Chapter 5, Operations Research (OR)

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1 Geometry of the simplex tableau

In the last chapter, we derived the simplex tableau associated with a basis of a linear program (P)

$$
\begin{aligned}
\text{Minimize } Z &= c^T x \\
\text{s.t.} & \\
Ax &= b, \qquad \text{(P)} \\
x &\geq 0_n.
\end{aligned}
$$

Throughout this section, we consider a fixed basis $B \subseteq \{1, 2, \ldots, n\}$ for (P), and the corresponding set $N := \{1, 2, \ldots, n\} \setminus B$ of non-basic variables. The simplex tableau from B is given by the equalities

$$Z = c^T x^B + \sum_{j \in N} r_j x_j \tag{1}$$

$$x^B_{j(i)} = x_{j(i)} + \sum_{j \in N} \bar{a}_{i,j} x_j \quad \text{for } i = 1, 2, \ldots, m, \tag{2}$$

where the quantities involved in the above equalities are given as follows. Recall that A_B denotes the (column induced) submatrix of A generated by B.

(i) The basic solution x^B for the problem (P).

(ii) The reduced costs $r^B_j := c_j - (a_{.j})^T y^b$, where y^B is the dual basic solution, and $a_{.j}$ denotes the j^{th} column of A.

(iii) The one-to-one mapping $j() : \{1, 2, \ldots, m\} \to B$. Given $i \in \{1, 2, \ldots, m\}$, $j(i)$ denotes the basic variable $j(i) \in B$ such that $(A_B)^{-1} a_{.j(i)}$ is the i^{th} unit vector.

(iv) For every non-basic variable $j \in N$, the m-dimensional vector $\bar{a}_{.j} := (A_B)^{-1} a_{.j}$.

Observe that the mapping $j()$ essentially answers the question

Which basic variable is involved in the i^{th} row of the simplex tableau?

The answer to this question is, of course, the variable $x_{j(i)}$. We now give the geometric intuition behind the simplex tableau. Suppose we add the following redundant/obvious equalities to the system (1)-(2).

$$0 = x_j - x_j \quad \text{for } j \in N. \tag{3}$$

The equality system (1)-(3) has one equality for every variable of (P), and (1)-(3) can therefore be written, in compact form, as

$$x = x^B + \sum_{j \in N} d^j x_j, \tag{4}$$

where the n-dimensional vectors $d^j \in \mathbb{R}^n$ for $j \in N$ are defined as follows.

$$d_k^j := \begin{cases} 1, & \text{if } k = j, \\ 0, & \text{if } k \in N \setminus \{j\}, \\ -\bar{a}_{i,j}, & \text{if } k \in B \text{ and } i \in \{1, 2, \ldots, m\} \text{ is such that } j(i) = k. \end{cases} \tag{5}$$

Consider an arbitrary basic variable $k \in B$, and let $i \in \{1, 2, \ldots, m\}$ be such that $k = j(i)$. The k^{th} equality of system (4) reads

$$x_k = x_k^B + \sum_{j \in N} d_k^j x_j,$$

or equivalently

$$x_{j(i)} = x_{j(i)}^B - \sum_{j \in N} \bar{a}_{i,j} x_j,$$

which is exactly the i^{th} row of the simplex tableau. Similarily, if $j \in N$ is a non-basic variable, then the j^{th} equality of (4) reduces to (3). The equalities (4) can be used to reformulate the problem (P) wrt. the basis B as the following linear program (P').

$$\begin{aligned} &\text{Minimize } Z = c^T x^B + \sum_{j \in N} r_j x_j \\ &\text{s.t.} \\ &\quad x = x^B + \sum_{j \in N} d^j x_j, \qquad \text{(P')} \\ &\quad x \geq 0_n. \end{aligned}$$

The problem (P') is exactly the same as the problem (P) in terms of feasible solutions and optimal objective value (in other words, (P') is just a reformulation of (P)).

So what is the geometric interpretation of the this linear program? Firstly, if B is a feasible basis, then a feasible solution can be obtained for (P') by simply setting all nonbasic variables $j \in N$ to the value zero giving $x = x^B$. The vectors d^j are potential *directions* along which we can move away from the "current" point x^B. The reduced reduced cost r_j on the direction d^j gives the potential reduction in the cost of moving in the direction d^j. The problem (P') is therefore the problem of finding a point of the form $y = \sum_{j \in N} d^j x'_j$ in the cone spanned by the directions $\{d^j\}_{j \in N}$, such that $y + x^B \geq 0_n$ and the weighted sum of the combinations used $\sum_{j \in N} r_j x'_j$ is as small as possible.

