Chapter 6, Operations Research (OR)

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1 The dual simplex algorithm

As in the last chapter, we consider a linear program (P) of the form

$$\begin{aligned} \text{Minimize } Z_P = c^T x \\ \text{s.t.} \\ Ax = b, \qquad \text{(P)} \\ x \geq 0_n. \end{aligned}$$

and its dual (D)

$$\begin{aligned} \text{Maximize } Z_D = b^T y \\ \text{s.t.} \qquad \text{(D)} \\ A^T y + s = c, \\ s \geq 0_n. \end{aligned}$$

where we have here chosen to add slack variables $s \geq 0_n$. The simplex method, as presented previously, works by starting with an initial basic feasible solution to (P) (which can be found via a phase 1 problem). However, what if a dual feasible basis is immediately available, whereas a basic feasible solution to (P) is harder to obtain? Can this fact be exploited in an algorithm? The *dual simplex method* starts with a basic feasible solution to (D) instead exploits such a situation.

Given a basis B, we showed in Chapter 5 that the problem (P) can be reformulated as the following linear program

$$\begin{aligned} \text{Minimize } Z = c^T x^B + \sum_{j \in N} r_j x_j \\ \text{s.t.} \\ x = x^B + \sum_{j \in N} d^j x_j, \qquad \text{(P')} \\ x \geq 0_n. \end{aligned}$$

If the basis B is *not* feasible for (P), then we have $x^B_j < 0$ for some $j \in B$. Define the vector $s^B \in \mathbb{R}^n$ as $s^B := c - A^T y^b$ for $j \in B$, where y^B denotes the dual basic solution corresponding to B (in other words, s^B is the slack vector in the dual constraints for the dual basic solution y^B). Observe that y^B is dual feasible exactly when $s^B \geq 0_n$, and that s^B_j for $j \in N$ is exactly the reduced cost r^B_j on the variable x_j.

We now dualize the problem (P'). By recalling the definition of the directions d^j for $j \in N$, the problem (P') can be written as follows.

$$\text{Minimize } Z_P = c^T x^B + \sum_{j \in N} r_j x_j$$

$$\text{s.t.}$$

$$-x_{j(i)} - \sum_{j \in N} \bar{a}_{i,j} x_j = -x^B_{j(i)}, \qquad \text{for } i \in \{1, 2, \dots, m\} \qquad (y_i)$$

$$x \geq 0_n. \qquad (s)$$

where the name of the dual variables have been written next to the constraints. The dual (D') of (P') can now be written as follows (recall that $c^T x^B$ is a constant for a given basis B).

$$\text{Maximize } Z_D = c^T x^B - \sum_{i=1}^{m} x^B_{j(i)} y_i$$

$$\text{s.t.}$$

$$y_i - s_{j(i)} = 0 \qquad \text{for } i \in \{1, 2, \dots, m\}$$

$$-\sum_{i=1}^{m} y_i \bar{a}_{i,j} + s_j = r_j, \qquad \text{for } j \in N \qquad \text{(D')}$$

$$s \geq 0_n.$$

The first set of equalities in (D') can be used to substitute the variables y_i for $i \in \{1, 2, \dots, m\}$ out of the problem (D'). Furthermore, as shown earlier, we have $c^T x^B = b^T y^B$, or in other words, the objective value $c^T x^B$ of the basic solution x^B in the problem (P) is the same as the objective value $b^T y^B$ of the dual objective value of the dual basic solution y^B for the problem (D). We can therefore rewrite the problem (D') as follows.

$$\text{Maximize } Z_D = b^T y^B - \sum_{j \in B} x^B_j s_j$$

$$\text{s.t.}$$

$$s_j = s^B_j + \sum_{i=1}^{m} s_{j(i)} \bar{a}_{i,j}, \text{ for } j \in N \qquad \text{(D')}$$

$$s \geq 0_n.$$

Now, similar to the construction of the problem (P'), we can enrich the problem (D') by adding redundant/obvious inequalities $s_j = s_j$ for $j \in B$.

