

# Chapter 2, Operations Research (OR)

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## 1 Linear Programs

In the last chapter, we formulated a mathematical model for allocating resources to activities in such a way that the total profit is maximized. Specifically, this mathematical model was the problem of choosing levels  $x_1, x_2, \dots, x_n$  for  $n$  activities so as to

$$\begin{aligned} & \text{Maximize } Z = c_1x_1 + c_2x_2 + \dots + c_nx_n \\ & \text{subject to the restrictions that} \\ & \quad a_{i,1}x_1 + a_{i,2}x_2 + \dots + a_{i,n}x_n \leq b_i \quad \text{for } i = 1, 2, \dots, m \\ & \quad x_1 \geq 0, x_2 \geq 0, \dots, x_n \geq 0 \quad \text{(Non-negativity)} \end{aligned}$$

In the above model,  $x_1, x_2, \dots, x_n$  are the *variables*,  $Z = c_1x_1 + c_2x_2 + \dots + c_nx_n$  denotes the total profit, and the inequality  $a_{i,1}x_1 + a_{i,2}x_2 + \dots + a_{i,n}x_n \leq b_i$  states that the  $i^{\text{th}}$  resource can not be over used.

Any mathematical model, which can be written in the above for some parameters  $c_j$ ,  $b_i$  and  $a_{i,j}$  (where  $i \in \{1, 2, \dots, m\}$  and  $j \in \{1, 2, \dots, n\}$ ) is called a *Linear Program* (often abbreviated LP). Observe that the WGC problem is a linear program with  $m = 3$  (three resources) and  $n = 2$  (two activities). In the following, we will talk about linear programs as a *mathematical object*. In other words, we drop the analogy with any specific application.

In general, the numbers  $c_j$  for  $j = 1, 2, \dots, n$  are called *objective function coefficients*, and the number  $b_i$  for  $i = 1, 2, \dots, m$  are called *right hand sides*. Also, the restrictions on the variables are commonly called *constraints*, and in particular, constraints of the form  $x_j \geq 0$  are called *non-negativity constraints*.

### 1.1 Other forms of linear programs

In the above definition of a linear program, all constraints were in the form of “less than or equal” inequalities, all variables were non-negative, and the objective was to maximize. The essential feature of a linear program is *linearity*, and not a particular form. We now discuss other forms, and show how it is possible to transform one form into another form.

(1) Minimization:

If the objective is to minimize, the problem can be turned into a maximization problem with the substitution  $Z' = -c_1x_1 - c_2x_2 - \dots - c_nx_n$ . The same trick can be used to transform a maximization form into a minimization form.

(2) Greater-than-equal-to-constraints:

A constraint of the form  $a_{i,1}x_1 + a_{i,2}x_2 + \dots + a_{i,n}x_n \geq b_i$  can be made a less-than-or-equal-to inequality by multiplying with  $-1$  giving  $a_{i,1}x_1 - a_{i,2}x_2 - \dots - a_{i,n}x_n \leq -b_i$ .

(3) Equalities:

If a model contains an equality constraint  $a_{i,1}x_1 + a_{i,2}x_2 + \dots + a_{i,n}x_n = b_i$ , the model can be transformed into “inequality form” by replacing the equality with *two* inequalities  $a_{i,1}x_1 + a_{i,2}x_2 + \dots + a_{i,n}x_n \leq b_i$  and  $a_{i,1}x_1 + a_{i,2}x_2 + \dots + a_{i,n}x_n \geq b_i$ . Conversely, a less-than-or-equal-to inequality  $a_{i,1}x_1 + a_{i,2}x_2 + \dots + a_{i,n}x_n \leq b_i$  can be replaced by the system:

$$\begin{aligned} a_{i,1}x_1 + a_{i,2}x_2 + \dots + a_{i,n}x_n + s_i &= b_i, \\ s_i &\geq 0. \end{aligned}$$

The extra variable  $s_i$  is called a *slack* variable. Similarly, a greater-than-or-equal-to inequality  $a_{i,1}x_1 + a_{i,2}x_2 + \dots + a_{i,n}x_n \geq b_i$  can be replaced by the system:

$$\begin{aligned} a_{i,1}x_1 + a_{i,2}x_2 + \dots + a_{i,n}x_n - s_i &= b_i, \\ s_i &\geq 0. \end{aligned}$$

When obtained from a greater-than-or-equal-to inequality, the extra variable  $s_i$  is called a *surplus* variable. Observe that, to go from inequality forms to equality forms, it is necessary to introduce extra variables.