A pivot, or a change of basis, can now be interpreted geometrically as follows. Suppose B is a feasible basis, or in other words that $x^B \geq 0$. If $r^j \geq 0$ for every nonbasic variable $j \in N$, then we know that x^B is an optimal solution for (P), so suppose we can find a non-basic variable $\bar{j} \in N$ such that $r_{\bar{j}} < 0$. We now know that the cost can be reduced by moving along the direction $d^{\bar{j}}$. The question is, how far can we move along the direction $d^{\bar{j}}$ and still have a feasible solution to (P), or in other words, how large can we make $\delta \geq 0$, such that

$$x^B + \delta d^{\bar{j}} \geq 0_n.$$

Now, using the definition of the direction $d^{\bar{j}}$, the above restrictions on δ reduce to

$$x^B_{j(i)} - \bar{a}_{i,\bar{j}}\delta \geq 0 \text{ for } i = 1, 2, \ldots, m.$$

For those $i \in \{1, 2, \ldots, m\}$ for which $\bar{a}_{i,\bar{j}} \leq 0$, the above inequalities do not put any restriction on δ. Hence, the largest possible value δ^* of $\delta \geq 0$ that satisfies all of the above inequalities is given by

$$\delta^* = \min\{\frac{x^B_{j(i)}}{\bar{a}_{i,\bar{j}}} : i \in \{1, 2, \ldots, m\} \text{ and } \bar{a}_{i,\bar{j}} > 0\}.$$

In computing the number δ^*, two things might happen.

(i) $\bar{a}_{i,\bar{j}} \leq 0$ for all $i \in \{1, 2, \ldots, m\}$. This means all points on the line from x^B in the direction $d^{\bar{j}}$ are feasible for (P). In this case, we have shown that (P) is unbounded, since $r_{\bar{j}} < 0$.

(ii) $\delta^* = 0$, or in other words, $x^B_{j(\bar{i})} = 0$ for some $\bar{i} \in \{1, 2, \ldots, m\}$ satisfying $\bar{a}_{i,\bar{j}} > 0$. In that case, we can perform a pivot, but the basic solution does not change. We call such a pivot a *degenerate* pivot.

One might wonder whether it is always possible to detect that (P) is unbounded in the way described in (i). The following lemma shows this is indeed possible.

Lemma 1 *Suppose (P) is unbounded. Then there exists a feasible basis B, and a non-basic variable $j \in \{1, 2, \ldots, n\} \setminus B$ such that $r^B_j < 0$ and $\bar{a}^B_{i,j} \leq 0$ for all $i \in \{1, 2, \ldots, m\}$.*

Example 1 *Consider, again, the linear program that models the WGC problem, which we here call (P)*

$$\begin{aligned} \textit{Minimize } Z &= -3x_1 - 5x_2 \\ \textit{subject to} & \\ x_1 + x_3 &= 4 \\ 2x_2 + x_4 &= 12 \\ 3x_1 + 2x_2 - x_5 &= 18 \\ x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 &\geq 0, x_5 \geq 0 \end{aligned}$$

and recall that the set $B = \{1, 2, 4\}$ constitutes a feasible basis with the corresponding basic solution given by $x^B = (4, 3, 0, 6, 0)$. The simplex tableau is given by

$$\begin{aligned} Z &= -27 - 4\frac{1}{2}x_3 + 2\frac{1}{2}x_5, \\ 4 &= x_1 + x_3, \\ 3 &= x_2 - \frac{3}{2}x_3 + \frac{1}{2}x_5, \\ 6 &= x_4 + 3x_3 - x_5, \end{aligned}$$

or equivalently, by adding the redundant equalities $0 = x_3 - x_3$ and $0 = x_5 - x_5$

$$Z = -27 - 4\frac{1}{2}x_3 + 2\frac{1}{2}x_5,$$
$$\begin{pmatrix} 4 \\ 3 \\ 0 \\ 6 \\ 0 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} + \begin{pmatrix} 1 \\ -\frac{3}{2} \\ -1 \\ 3 \\ 0 \end{pmatrix} x_3 + \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \\ -1 \\ -1 \end{pmatrix} x_5,$$

and finally by reordering we get

$$Z = -27 - 4\frac{1}{2}x_3 + 2\frac{1}{2}x_5,$$
$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \\ 0 \\ 6 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 \\ \frac{3}{2} \\ 1 \\ -3 \\ 0 \end{pmatrix} x_3 + \begin{pmatrix} 0 \\ -\frac{1}{2} \\ 0 \\ 1 \\ 1 \end{pmatrix} x_5.$$