$$\text{Maximize } Z_D = b^T y^B - \sum_{j \in B} x^B_j s_j$$

$$\text{s.t.}$$

$$s_j = s^B_j + \sum_{i=1}^{m} s_{j(i)} \bar{a}_{i,j}, \text{ for } j \in N \qquad \text{(D')}$$

$$s_j = s_j, \qquad \text{for } j \in B$$

$$s \geq 0_n.$$

We finally, as in the problem (P'), define some *dual directions* $\tilde{d}^i$ for $i \in \{1, 2, \dots, m\}$ as follows.

$$\tilde{d}^i_k := \begin{cases} 1, & \text{if } k \in B \text{ and } k = j(i), \\ 0, & \text{if } k \in B \text{ and } k \neq j(i), \\ \bar{a}_{i,k}, & \text{if } k \in N. \end{cases} \tag{1}$$

The dual problem (D') of (P') can now be written, compactly, as follows.

$$\begin{aligned} \text{Maximize } Z_D = b^T y^B - \sum_{j \in B} x^B_j s_j \\ \text{s.t.} \\ s = s^B + \sum_{i=1}^{m} \tilde{d}^i s_{j(i)}, \qquad \text{(D')} \\ s \geq 0_n. \end{aligned}$$

Note the similarity between the problem (D') and the problem (P'). The geometric interpretation of the problem (D') is also similar to the geometric interpretation of the problem (P'). When $s_j = 0$ for all $j \in B$ (or equivalently $s_{j(i)} = 0$ for all $i \in \{1, 2, \ldots, m\}$), the solution to (D') is simply $s = s^B$, which corresponds to the dual basic solution y^B. (The actual values of y^B can be obtained by solving the system $c_j - a^T_{\cdot j} y = s^B_j = 0$ for $i \in B$). The dual directions $\tilde{d}^i$ then indicate how the dual solution $s = s^B$ changes as the values s^B_j for $j \in B$ are increased from their current value of zero, and the term "$\sum_{j \in B} x^B_j s_j$" shows how moving away from the "current" point s^B changes the objective value. Observe that, if (y^B, s^B) is dual feasible, and $x^B_j \geq 0$ for all $j \in B$ (which simply means that x^B is feasible for (P)), then (y^B, s^B) is optimal for (D'). Also note that, in the dual (D') of (P'), the value $-x^B_k$ for $k = j(i) \in B$ acts as a "reduced cost" on the variable s_k for the problem (D'), or in other words, if $-x^B_k > 0$, then it pays off to move in the direction $\tilde{d}^i$.

The above discussion motivates an algorithm that starts with a basis B which is dual feasible instead of feasible for (P). Suppose B is dual feasible. This means that $s^B \geq 0_n$. If also x^B is feasible for (P), then we know that B is an optimal basis for both (P) and (D). Otherwise we can find a basic variable $\bar{j} \in B$ such that $x^B_{\bar{j}} < 0$ (since we have assumed that B is dual feasible). Furthermore, since B is dual feasible, we have that $s^B \geq 0_n$, and the solution $s = s^B$ is feasible for (D'). Let $\bar{i} \in \{1, 2, \ldots, m\}$ be such that $\bar{j} = j(\bar{i})$.