(4) Free variables:

If, for some variable  $x_j$ , the non-negative constraint  $x_j \geq 0$  does not appear in the model, then  $x_j$  is called a *free* variable. Free variables appear naturally in models, where some variables model changes in certain quantities. We can replace  $x_j$  with two non-negative variables  $x_j^+$  and  $x_j^-$ , where  $x_j^+ \geq 0$ ,  $x_j^- \geq 0$  and  $x_j = x_j^+ - x_j^-$ . Note that  $x_j^+$  denotes the positive part of  $x_j$ , and that  $x_j^-$  denotes the negative part.

**Example 1** Consider the linear program that models the problem of the WGC

$$\begin{aligned} \text{Maximize } Z &= 3x_1 + 5x_2 \\ \text{subject to} \\ x_1 &\leq 4 \\ 2x_2 &\leq 12 \\ 3x_1 + 2x_2 &\leq 18 \\ x_1 \geq 0, x_2 &\geq 0 \end{aligned}$$

We can reformulate the above linear program by introducing slack variables  $s_1$ ,  $s_2$  and  $s_3$  in the above constraints

$$\begin{aligned} \text{Maximize } Z &= 3x_1 + 5x_2 \\ \text{subject to} \\ x_1 + s_1 &= 4 \\ 2x_2 + s_2 &= 12 \\ 3x_1 + 2x_2 + s_3 &= 18 \\ s_1 \geq 0, s_2 \geq 0, s_3 \geq 0, x_1 \geq 0, x_2 &\geq 0 \end{aligned}$$

It is also possible to transform this formulation into a minimization form by instead minimizing  $Z' = -3x_1 - 5x_2$ . The WGC example does not contain free variables.

## 1.2 Matrix form of a linear program

Consider again the linear program we formulated earlier

$$\begin{aligned} & \text{Maximize } Z = c_1x_1 + c_2x_2 + \dots + c_nx_n \\ & \text{subject to} \\ & a_{i,1}x_1 + a_{i,2}x_2 + \dots + a_{i,n}x_n \leq b_i \qquad \text{for } i = 1, 2, \dots, m \\ & x_1 \geq 0, x_2 \geq 0, \dots, x_n \geq 0 \qquad \text{(Non-negativity)} \end{aligned}$$

We can write this linear program in terms of matrices and vectors. Let  $c := (c_1, c_2, \dots, c_n)^T$  be an  $n$ -dimensional (column) vector that contains the objective function coefficients (the symbol “T” is for transpose), and let  $b := (b_1, b_2, \dots, b_m)^T$  be an  $m$ -dimensional (column) vector containing the right hand sides. We collect the multipliers  $a_{i,j}$  on the variables in the constraints into an  $m \times n$  matrix  $A$ . In other words, the coefficient  $a_{i,j}$  on the variable  $x_j$  in the  $i^{\text{th}}$  constraint is placed in the  $i^{\text{th}}$  row and  $j^{\text{th}}$  column of  $A$ . Finally, we collect the variables in an  $n$ -dimensional (column) vector  $x = (x_1, x_2, \dots, x_n)^T$ . The above problem can be formulated as follows.

$$\begin{aligned} & \text{Maximize } Z = c^T x \\ & \text{s.t.} \\ & Ax \leq b \\ & x \geq 0_n, \end{aligned}$$

where  $0_n$  denotes the  $n$ -dimensional vector of all zeros.

**Example 2** Consider, again, the linear program that models the problem of the WGC

$$\begin{aligned} & \text{Maximize } Z = 3x_1 + 5x_2 \\ & \text{subject to} \\ & x_1 \leq 4 \\ & 2x_2 \leq 12 \\ & 3x_1 + 2x_2 \leq 18 \\ & x_1 \geq 0, x_2 \geq 0 \end{aligned}$$

*This problem can be written as*

$$\begin{aligned} & \text{Maximize } Z = c^T x \\ & \text{s.t.} \\ & Ax \leq b, \\ & x \geq 0_n, \end{aligned}$$

with  $n = 2$ ,  $m = 3$  and

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \\ 3 & 2 \end{pmatrix}, \quad b = \begin{pmatrix} 4 \\ 12 \\ 18 \end{pmatrix}, \quad c = \begin{pmatrix} 3 \\ 5 \end{pmatrix} \quad \text{and } x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

## 2 Theory of linear programs

In this section, we consider a linear program in equality form, which we call (P)

$$\begin{aligned} &\text{Minimize } Z = c^T x \\ &\text{s.t.} \\ &Ax = b, \quad (\text{P}) \\ &x \geq 0_n, \end{aligned}$$

where  $A$  is an  $m \times n$  matrix,  $b$  is an  $m$ -dimensional vector of right hand sides,  $c$  is an  $n$  dimensional vector of objective function coefficients and  $x$  is an  $n$ -dimensional vector of variables. Hence, the problem (P) has  $n$  variables and, not counting the  $n$  non-negative constraints, (P) has  $m$  constraints. We use the symbol  $\mathcal{X}$  to denote the set of *feasible solutions* to (P)

$$\mathcal{X} := \{x \in \mathbb{R}^n : Ax = b \text{ and } x \geq 0\}.$$

First note that the problem (P) must fall into one of the following three categories.