We therefore have $d^3 = (-1, \frac{3}{2}, -1, -3, 0)^T$ *and* $d^5 = (0, -\frac{1}{2}, 0, 1, -1)$, $r_3 = -4\frac{1}{2}^T$ *and* $r_5 = \frac{1}{2}$. *Since* $r_3 < 0$, *we might be able to get a better solution than* x^B *by increasing the value of* x_3 *from its current value of zero. The maximum value of* δ *for which* $x^B + \delta d^3$ *is feasible for (P) is then given by*

$$\delta^* = \min\{\frac{x_1^B}{\bar{a}_{1,3}}, \frac{x_4^B}{\bar{a}_{3,3}}\} = \min\{\frac{4}{1}, \frac{6}{3}\} = 2.$$

We therefore have $\delta^* = 2$, *and* x_4 *is a variable which becomes zero first when increasing the value of* x_3. *Hence the new basis is* $B' = \{1, 2, 3\}$. *The new basic solution is* $x^B + \delta^* d^3$, *or* $x^{B'} = (2, 6, 2, 0, 0)^T$

2 The simplex method revisited

We have now analyzed each step of the simplex method in detail. We therefore now give a more detail version of the simplex method.

Step 1: Find an initial feasible basis for (P). This can be done by applying the simplex method on the two-phase problem corresponding to (P), where an initial basis is known. If not all the artificial variables have been eliminated in phase 1 - STOP. The problem is infeasible. Otherwise let B be a feasible basis for (P).

Step 2: If x^B is feasible for (P) and y^B is feasible for the dual (D) - STOP. B is an optimal basis.

Step 3: Let $\bar{j} \in N$ be a non-basic variable with negative reduced cost, or in other words $r_{\bar{j}} < 0$. Also, let $d^{\bar{j}}$ be the corresponing direction. If $x^B + \delta d^{\bar{j}}$ is feasible for all $\delta \geq 0$ - STOP. The problem (P) is unbounded. Otherwise compute the value

$$\delta^* = \min\{\frac{x^B_{j(i)}}{\bar{a}_{i,\bar{j}}} : i \in \{1, 2, \ldots, m\} \text{ and } \bar{a}_{i,\bar{j}} > 0\}.$$

Let $\bar{i} \in \{1, 2, \ldots, m\}$ be such that $\delta^* = \frac{x^B_{j(\bar{i})}}{\bar{a}_{\bar{i},\bar{j}}}$. Update the basis B to $B := (B \setminus \{j(\bar{i})\}) \cup \{\bar{j}\}$ and return to Step 2.

3 Pivoting by using matrix operations

We now discuss how to do the pivots from basis to basis most efficiently. This is best done "by example", so consider again the WGC problem for the feasible basis $B = \{1, 2, 4\}$. Recall that the simplex tableau is given by

$$
\begin{aligned}
-27 &= Z + 4\frac{1}{2}x_3 - 2\frac{1}{2}x_5, \\
4 &= x_1 + x_3, \\
3 &= x_2 - \frac{3}{2}x_3 + \frac{1}{2}x_5, \\
6 &= x_4 + 3x_3 - x_5,
\end{aligned}
$$

The information in the above simplex tableau can be represented compactly in a table as follows.

Basic variable	Z	x_1	x_2	x_3	x_4	x_5	Right hand side
Z	1	0	0	$4\frac{1}{2}$	0	$-2\frac{1}{2}$	-27
x_1	0	1	0	1	0	0	4
x_2	0	0	1	$-\frac{3}{2}$	0	$\frac{1}{2}$	3
x_4	0	0	0	3	1	-1	6

Table 1:

A pivot can be performed directly on the above table as follows. First we select the entering basic variable. The reduced cost $r_3 = -4\frac{1}{2} < 0$ of x_3 can be read directly from the above table. In our case, there is only one variable with a negative reduced cost, but in general, there might be several, so this is a choice. (An often used rule is to select the variable with the most negative reduced cost because of the geometric interpretation).