Since $x^B_{\bar{j}} < 0$, we know that it "pays off" to move from s^B in the dual direction $\tilde{d}^{\bar{i}}$. The question is how far we can move in this direction and still have a dual feasible solution. In other words, we wish to find the largest possible value δ^* of $\delta \geq 0$ such that

$$s^B + \delta \tilde{d}^{\bar{i}} \geq 0_n.$$

Recalling the definition of the dual direction $\tilde{d}^{\bar{i}}$, this means that we need to find the largest possible value δ^* of $\delta \geq 0$ such that

$$s^B_j + \delta \bar{a}_{\bar{i},j} \geq 0 \text{ for } j \in N.$$

For those $j \in N$ for which $\bar{a}_{\bar{i},j} \geq 0$, the above inequalities do not put any restriction on δ. Hence, the largest possible value δ^* of δ is given by

$$\delta^* = \min\{-\frac{s^B_j}{\bar{a}_{\bar{i},j}} : j \in N \text{ and } \bar{a}_{\bar{i},j} < 0\}$$

Recall that $s^B_j = r^B_j \geq 0$ for $j \in N$, where $r^B_j \geq 0$ denotes the reduced cost of the variable x_j in the basis B. Also, let $\bar{k} \in N$ be a non-basic variable that satisfies $\delta^* = -\frac{s^B_{\bar{k}}}{\bar{a}_{\bar{i},\bar{k}}}$. We have shown that the basis $B' = (B \setminus \{\bar{j}\}) \cup \{\bar{k}\}$ is a better basis for (D) than B, and we therefore let $x_{\bar{j}}$ exit the basis, and $x_{\bar{k}}$ enters the basis.

The dual simplex method can now be summarized as follows.

Step 1: Let B be an initial feasible basis for (D). (This can be done by applying the simplex method on a two-phase problem corresponding to (D), where an initial basis is known).

Step 2: If x^B is feasible for (P) and y^B is feasible for the dual (D) - STOP. B is an optimal basis.

Step 3: Let $\bar{j} \in N$ be a basic variable that satisfies $x^B_{\bar{j}} < 0$, or in other words the value of $x_{\bar{j}}$ is negative in the basic solution corresponding to B. Also, let $\bar{i} \in \{1, 2, \ldots, m\}$ be such that $\bar{j} = j(\bar{i})$, and let $\widetilde{d^{\bar{i}}}$ be the corresponding dual direction. If $s^B + \delta\widetilde{d^{\bar{i}}}$ is feasible for (D) for all $\delta \geq 0$ - STOP. The problem (D) is unbounded. Otherwise compute the value

$$\delta^* = \min\{-\frac{s^B_j}{\bar{a}_{\bar{i},j}} : j \in N \text{ and } \bar{a}_{\bar{i},j} < 0\}$$

Let $\bar{k} \in N$ be such that $\delta^* = -\frac{s^B_{\bar{k}}}{\bar{a}_{\bar{i},\bar{k}}}$. Update the basis B to $B := (B \setminus \{\bar{j}\}) \cup \{\bar{k}\}$ and return to Step 2.

Example 1 *Consider, again, the linear program (P) that models the WGC problem*

$$\begin{aligned} &\textit{Minimize } Z = -3x_1 - 5x_2 \\ &\textit{subject to} \\ &x_1 + x_3 = 4 && (y_1) \\ &2x_2 + x_4 = 12 && (y_2) \\ &3x_1 + 2x_2 + x_5 = 18 && (y_3) \\ &x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0, x_5 \geq 0 \end{aligned}$$

and consider the dual linear program (D)

$$\begin{aligned} &\textit{Maximize } Z' = 4y_1 + 12y_2 + 18y_3 \\ &\textit{subject to} \\ &y_1 + 3y_3 + s_1 = -3 && (x_1) \\ &2y_2 + 2y_3 + s_2 = -5 && (x_2) \\ &y_1 + s_3 = 0, && (x_3) \\ &y_2 + s_4 = 0, && (x_4) \\ &y_3 + s_5 = 0. && (x_5) \\ &s_1 \geq 0, s_2 \geq 0, s_3 \geq 0, s_4 \geq 0, s_5 \geq 0 \end{aligned}$$

where we have added the slack variables s_1, s_2, s_3, s_4, s_5. Consider the basis $B = \{2, 3, 4\}$. We have $x^B = (0, 9, 4, -6, 0)$, $y^B = (0, 0, -2\frac{1}{2})$, and therefore $s^B = (4\frac{1}{2}, 0, 0, 0, 2\frac{1}{2})$. Since $s^B \geq 0_5$, this basic solution is dual feasible. However, since $x^B_4 = -6 < 0$, B is not a feasible basis for (P).