- (a) The set  $\mathcal{X}$  is empty. In this case, there are no feasible solutions to (P), and we call the problem (P) *infeasible*
- (b) (P) is feasible, and there is a real number  $\underline{Z} \in \mathbb{R}$  such that  $c^T x \geq \underline{Z}$  for every feasible solution  $x \in \mathcal{X}$ . In this case, we say the problem (P) is *bounded*.
- (c) (P) is feasible, and there exists a sequence of points  $y^1, y^2, \dots$  contained in  $\mathcal{X}$ , such that  $c^T y^k$  goes to minus infinity as  $k$  goes to infinity. In this case, we say the problem (P) is *unbounded*.

### 2.1 The dual linear program

We now introduce another linear program, which is defined from the same data  $A$ ,  $b$  and  $c$  as the linear program (P), and we show that this linear program is closely related to the problem (P). The derivation of this linear program, which is called the *dual* of (P), is motivated by the problem of finding a *lower bound* for the problem (P). A lower bound on (P) is a real number  $\underline{Z} \in \mathbb{R}$  that satisfies

$$c^T x \geq \underline{Z} \text{ for all } x \in \mathcal{X}.$$

Observe that (b) above states that the problem (P) is bounded, if (P) is feasible, and there exists a lower bound for (P). Also, (c) states that (P) is unbounded, if (P) is feasible and does not have a lower bound.

We next argue that an  $m$ -dimensional vector  $y \in \mathbb{R}^m$  which satisfies  $A^T y \leq c$  can be used to derive a lower bound on (P). Indeed, suppose  $y \in \mathbb{R}^m$  satisfies  $A^T y \leq c$ . Then for any  $x \in \mathcal{X}$ , we have

$$\begin{aligned} c^T x &\geq (A^T y)^T x \Rightarrow (\text{ since } x \geq 0_n) \\ c^T x &\geq y^T (Ax) \Rightarrow \\ c^T x &\geq b^T y. \quad (\text{ since } Ax = b) \end{aligned}$$

The above derivation shows that, if we can find a vector  $y \in \mathbb{R}^m$  that satisfies  $A^T y \leq c$ , then  $\underline{Z} := b^T y$  is a lower bound for (P). Now consider the linear program (D) of finding the vector  $y$  that gives the tightest lower bound for (P) of this form

$$\begin{aligned} &\text{Maximize } \underline{Z} = b^T y \\ &\text{s.t.} \qquad \qquad \qquad (D) \\ &A^T y \leq c, \end{aligned}$$

The linear program (D) is called the *dual* of the linear program (P). Observe that, in the linear program (D), *all* variables in the vector  $y$  are free. We have just proved the following facts about the linear program (P) and its dual (D).

**Lemma 1** *The following relations between the linear programs (P) and (D) hold.*

- (i) *If (D) is unbounded, then (P) is infeasible.*
- (ii) *If (P) is unbounded, then (D) is infeasible.*
- (iii) *(Weak duality): If  $x$  is feasible for (P) and  $y$  is feasible for (D), then  $c^T x \geq b^T y$ .*

**Example 3** *Consider, again, the linear program that models the problem of the WGC*

$$\begin{aligned} &\text{Maximize } Z = 3x_1 + 5x_2 \\ &\text{subject to} \\ &\quad x_1 + s_1 = 4 \\ &\quad 2x_2 + s_2 = 12 \\ &\quad 3x_1 + 2x_2 + s_3 = 18 \\ &s_1 \geq 0, s_2 \geq 0, s_3 \geq 0, x_1 \geq 0, x_2 \geq 0, \end{aligned}$$

where we used the version of this problem obtained after adding slack variables in the first three constraints. In the above formulation, we therefore have  $n = 5$  variables and  $m = 3$ . In matrix form, and switching from “Maximize” to “Minimize”, this problem can be written as

$$\begin{aligned} &\text{Minimize } Z' = c^T x \\ &\text{s.t.} \\ &Ax = b, \\ &x \geq 0_5, \end{aligned}$$

where  $A$ ,  $b$ ,  $c$  and  $x$  are given by

$$A = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 \\ 3 & 2 & 0 & 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 4 \\ 12 \\ 18 \end{pmatrix}, \quad c = \begin{pmatrix} -3 \\ -5 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad x = \begin{pmatrix} x_1 \\ x_2 \\ s_1 \\ s_2 \\ s_3 \end{pmatrix}$$