Next, to find the leaving basic variable, observe that we only need to consider the column corresponding to x_3 and the "right hand side" column. The leaving basic variable is found from the formula

$$\delta^* = \min\{\frac{x^B_{j(i)}}{\bar{a}_{i,j}} : i \in \{1, 2, \dots, m\} \text{ and } \bar{a}_{i,j} > 0\},$$

where we here have $j = 3$ and $i = 1, 2, 3$. The numbers $\bar{a}_{i,3}$ for $i = 1, 2, 3$ are all given in the column corresponding to x_3 in Table 1. Similarily, the number $x^B_{j(1)}$, $x^B_{j(2)}$ and $x^B_{j(3)}$ are given in the column headed "right hand sides" in Table 1. The ratios between the numbers in these two columns are then computed as indicated in the following table, and the element with the minimum ratio is chosen.

Basic variable	Z	x_1	x_2	x_3	x_4	x_5	Right hand side	
Z	1	0	0	$4\frac{1}{2}$	0	$-2\frac{1}{2}$	-27	
x_1	0	1	0	1	0	0	4	$\frac{4}{1} = 4$
x_2	0	0	1	$-\frac{3}{2}$	0	$\frac{1}{2}$	3	entry ≤ 0
x_4	0	0	0	3	1	-1	6	$\frac{6}{3} = 2 \leftarrow$ minimum

Table 2:

It follows from this computation that the number "3" in the column correponding to x_3, and the row corresponding to x_4, is the one that given the minimum. This element is called the *pivot element*. As argued earlier, this means that x_4 leaves the basis and x_3 enters the basis resulting in the basis $B' = \{1, 2, 3\}$.

We next construct the simplex tableau corresponding to the basis B' by doing Gaussian elim-iniation on Table 2. Observe that the columns of Table 2 corresponding to basic variables are unit vectors. We will make the column corresponding to x_3 a unit vector by using Gaussian elimination. First, we multiply the last row of Table 2 with the number $\frac{1}{3}$, and this gives Table 3.

Basic variable	Z	x_1	x_2	x_3	x_4	x_5	Right hand side
Z	1	0	0	$4\frac{1}{2}$	0	$-2\frac{1}{2}$	-27
x_1	0	1	0	1	0	0	4
x_2	0	0	1	$-\frac{3}{2}$	0	$\frac{1}{2}$	3
x_4	0	0	0	1	$\frac{1}{3}$	$-\frac{1}{3}$	2

Table 3:

Next, we multiply the last row of Table 3 with the number $\frac{3}{2}$ and add the result to the row of Table 3 correponding to x_2. This gives Table 4.

Basic variable	Z	x_1	x_2	x_3	x_4	x_5	Right hand side
Z	1	0	0	$4\frac{1}{2}$	0	$-2\frac{1}{2}$	-27
x_1	0	1	0	1	0	0	4
x_2	0	0	1	0	$\frac{1}{2}$	0	6
x_4	0	0	0	1	$\frac{1}{3}$	$-\frac{1}{3}$	2

Table 4:

The next step is to multiply the last row Table 4 with (-1) and add the result to the row of Table 4 corresponding to x_1. This gives Table 5.

Basic variable	Z	x_1	x_2	x_3	x_4	x_5	Right hand side
Z	1	0	0	$4\frac{1}{2}$	0	$-2\frac{1}{2}$	-27
x_1	0	1	0	0	$-\frac{1}{3}$	$\frac{1}{3}$	2
x_2	0	0	1	0	$\frac{1}{2}$	0	6
x_4	0	0	0	1	$\frac{1}{3}$	$-\frac{1}{3}$	2

Table 5:

Finally, we multiply the last row of Table 5 with $-4\frac{1}{2}$ and add the result to the row of Table 5 corresponding to Z. This gives Table 6, where the basic variable in the last row is now x_3.

