Chapter 7, Operations Research (OR)

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1 Sensitivity analysis (revisited)

As usual, we consider a linear program (P) of the form

$$\begin{aligned} \text{Minimize } Z_P &= c^T x \\ \text{s.t.} & \\ Ax &= b, \qquad \text{(P)} \\ x &\geq 0_n. \end{aligned}$$

and its dual (D)

$$\begin{aligned} \text{Maximize } Z_D &= b^T y \\ \text{s.t.} & \qquad \text{(D)} \\ A^T y &\leq c, \\ s &\geq 0_n. \end{aligned}$$

We assume we have solved the problems (P) and (D), and that we have an optimal basis $B \subseteq \{1, 2, \ldots, n\}$. *Sensitivity analysis* then refers to using to the basis B to solve a modified version (P') and (D') of the problems (P) and (D) from modified data A', b' and c' (whenever this information is useful). This is best done by example. Note that, if the optimal basic matrix A_B remains unchanged, then B remains a valid basis for the problems (P') and (D'). This is the main property that is used in sensitivity analysis.

Example 1 *In the following, we consider the linear program (P) that models the WGC problem in the following format*

$$\begin{aligned} \textit{Minimize } Z &= -3x_1 - 5x_2 \\ \textit{subject to} & \\ x_1 + x_3 &= 4 \qquad (y_1) \\ 2x_2 + x_4 &= 12 \qquad (y_2) \\ 3x_1 + 2x_2 + x_5 &= 18 \qquad (y_3) \\ x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 &\geq 0, x_5 \geq 0 \end{aligned}$$

and the corresponding dual linear program (D)

$$
\begin{aligned}
&\textit{Maximize } Z' = 4y_1 + 12y_2 + 18y_3 \\
&\quad \textit{subject to} \\
&\quad y_1 + 3y_3 \le -3 \qquad\qquad (x_1) \\
&\quad 2y_2 + 2y_3 \le -5 \qquad\qquad (x_2) \\
&\quad y_1 \le 0, \qquad\qquad (x_3) \\
&\quad y_2 \le 0, \qquad\qquad (x_4) \\
&\quad y_3 \le 0. \qquad\qquad (x_5)
\end{aligned}
$$

Recall that $B = \{1, 2, 3\}$ *is an optimal basis for the problems (P) and (D). The optimal basis* A_B*, and its inverse* A_B^{-1} *are given by*

$$
A_B = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 3 & 2 & 0 \end{pmatrix} \textit{ and } A_B^{-1} = \begin{pmatrix} 0 & -\frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{2} & 0 \\ 1 & \frac{1}{3} & -\frac{1}{3} \end{pmatrix}.
$$

Furthermore, the optimal dual basic solution is given by the unique solution to the system $a_{.1}^T y = c_1$*,* $a_{.2}^T y = c_2$ *and* $a_{.3}^T y = c_3$*, or equivalently the system* $y_1 + 3y_3 = -3$*,* $2y_2 + 2y_3 = -5$ *and* $y_1 = 0$*. This implies* $y^B = (0, -1\frac{1}{2}, -1)$*.*

Now, the reduced costs on the non-basic variables x_4 *and* x_5 *are given by* $c_4 - a_{.4}^T y^B = 0 - (-y_2^B) = 1\frac{1}{2}$ *and* $c_5 - a_{.5}^T y^B = 0 - (-1) = 1$*. Since all reduced costs are non-negative, this shows* B *is an optimal basis. Finally, the updated columns on the non-basic variables* $\bar{a}_{.4} = A_B^{-1} a_{.4}$ *and* $\bar{a}_{.5} = A_B^{-1} a_{.5}$ *are given by*

$$
\bar{a}_{.4} = \begin{pmatrix} -\frac{1}{3} \\ \frac{1}{2} \\ \frac{1}{3} \end{pmatrix} \textit{ and } \bar{a}_{.5} = \begin{pmatrix} \frac{1}{3} \\ 0 \\ -\frac{1}{3} \end{pmatrix}.
$$