The dual linear program of the above linear program can now be written as

$$\begin{aligned} &\text{Maximize } \underline{Z} = 4y_1 + 12y_2 + 18y_3 \\ &\text{subject to} \\ &\quad y_1 + 3y_3 \leq -3 \\ &\quad 2y_2 + 2y_3 \leq -5 \\ &y_1 \leq 0, y_2 \leq 0, y_3 \leq 0 \end{aligned}$$

Consider the feasible solution  $(y_1^*, y_2^*, y_3^*) = (0, -\frac{3}{2}, -1)$  to the dual linear program. We have  $b^T y^* = -36$ . As mentioned earlier, the solution  $(x_1^*, x_2^*, s_1^*, s_2^*, s_3^*) = (2, 6, 2, 0, 0)$  is a feasible solution to the WGC problem with  $-3x_1^* - 5x_2^* = -36$ . It follows from Lemma 1 (weak duality) that  $(x_1^*, x_2^*, s_1^*, s_2^*, s_3^*)$  must be an optimal solution to the WGC problem. Furthermore, we have that  $(y_1^*, y_2^*, y_3^*)$  is an optimal solution to the dual problem above.

## 2.2 Dual problems of other forms

In Sect. 2.1, we defined the dual problem (D) relative to a problem (P) in “equality form”. In other words, the only constraints that were not equalities were the non-negativity constraints. Since a linear program can be written in many different forms (with or without slack/surplus variables etc.), the dual of a linear program can also take different forms. It is therefore convenient to have a direct method for writing down the dual of a problem also in the case when the problem is not necessarily in equality form. We now give the dual problem of some other forms of a linear program.

**Lemma 2** *The dual of the linear program (P')*

$$\begin{aligned} \text{Minimize } Z &= c^T x \\ \text{s.t.} \\ Ax &\geq b, \\ x &\geq 0_n, \end{aligned}$$

is the linear program (D')

$$\begin{aligned} \text{Maximize } Z' &= b^T y \\ \text{s.t.} \\ A^T y &\leq c, \\ y &\geq 0_m, \end{aligned}$$

*Proof:* Introducing surplus variables in the problem (P') gives the problem

$$\begin{aligned} \text{Minimize } Z &= [c^T, 0_m^T] \begin{bmatrix} x \\ s \end{bmatrix} \\ \text{s.t.} \\ [A, -I_m] \begin{bmatrix} x \\ s \end{bmatrix} &= b, \\ x \geq 0_n, s &\geq 0_m, \end{aligned}$$

where  $I_m$  denotes the  $m \times m$  identity matrix. Taking the dual of this problem then gives the problem

$$\begin{aligned} \text{Maximize } Z' &= b^T y \\ \text{s.t.} \\ \begin{bmatrix} A^T \\ -I_m \end{bmatrix} y &\leq \begin{bmatrix} c \\ 0_m \end{bmatrix}. \end{aligned}$$

■

**Lemma 3** *The dual of the linear program (D)*

$$\begin{aligned} & \text{Maximize } Z' = b^T y \\ & \text{s.t.} \\ & A^T y \leq c, \end{aligned}$$

is the linear program (P)

$$\begin{aligned} & \text{Minimize } Z = c^T x \\ & \text{s.t.} \\ & Ax = b, \\ & x \geq 0_n, \end{aligned}$$

In other words, taking the dual of the dual of a linear program gives the original linear program.