We next construct the problem (D') for this basis. The basis matrix A_B and its inverse are given by

$$A_B = \begin{pmatrix} 0 & 1 & 0 \\ 2 & 0 & 1 \\ 2 & 0 & 0 \end{pmatrix} \textit{ and } A_B^{-1} = \begin{pmatrix} 0 & 0 & \frac{1}{2} \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{pmatrix}.$$

Since $A_B^{-1}a_{.2} = e_1$, $A_B^{-1}a_{.3} = e_2$ *and* $A_B^{-1}a_{.4} = e_3$, *we have* $j(1) = 2$, $j(2) = 3$ *and* $j(3) = 4$. *Furthermore, we have* $b^T y^B = -45$ *(hence this is a super-optimal solution). Finally, to obtain the problem (D'), we need the vectors* $\bar{a}_{.1}$ *and* $\bar{a}_{.5}$

$$\bar{a}_{.1} = A_B^{-1}a_{.1} = \begin{pmatrix} \frac{3}{2} \\ 1 \\ -3 \end{pmatrix} \text{ and } \bar{a}_{.5} = A_B^{-1}a_{.5} = \begin{pmatrix} \frac{1}{2} \\ 0 \\ -1 \end{pmatrix}.$$

We now have all the information for finding the vectors $\tilde{d}^1$, $\tilde{d}^1$ *and* $\tilde{d}^1$

$$\tilde{d}^1 = \begin{pmatrix} \frac{3}{2} \\ 1 \\ 0 \\ 0 \\ \frac{1}{2} \end{pmatrix}, \tilde{d}^2 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \text{ and } \tilde{d}^3 = \begin{pmatrix} -3 \\ 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}.$$

The problem (D') associated with the basis $B = \{2, 3, 4\}$ *can now be formulated as follows.*

$$\text{Maximize } Z' = -45 - 9s_2 - 4s_3 + 6s_4,$$

$$\text{subject to}$$

$$\begin{pmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \\ s_5 \end{pmatrix} = \begin{pmatrix} 4\frac{1}{2} \\ 0 \\ 0 \\ 0 \\ 2\frac{1}{2} \end{pmatrix} + \begin{pmatrix} \frac{3}{2} \\ 1 \\ 0 \\ 0 \\ \frac{1}{2} \end{pmatrix} s_2 + \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} s_3 + \begin{pmatrix} -3 \\ 0 \\ 0 \\ 1 \\ -1 \end{pmatrix} s_4.$$

$$s \geq 0_5.$$

Since the objective coefficient on s_4 *is positive, the basis* B *is not optimal, and the variable* x_4 *leaves the basis. To figure out which variable enters the basis, we need to find the largest possible value* δ^* *of* δ *that satisfies*

$$s^B + \delta\tilde{d}^3 \geq 0_5.$$

We obtain that $\delta^* = 1\frac{1}{2}$, *and that* s_1 *is the variable that becomes zero first when increasing the value of* s_4 *from its current value of zero. Hence,* x_1 *enters the basis, and the new basis becomes* $B' = \{1, 2, 3\}$. *The value of* Z' *increases from* -45 *to* $-45 + 9 \times 1\frac{1}{2} = -36$, *which we know is optimal. Furthermore, we have* $s^{B'} = s^B + 1\frac{1}{2}\tilde{d}^3$. *Finally, given* $s^{B'}$, *it is possible to solve for* $x^{B'}$ *and* $y^{B'}$, *and verify that we know have* $x^{B'} \geq 0$, *which means that* $x^{B'}$ *and* $y^{B'}$ *are optimal for (P) and (D) respectively.*