In summary, the information available can be summarized in Table 1, which simply contains the information of the simplex tableau from the basis B*:*

$$
\begin{aligned}
-36 &= Z - \frac{3}{2}x_4 - x_5, \\
2 &= x_1 - \frac{1}{3}x_4 + \frac{1}{3}, \\
6 &= x_2 + \frac{1}{2}x_4, \\
2 &= x_3 + \frac{1}{3}x_4 - \frac{1}{3}x_5,
\end{aligned}
$$

Basic variable	Z	x_1	x_2	x_3	x_4	x_5	Right hand side
Z	1	0	0	0	$-\frac{3}{2}$	-1	-36
x_1	0	1	0	0	$-\frac{1}{3}$	$\frac{1}{3}$	2
x_2	0	0	1	0	$\frac{1}{2}$	0	6
x_3	0	0	0	1	$\frac{1}{3}$	$-\frac{1}{3}$	2

Table 1:

We next consider various changes in the WGC model.

Example 2 *(Change of right-hand-side)*

Recall that, in the inequality $2x_2 \leq 12$ *of the WGC model, the number "12" refers to a capacity of a plant in a production company. What happens if if the capacity of this plant is increased by* δ*, or in other words, if the inequality* $2x_2 \leq 12$ *is changed to* $2x_2 \leq 12+\delta$*? Let (P') and (D') denote the new linear programs, and let* b' *denote the new right-hand-side vector (*$A' = A$ *and* $c' = c$*).*

A first thing to observe is that the matrix A_B *remains non-singular (since* A_B *is not changed), and therefore* B *remains a basis for the new problem (P').*

Secondly, the dual basic solution corresponding to the basis B *is given as the unique solution* y^B *to the system* $a_{.j}^T y = c_j$ *for* $j \in B$*. Hence, since* $A' = A$ *and* $c' = c$*, the "old"* y^B *remains a dual feasible basic solution for the problem (D').*

It follows that, if B *is feasible for (P'), then* B *is also an optimal basis for (P') and (D').*

To compute the "new" basic solution, we need to solve the system $\sum_{j\in B} a_{j.}x_j = b'$*. or in other words*

$$A_B \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = b + \begin{pmatrix} 0 \\ \delta \\ 0 \end{pmatrix} \Longrightarrow$$

$$\begin{pmatrix} x_1^B \\ x_2^B \\ x_3^B \end{pmatrix} = a_B^{-1} b + a_B^{-1} \begin{pmatrix} 0 \\ \delta \\ 0 \end{pmatrix} \Longrightarrow$$

$$\begin{pmatrix} x_1^B \\ x_2^B \\ x_3^B \end{pmatrix} = \begin{pmatrix} 2 \\ 6 \\ 2 \end{pmatrix} + \delta \begin{pmatrix} -\frac{1}{3} \\ \frac{1}{2} \\ \frac{1}{3} \end{pmatrix} = \begin{pmatrix} 2-\delta\frac{1}{3} \\ 6+\delta\frac{1}{2} \\ 2+\delta\frac{1}{3} \end{pmatrix}$$

Observe that, as long as $-6 \leq \delta \leq 6$*, the "new" basic solution is feasible for (P'). It follows that* B *is an optimal basis for (P') if* $-6 \leq \delta \leq 6$*.*

What is the objective value of this "new" basic solution in the case when $-6 \leq \delta \leq 6$*? This change can be calculated as* $(b')^T y^B = b^T y^B + [0,\delta,0]y^B = b^T y^B + \delta y_2^B$*. In our case, we have* $y_2^B = -1\frac{1}{2}$ *which indicates that the cost can be decreased by* $1\frac{1}{2}$ *for every extra unit of the second resource. This is why the optimal dual variables are also called* shadow prices *(implicit prices on the resources).*