*Proof:* Introducing slack variables in the problem (D), changing from maximize to minimize and introducing positive and negative parts of  $y$  with  $y = y^+ - y^-$  gives the problem

$$\begin{aligned} & \text{Minimize } Z = [-b^T, b^T, 0_n^T] \begin{bmatrix} y^+ \\ y^- \\ s \end{bmatrix} \\ & \text{s.t.} \\ & [-A^T, A^T, -I_n] \begin{bmatrix} y^+ \\ y^- \\ s \end{bmatrix} = -c, \\ & y^+ \geq 0_m, y^- \geq 0_m, s \geq 0_m, \end{aligned}$$

where  $I_n$  denotes the  $n \times n$  identity matrix. Taking the dual of this problem then gives the problem

$$\begin{aligned} & \text{Maximize } Z = -c^T x \\ & \text{s.t.} \\ & \begin{bmatrix} -A \\ A \\ -I_n \end{bmatrix} x \leq \begin{bmatrix} -b \\ b \\ 0_n \end{bmatrix}. \end{aligned}$$

Now, changing from maximize to minimize exactly gives the problem (P). ■

**Example 4** *Consider, again, the linear program of the WGC written without adding slack variables*

$$\begin{aligned} & \text{Maximize } Z = c^T x \\ & \text{s.t.} \\ & Ax \leq b, \\ & x \geq 0_n, \end{aligned}$$

with  $n = 2$ ,  $m = 3$  and

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \\ 3 & 2 \end{pmatrix}, \quad b = \begin{pmatrix} 4 \\ 12 \\ 18 \end{pmatrix}, \quad c = \begin{pmatrix} 3 \\ 5 \end{pmatrix} \quad \text{and} \quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Note that this form of the linear program of the WGC “fits the profile” of the problem (D’) in Lemma 2 (with  $b$  and  $c$ , and  $m$  and  $n$  interchanged). Since the dual of (D’) is the problem (P’) (here we apply Lemma 3), the dual is

$$\begin{aligned} \text{Minimize } Z &= b^T y \\ \text{s.t.} \\ A^T y &\geq c, \\ y &\geq 0_m, \end{aligned}$$

or equivalently

$$\begin{aligned} \text{Minimize } Z &= 4y_1 + 12y_2 + 18y_3 \\ \text{subject to} \\ y_1 + 3y_3 &\geq 3 \\ 2y_2 + 2y_3 &\geq 5 \\ y_1 \geq 0, y_2 \geq 0, y_3 &\geq 0 \end{aligned}$$

In Example 3 we obtained the following linear program as the dual of the WGC problem

$$\begin{aligned} \text{Maximize } \underline{Z} &= 4y_1 + 12y_2 + 18y_3 \\ \text{subject to} \\ y_1 + 3y_3 &\leq -3 \\ 2y_2 + 2y_3 &\leq -5 \\ y_1 \leq 0, y_2 \leq 0, y_3 &\leq 0 \end{aligned}$$

so what is the reason for the difference between the two problems? The difference is that the dual derived in Example 3 was derived from a different form of the WGC problem (by introducing slack variables and changing maximization to minimization). Indeed, if we make the substitution  $y'_i := -y_i$  for  $i = 1, 2, \dots, m$  in the above linear program, we obtain

$$\begin{aligned} \text{Maximize } \underline{Z}' &= -4y'_1 - 12y'_2 - 18y'_3 \\ \text{subject to} \\ -y'_1 - 3y'_3 &\leq -3 \\ -2y'_2 - 2y'_3 &\leq -5 \\ y'_1 \geq 0, y'_2 \geq 0, y'_3 &\geq 0 \end{aligned}$$

which is clearly equivalent to the dual problem of the WGC problem we obtained from Lemma 2. The key in the difference is that, in Example 3, we changed from maximization to minimization.

### 2.3 Strong duality

Lemma 1 shows that there is a very close relationship between the linear programs (P) and (D). However, we still don’t know whether the inequality in Lemma 1.(iii) is tight. In other words, assuming both linear programs (P) and (D) are feasible and bounded, does there always exist a feasible solution  $x^* \in \mathcal{X}$  for (P), and a feasible solution  $y^*$  for (D) (that is a vector  $y^* \in \mathbb{R}^m$  satisfying  $A^T y^* \leq b$ ) such that  $c^T x^* = b^T y^*$ ? The *strong duality theorem* states that this is indeed the case.

**Theorem 1** *Suppose the linear programs (P) and (D) are both feasible and bounded. Then there is a feasible solution  $x^* \in \mathcal{X}$  for (P), and a feasible solution  $y^*$  for (D), such that  $c^T x^* = b^T y^*$ .*

This theorem is a consequence of the so-called “Farkas Lemma”.

**Lemma 4** *(Farkas Lemma): One of the following two conditions hold, but not both.*

- (i)  $\mathcal{X} \neq \emptyset$ , or in other words, there exists  $x \in \mathbb{R}^n$  such that  $x \geq 0_n$  and  $Ax = b$ .*
- (ii) There exists  $y \in \mathbb{R}^m$  such that  $A^T y \leq 0$  and  $b^T y > 0$ .*

We will not prove Farkas’s Lemma, since the proof is fairly technical.