Now, in Table 6, the column corresponding to x_3 is a unit vector. In fact, the simplex tableau corresponding to the basis $B' = \{1, 2, 3\}$ can be read directly from Table 6 as follows.

$$\begin{aligned}
-36 &= Z - \frac{3}{2}x_4 - x_5, \\
2 &= x_1 - \frac{1}{3}x_4 + \frac{1}{3}x_5, \\
6 &= x_2 + \frac{1}{2}x_4, \\
2 &= x_3 + \frac{1}{3}x_4 - \frac{1}{3}x_5,
\end{aligned}$$

From Table 6 we can further read that, in the basis B', the reduced cost on x_4 is $r_4 = \frac{3}{2}$, and the reduced cost on x_5 is $r_5 = 1$. Further, we can see that $x_1^{B'} = 2$, $x_2^{B'} = 6$ and $x_3^{B'} = 2$. Since x_4 and x_5 are nonbasic in the basis B', we have $x_4^{B'} = 0$ and $x_5^{B'} = 0$. Since this basic solution is feasible, and all reduced costs are non-negative, the basis B' is optimal.

Basic variable	Z	x_1	x_2	x_3	x_4	x_5	Right hand side
Z	1	0	0	0	$-\frac{3}{2}$	-1	-36
x_1	0	1	0	0	$-\frac{1}{3}$	$\frac{1}{3}$	2
x_2	0	0	1	0	$\frac{1}{2}$	0	6
x_3	0	0	0	1	$\frac{1}{3}$	$-\frac{1}{3}$	2

Table 6:

4 Sensitivity analysis

The solution of a linear program assumes that the data (A, b and c) is known with certainty. However, in practice, data is uncertain, and often the parameters A, b and c are the result of estimation, which means that their values are often revised after solving the problem from the given (initial) data. We now examine how an optimal solution to the problem (P) changes when the parameters b and c change (a change in the values in the matrix A can be analyzed in a similar fashion).

Let B denote an optimal basis for (P), and let (x^B, y^B) denote the corresponding optimal solutions to (P) and (D) respectively. Also, let r_j^B for $j \in N := \{1, 2, \ldots, n\} \setminus B$ denote the reduced costs of the nonbasic variables.

First suppose we modify the i^{th} component of the right hand side

$$b' = b + \sigma e_i, \tag{6}$$

where e_i denotes the i^{th} unit vector in $\mathbb{R}^m$. Note that, if B remains an optimal basis, then, since the right hand side does not appear in the constraints of the dual (D) of (P), the dual solution corresponding to B does not change. Hence the dual objective, when B remains an optimal basis, is given by

$$(y^B)^T b' = (y^B)^T b + \sigma y_i^B.$$

Hence the value of the dual variable y_i^B can be interpreted as the marginal value of having one extra unit of the the resource b_i. Since y^B remains feasible for (D) when we change b, the basis B remains optimal if the new basic solution remains feasible for (P). Let $\bar{b}^i := (A_B)^{-1} e_i$ denote the i^{th} The basis B remains optimal when we change b to b' if

$$(A_B)^{-1} b' \geq 0_m \iff x^B + \sigma \bar{b}^i \geq 0_m.$$

Similarily, if we modify the cost function c to

$$c' = c + \sigma e_j, \tag{7}$$

where e_j now denotes the j^{th} unit vector in $\mathbb{R}^n$. If B remains an optimal basis, then, since c does not appear in the constraints of (P), the solution x^B to (P) remains the same. The change in the objective value of (P), when B remains an optimal basis, is therefore given by

$$(x^B)^T c' = (x^B)^T c + \sigma x_j^B.$$

In particular, if $\sigma > 0$ and $j \in N$ is a nonbasic variable, then the change in c_j only increases the value of the reduced cost r_j associated with x_j, and therefore the basis remains optimal. Furthermore, if $\sigma < 0$ and $j \in N$ is nonbasic, the basis B remains optimal as long as $-\sigma \in [0, r_j]$.

Now, if $j \in B$ is a basic variable, and c is changed to c', the dual solution is changed. Let $i \in \{1, 2, \ldots, m\}$ be such that $j = j(i)$. The new dual basic solution is the unique solution y' to the system

$$
\begin{aligned}
(a_{.k})^T y &= c_k \text{ for } k \in B \setminus \{j\} \\
(a_{.j})^T y &= c_j + \sigma
\end{aligned}
$$

which is given by $y' = y^B + \sigma \tilde{b}^i$, where $\tilde{b}^i$ denotes the i^{th} row of the matrix A_B^{-1}. Hence, the basis B remains optimal, if

$$(a_{.k})^T y' \leq c_k, \text{ for all } k \in N.$$