Example 3 *(Changing coefficients on a non-basic variable)*

Consider now a non-basic variable $\bar{j} \in \{1,2,\ldots,n\} \setminus B$*, and suppose we change* $c_{\bar{j}}$ *to* $c'_{\bar{j}}$ *and* $a_{.\bar{j}}$ *to* $a'_{.\bar{j}}$*. Observe that this change does not change the fact that* B *is a feasible basis (since* $a'_{.j} = a_{.j}$ *for every* $j \in B$ *and* $b' = b$*). Also, since* $c'_j = c_j$ *for every* $j \in B$*, the dual basic solution corresponding to* B *remains the same. Finally, since* $a'_{.j} = a_{.j}$ *and* $c'_j = c_j$ *for every non-basic variable* x_j *different from* $x_{\bar{j}}$*, we have that* y^B *satisfies all dual constraints except perhaps the inequality* $(a'_{.\bar{j}})^T y \leq c'_{\bar{j}}$*.*

Therefore, the question is whether y^B *remains feasible when* $a_{.\bar{j}}^T y \leq c_{\bar{j}}$ *is changed to* $(a'_{.\bar{j}})^T y \leq c'_{\bar{j}}$*. If yes, then* x^B *remains optimal, and if no, the reduced cost on the variable* $x_{\bar{j}}$ *becomes* $c'_{\bar{j}} - (a'_{.\bar{j}})^T y^B < 0$*, and the variable* $x_{\bar{j}}$ *becomes a candidate (in fact the only candidate) for entering the basis.*

For example, suppose in the WGC problem we change the cost coefficient c_4 *on* x_4 *from its current value* $c_4 = 0$ *to* $c'_4 = -1$ *(making the variable* x_5 *"more attractive"). Does* $y^B = (0,-1,-1\frac{1}{2})$ *satisfy* $a_{.4}^T y \leq c'_4$*. The answer is yes, since* $a_{.4}^T y^B = -1\frac{1}{2}$*. Hence, the optimal solution* $x^B = (2,6,2,0,0)$ *remains unchanged when* $c_4 = 0$ *is changed to* $c'_4 = -1$*.*

Example 4 *(Adding a new variable)*

Suppose we add a new variable x_{new} *to the WGC problem as follows*

$$\text{Minimize } Z = -3x_1 - 5x_2 - 4x_{\text{new}}$$
$$\text{subject to}$$
$$x_1 + x_3 + 2x_{\text{new}} = 4$$
$$2x_2 + x_4 + 3x_{\text{new}} = 12$$
$$3x_1 + 2x_2 + x_5 + x_{\text{new}} = 18$$
$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0, x_5 \geq 0, x_{\text{new}} \geq 0$$

We have $c_{\text{new}} = -4$ *and* $a_{.\text{new}} = (2,3,1)^T$. *As in example 3, the basis B is a feasible basis for the modified problem (P'), and the question is whether this basis is still optimal. This question is answered by checking whether or not* y^B *satisfies the new constraint* $(a_{.\text{new}})^T y \leq c_{\text{new}}$. *Since we have* $(a_{.\text{new}})^T y^B = -5\frac{1}{2}$ *and* $c_{\text{new}} = -4$, *the new dual constraint is indeed satisfied. This implies that the reduced cost* r_{new} *on* x_{new} *is positive, and therefore B is still an optimal basis for the new problem. If the reduced cost on* x_{new} *had been negative, the simplex method could be continued from the basis B by letting* x_{new} *enter the basis (this is called re-optimization).*

