frac{1}{7})$

16.c) $x = (0, 0, -\frac{2}{7}, \frac{10}{7}, 1)$

16.d) $x = (\frac{20}{7}, 3, \frac{22}{7}, 0, 0)$

16.e) $x = (-\frac{59}{7}, 0, 0, \frac{4}{7}, \frac{1}{7})$

16.f) $x = (1, 1, 0, 0, 1)$

□

It is now wished to solve the problem to integer optimality. Derive a Gomory cut from the first constraint in the simplex tableau.

Q 17 Which of the following inequalities appears:

17.a) $\frac{1}{7}x_3 + \frac{1}{7}x_4 \geq \frac{6}{7}$

17.b) $x_1 + \frac{1}{7}x_3 + \frac{2}{7}x_4 \geq \frac{20}{7}$

17.c) $\frac{5}{7}x_1 + \frac{1}{7}x_2 \geq 1$

17.d) $\frac{2}{7}x_1 + \frac{1}{7}x_2 + \frac{1}{3}x_3 \geq \frac{1}{3}$

17.e) $x_3 + 2x_4 \geq 6$

□

Modular arithmetics

Professor Wolsey also worked at Company XYZ during one summer, where he was making home-made *fettucine* and *tagliatelle*. Looking at the endless rows of equidistant slices inspired him to develop the theory of modular arithmetics.

In an IP problem we have the constraint

$$21x_1 + 43x_2 + 7x_3 = 35$$

Q 18 Which of the following inequalities can be derived using modular arithmetics

18.a) $x_1 + x_2 + x_3 \geq 7$

18.b) $3x_1 + x_2 + x_3 \geq 5$

18.c) $3x_1 + x_2 + x_3 \geq 6$

18.d) $3x_1 + 2x_2 + x_3 \geq 7$

18.e) $x_1 + 3x_2 + x_3 \geq 6$

□

Spaghetti Bolognese (text questions)

The only place in the world, where you cannot order *Spaghetti Bolognese* is actually in Bologna, where this dish is considered to be too simple. Anyhow, to make a good meat sauce for spaghetti Bolognese, you need to mix a quantity of onion x_1 and minced beef x_2 according to the following constraints:

$$\begin{array}{rrcrcr} z = \max & 2x_1 & + & 7x_2 & & \\ \text{s.t.} & x_1 & + & 4x_2 & \leq & 12 \\ & 2x_1 & + & x_2 & \leq & 7 \\ & & & x_2 & \geq & 1 \\ & x_1 & & & \geq & 1 \\ & \multicolumn{5}{l}{x_1, x_2 \geq 0, \text{integer}} \end{array}$$

Q 19 Relax the first constraint $x_1 + 4x_2 \leq 12$ using a lagrangian multiplier λ. Write down the relaxed problem. □

Q 20 For which value of λ do we obtain the smallest solution value of the lagrangian relaxed problem? □