Example 5 *(Changing the coefficients on a basic variable)*

Consider, now, a basic variable $\bar{j} \in B$, *and suppose we change* $c_{\bar{j}}$ *to* $c'_{\bar{j}}$ *and* $a_{.\bar{j}}$ *to* $a'_{.\bar{j}}$. *Now, since we have changed a column of the matrix* A_B, *the matrix* A_B *may no longer be non-singular, and therefore B might not be a basis. However, it is still possible to use the information obtained from the basis B. Let* A_B *denotes the basis matrix* before *the change, and define* $\bar{a}'_{.\bar{j}} := A_B^{-1} a'_{.\bar{j}}$ *and* $r'_{\bar{j}} := c_{\bar{j}} - (a'_{\bar{j}})^T y^B$. *We have the equality system*

$$Z = c^T x^B + r'_{\bar{j}} x_{\bar{j}} + \sum_{j \in N \setminus \{\bar{j}\}} r_j x_j \tag{1}$$

$$A_B^{-1} b = \bar{a}'_{.\bar{j}} x_{\bar{j}} + \sum_{j \in N \setminus \{\bar{j}\}} \bar{a}_{.j} x_j \tag{2}$$

This system is almost a simplex tableau for the new problem, except that we might have that $r'_{\bar{j}} \neq 0$ *and that* $\bar{a}'_{.\bar{j}}$ *might not be a unit vector. If changing* $a_{.\bar{j}}$ *to* $a'_{.\bar{j}}$ *still gives a basis for the new problem, then it is possible to make the column corresponding to* $x_{\bar{j}}$ *a unit vector by using Gaussian elimination, and thereby create a simplex tableau for the new problem for the basis B. We illustrate this on an example.*

Consider, again, the WGC problem, and the optimal basis for this problem. Recall that the optimal simplex tableau is given in Table 1. Suppose we change the coefficients on the basic variable x_2 *by changing* $c_2 = -5$ *to* $c'_2 = -3$ *and* $a_{.2} = (0,2,2)^T$ *to* $a'_{.2} = (0,2,3)^T$. *This then gives* $r'_2 = c'_2 - (a'_{.2})^T y^B = 3$ *and* $\bar{a}'_{.2} = A_B^{-1} a'_{.2} = (\frac{1}{3}, 1, -\frac{1}{3})^T$. *Therefore, if we modify the column corresponding to* x_2 *in Table 1 accordingly, we obtain Table 2.*

Basic variable	Z	x_1	x_2	x_3	x_4	x_5	Right hand side
Z	1	0	-3	0	$-\frac{3}{2}$	-1	-36
x_1	0	1	$\frac{1}{3}$	0	$-\frac{1}{3}$	$\frac{1}{3}$	2
x_2	0	0	1	0	$\frac{1}{2}$	0	6
x_3	0	0	$-\frac{1}{3}$	1	$\frac{1}{3}$	$-\frac{1}{3}$	2

Table 2:

To obtain a simplex tableau for the new problem, we need to make the column corresponding to x_2 *a unit vector. First, we multiply the row of Table 2 corresponding to* x_2 *with 3, and add the result to the row corresponding to Z. This gives Table 3.*

Basic variable	Z	x_1	x_2	x_3	x_4	x_5	Right hand side
Z	1	0	0	0	0	-1	-18
x_1	0	1	$\frac{1}{3}$	0	$-\frac{1}{3}$	$\frac{1}{3}$	2
x_2	0	0	1	0	$\frac{1}{2}$	0	6
x_3	0	0	$-\frac{1}{3}$	1	$\frac{1}{3}$	$-\frac{1}{3}$	2

Table 3:

Next, multiply the row of Table 3 corresponding to x_2 with $\frac{1}{3}$, and add the result to the row corresponding to x_3. This gives Table 4.

Basic variable	Z	x_1	x_2	x_3	x_4	x_5	Right hand side
Z	1	0	0	0	0	-1	-18
x_1	0	1	$\frac{1}{3}$	0	$-\frac{1}{3}$	$\frac{1}{3}$	2
x_2	0	0	1	0	$\frac{1}{2}$	0	6
x_3	0	0	0	1	$\frac{1}{2}$	$-\frac{1}{3}$	4

Table 4:

Finally, multiply the row of Table 4 corresponding to x_2 with $-\frac{1}{3}$, and add the result to the row corresponding to x_1. This gives Table 5.

Basic variable	Z	x_1	x_2	x_3	x_4	x_5	Right hand side
Z	1	0	0	0	0	-1	-18
x_1	0	1	0	0	$-\frac{1}{2}$	$\frac{1}{3}$	0
x_2	0	0	1	0	$\frac{1}{2}$	0	6
x_3	0	0	0	1	$\frac{1}{2}$	$-\frac{1}{3}$	4

Table 5:

The final table shows that the basic solution for the new problem from the basis $B = \{1, 2, 3\}$ is $x^B = (0, 6, 4, 0, 0)$. Furthermore, since the reduced cost on x_5 is $r_5 = 1$, and the reduced cost on x_4 is zero, this solution is optimal for the new problem.

2 The transportation problem

We now introduce a special case of a linear program. Suppose we are given m sources and n destinations. Let $S := \{1, 2, \ldots, m\}$ and $D := \{1, 2, \ldots, n\}$ index the sources and the destinations respectively. The transportaion problem involves transporting amounts of a certain good from the sources to the destinations. Each source $i \in S$ has s_i units of the good that needs to transported to the destinations. Furthermore, for every destination $j \in D$, exactly d_j units of the good needs to be shipped to destination j. Define the following variables:

$$x_{i,j} : \text{ amount of the good shipped from source } i \in S \text{ to destination } j \in D.$$

Finally, there is a cost associated with transporting the good from a source $i \in S$ to a destination $j \in D$:

$$c_{i,j} : \text{ cost of transporting one unit of the good from source } i \in S \text{ to destination } j \in D.$$

Observe that, for it to be possible to transport *all* the amounts of the good available at the sources to *exactly* meet the demands of the goods at the destinations, we need to make the following assumption:

$$\sum_{i \in S} s_i = \sum_{j \in D} d_j$$

The problem of transporting the total amount of the good $\sum_{i \in S} s_i = \sum_{j \in D} d_j$ available at the sources to the destinations can now be formulated as the following linear program:

$$\begin{aligned}
\text{Minimize } Z = \sum_{i \in S} \sum_{j \in D} c_{i,j} x_{i,j} & \\
\text{s.t.} & \\
\sum_{j \in D} x_{i,j} = s_i, & \qquad \text{for all } i \in S \\
\sum_{i \in S} x_{i,j} = d_j, & \qquad \text{for all } j \in D \\
x_{i,j} \geq 0. & \qquad \text{for all } i \in S, j \in D
\end{aligned}$$

The following fact is an important property of transportation problems.

Lemma 1 *If the numbers* $\{s_i\}_{i \in S}$ *and* $\{d_j\}_{j \in D}$ *are integers, then* every *basic feasible solution* x^B *to the transportation problem is such that* x^B *is integer.*

3 The assignment problem

Consider the special of a transportation problem, where $s_i = 1$ for all $i \in S$ and $d_j = 1$ for all $j \in D$

$$\begin{aligned}
\text{Minimize } Z = \sum_{i \in S} \sum_{j \in D} c_{i,j} x_{i,j} & \\
\text{s.t.} & \\
\sum_{j \in D} x_{i,j} = 1, & \qquad \text{for all } i \in S \\
\sum_{i \in S} x_{i,j} = 1, & \qquad \text{for all } j \in D \\
x_{i,j} \geq 0. & \qquad \text{for all } i \in S, j \in D
\end{aligned}$$

In this case, the necessary assumption for the above linear program to be feasible is that $m = n$, or equivalently $D = S$. Also, the fact of Lemma 1 that every basic feasible solution x^B is integer simply states that $x^B_{i,j} \in \{0, 1\}$ for all feasible bases B and all $i \in S$ and $j \in D$. Such variables are called *decision variables*. Each decision variable $x_{i,j} \in \{0, 1\}$ decides whether or not the source $i \in S$ is assigned to the destination $j \in D$. Hence, this special case of the transportation problem is called the *assignment problem*. Another name for the assignment problem is the *perfect matching problem*, since every source is matched with one and only one destination, and the problem is to find the cheapest perfect matching.